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ON THE REPLACEMENT OF SYSTEMS SUBJECT TO SHOCKS AND WEAR-DEPENDENT FAILURE

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Abstract

A system is discussed which is subject to a conditional Poisson stream of failures, whose intensity function representing the cumulative damage level is a Markov chain.

CUMULATIVE DAMAGE; PRODUCT DENSITIES; CONDITIONAL POISSON PROCESS; FAILURE COUNTING PROCESS

1. Introduction

Systems subject to a sequence of randomly occurring shocks, each of which increases the cumulative damage, thus degrading their performance, have been investigated by several authors (Abdel-Hameed (1984), (1986), Boland and Proschan (1983), Gottlieb and Levikson (1984) to name a few). We consider a one-unit system with three possible states $X(t)$: normal (0), satisfactory (1) and unsatisfactory (2), the cumulative damage level $\lambda(t)$ in each of these being 0, α and β respectively, $0<\alpha<\beta$. When the system is in states 0 or α, shocks occurring at random times with rates γ_1 and γ_2 respectively raise the damage level to the next higher state. The failure process is assumed to be a conditional Poisson process with intensity function $\lambda(t)$. If the damage level is 0 when the system is in the normal state, failure in that state is not possible, though this could be relaxed. On failure, an unplanned replacement is made and each such replacement increases the running cost by c units per unit of time. A periodic replacement is made after every T units of time, costing c_0 units. Apart from the normal cost per unit time, a, of running the system there is the additional cost per unit time a_i of running the system in state $i=0$, 1 and 2. This note derives the statistical characteristics of the failure process, and obtains the optimal period T^* of replacement.

2. Moments of the failure counting process

We consider the counting process $N(t)$, denoting the number of failures over an arbitrary time interval $(0,t]$. Define

$$p_i(n,t)=\Pr\{N(t)=n,\ X(t)=i\},\ i=0,\ 1 \text{ and } 2 \tag{1}$$

and the marginal probability $\Pi_i(t)=\Pr\{X(t)=i\}=\sum_{n=0}^{\infty}p_i(n,t)$ of the homogeneous Markov chain $\{X(t),\ t\geqq 0\}$. Using the forward Kolmogorov equations for $p_i(n,t)$, we obtain

$$\frac{d}{dt}\Pi_0(t)=-\gamma_1\Pi_0(t)+\alpha\Pi_1(t)+\beta\Pi_2(t), \tag{2}$$

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$$\frac{d}{dt}\Pi_1(t) = -(\alpha+\gamma_2)\Pi_1(t) + \gamma_1\Pi_0(t), \tag{3}$$

$$\frac{d}{dt}\Pi_2(t) = -\beta\Pi_2(t) + \gamma_2\Pi_1(t), \tag{4}$$

with $\sum_{i=0}^{2}\Pi_i(t) = 1$, subject to the initial conditions $\Pi_i(0) = \rho_i$, $i = 0, 1, 2$. The solution $\Pi_i\,(t)$ of the above system of equations is given by

$$\begin{aligned}\Pi_1(t) = {} & \frac{\gamma_1\beta}{\gamma_1\gamma_2 + \beta(\alpha+\gamma_1+\gamma_2)} + \frac{S_1\rho_1(S_1+\beta+\gamma_1) + (S_1\rho_0+\beta)\gamma_1}{S_1(S_1-S_2)}\exp(S_1 t) \\ & + \frac{S_2\rho(S_2+\beta+\gamma_1) + (S_2+\rho_0+\beta)\gamma_1}{S_2(S_2-S_1)}\exp(S_2 t)\end{aligned} \tag{5}$$

$$\begin{aligned}\Pi_2(t) = {} & [\gamma_1\gamma_2/[\gamma_1\gamma_2+\beta(\alpha+\gamma_1+\gamma_2)]] \\ & + [S_1^3\rho_2 + S_1^2(\rho_2(\alpha+\beta+\gamma_1)+\gamma_2) + (\beta\rho_2(\alpha+\gamma_1) + \gamma_2\beta(1-\rho_0) + \gamma_1\gamma_2)S_1 \\ & + \gamma_1\gamma_2\beta]\exp(S_1 t)/[S_1(S_1+\beta)(S_1-S_2)] \\ & + [\rho_2(S_2^3 + S_2^2(\alpha+\beta+\gamma_1) + S_2\beta(\alpha+\gamma_1)) + (1-\rho_0)(S_2+\beta)S_2\gamma_2 \\ & + \gamma_1\gamma_2(S_2+\beta)]\exp(S_2 t)/[S_2(S_2+\beta)(S_2-S_1)],\end{aligned} \tag{6}$$

$$\Pi_0(t) = 1 - \Pi_1(t) - \Pi_2(t), \tag{7}$$

where

$$S_{1,2} = -\tfrac{1}{2}(\alpha+\beta+\gamma_1+\gamma_2) \pm \tfrac{1}{2}[(\beta-\alpha-\gamma_1-\gamma_2)^2 + 4\gamma_1\gamma_2]^{\frac{1}{2}}. \tag{8}$$

The various statistical characteristics of the failure counting process are highlighted by the sequence of product densities $h_n(t_1, t_2, \cdots, t_n)\,dt_1\,dt_2\cdots dt_n$ (Srinivasan (1974)) where

$$\begin{aligned}h_n(t_1, t_2, \cdots, t_n) = {} & \lim_{\Delta_1, \Delta_2, \cdots, \Delta_n \to 0} \Pr\{N(t_1+\Delta_1) - N(t_1) = 1, \\ & N(t_2+\Delta_2) - N(t_2) = 1, \cdots, N(t_n+\Delta_n) - N(t_n) = 1\}/\Delta_1\Delta_2\cdots\Delta_n.\end{aligned} \tag{9}$$

Using elementary probability arguments, we find assuming $\rho_o = 1$, $\rho_1 = \rho_2 = 0$, that

$$h_1(t) = \alpha\Pi_1(t) + \beta\Pi_2(t), \tag{10}$$

and

$$h_2(t_1, t_2) = \alpha\Pi_1(t_1)[\alpha\Pi_{10}(t) + \beta\Pi_{20}(t)] + \beta\Pi_2(t_1)[\alpha\Pi_{10}(t) + \beta\Pi_{20}(t)], \tag{11}$$

where the

$$\Pi_{ij}(t) = \Pr\{X(t) = i/X(0) = j\},\ i, j = 0, 1, 2 \tag{12}$$

are given by Equations (5)–(7) and $t = (t_2 - t_1)$. The mean and variance of the number of failures in $(0, T]$ are given by

$$M(t) = E\{N(T)\} = \int_0^T h_1(t)\,dt = \frac{\beta\gamma_1(\alpha+\gamma_2)T}{\gamma_1\gamma_2 + \beta(\alpha+\gamma_1+\gamma_2)}, \tag{13}$$

$$\operatorname{Var}[N(T)] = \int_0^T h_1(t)\,dt + 2\int_0^T\int_0^T h_2(t_1, t_2)\,dt_1\,dt_2 - \left[\int_0^T h_1(t)\,dt\right]^2. \tag{14}$$

The steady state probabilities are

$$\Pi_0 = \frac{\beta(\alpha+\gamma_2)}{A},\ \Pi_1 = \frac{\gamma_1\beta}{A} \quad\text{and}\quad \Pi_2 = \frac{\gamma_1\gamma_2}{A} \tag{15}$$

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where $A=\gamma_1\gamma_2+\beta(\alpha+\gamma_1+\gamma_2)$. The stationary values of the first- and second-order product densities are readily found to be

$$h_1(t)=\frac{\gamma_1\beta(\alpha+\gamma_2)}{A}, \tag{16}$$

$$\begin{aligned} h_{\text{stat}}(t)=\lim_{\substack{t_1\to\infty\\ t_2\to\infty\\ |t_1-t_2|=t}} h_2(t_1,t_2)&=\frac{\beta\gamma_1(\alpha+\gamma_2)}{A}\left\{\frac{\beta\gamma_1(\alpha+\gamma_2)}{A}\right.\\ &\left.+\frac{\gamma_1(\alpha S_1+\alpha\beta+\gamma_2\beta)}{S_1(S_1-S_2)}\exp(S_1t)+\frac{\gamma_1(\alpha S_2+\alpha\beta+\eta_2\beta)}{S_2(S_2-S_1)}\exp(S_2t)\right\}. \end{aligned} \tag{17}$$

3. Cost analysis

The jumps of the counting process $\{N(t);t\geqq 0\}$ are of unit magnitude. Let $\{t_n\}$ be the sequence of instants t_i of the ith unplanned replacements. The total cost of running the system per period T is given by

$$\begin{aligned} &aT+c(t_2-t_1)+2c(t_3-t_2)+\cdots+c[N(T)-1](t_{N(T)}-t_{N(T)-1})\\ &+c_{N(T)}(T-t_{N(T)})+c_0+\sum_{i=0}^{2}a_iR_i(T), \end{aligned} \tag{18}$$

where $R_i(T)$ is the proportion of time spent by the system in state i during $(0,T)$. Thus the long-run expected cost per unit time, using standard renewal theory arguments, is

$$C(T)=\frac{c_0+aT+c\int_0^T M(t)\,dt+\sum_{i=0}^{2}\int_0^T \Pi_i(t)\,dt}{T}, \tag{19}$$

where $M(t)$ is the expected number of failures in $(0,t)$, assumed to be continuous, and $\Pi_i(t)=\Pi_i(t)$, with $\rho_o=1$, $\rho_1=p_2=0$ given by Equations (5)–(7). A finite optimal T^* of (19) exists and is unique if

$$i=\sum_0^2 a_i\Pi_i'(t)+c[\alpha\Pi_1(t)+\beta\Pi_2(t)]\geqq 0.$$

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