End-to-end Focal Chords in an Ellipse.

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§ 1. The relation between the eccentric and focal angles of P (Fig. 1) is

$$\cos\phi = \frac{e + \cos\theta}{1 + e\cos\theta},$$

or, more compactly, $\tan\dfrac{\phi}{2} = k\tan\dfrac{\theta}{2},$

where $k = \sqrt{\dfrac{1-e}{1+e}} = \tan\dfrac{\epsilon}{2},$ with $e = \cos\epsilon.$

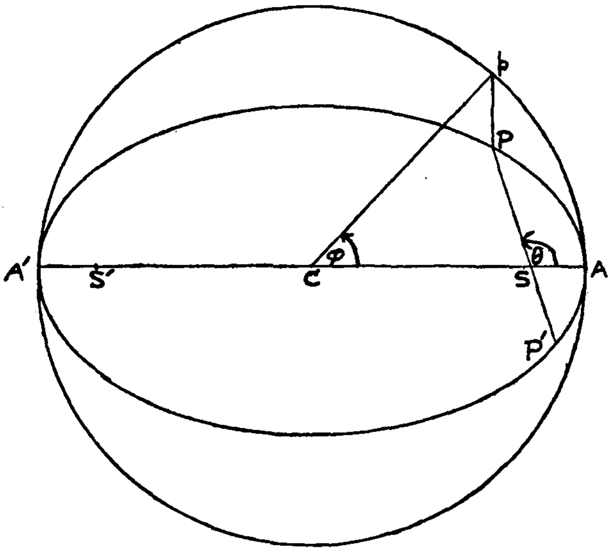

Fig. 1.

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If the eccentric angles of all points on the ellipse be reckoned from the same initial position CA, then, ϕ' being the eccentric angle of P′:

$$\phi' \text{ is negative, and } \cos\phi' = \frac{e+\cos(\theta-\pi)}{1+e\cos(\theta-\pi)},$$

whence
$$\tan\frac{\phi'}{2}\tan\frac{\theta}{2} = -k.$$

Thus for the extremities of a chord through S,

$$\tan\frac{\phi}{2}\tan\frac{\phi'}{2} = -k^2.$$

Likewise *for the extremities of a chord through* S′,

$$\tan\frac{\phi}{2}\tan\frac{\phi'}{2} = -\frac{1}{k^2}.$$

Now $t\left(=\tan\frac{\phi}{2}\right)$ being the parameter of any point, P, of the ellipse, the tangents at t, t' intersect at a point whose coordinates are given by

$$\frac{x}{a} = \frac{\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')} = \frac{1-tt'}{1+tt'}, \quad \frac{y}{b} = \frac{\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')} = \frac{t+t'}{1+tt'}.$$

Hence the tangents at the ends of a chord through S meet on the line

$$\frac{x}{a} = \frac{1+k^2}{1-k^2} = \frac{1}{\cos\epsilon} = \frac{1}{e},$$

and those at the ends of a chord through S′ meet on the line

$$\frac{x}{a} = \frac{k^2+1}{k^2-1} = -\frac{1}{e};$$

and these are the two directrices. This theorem and method will be generalised.

Let a system of chords be drawn, starting at any point P_0, and passing alternately through S and S′: say

$$P_0SP_1,\ P_1S'P_2,\ \ldots P_{2r}SP_{2r+1},\ P_{2r+1}S'P_{2r+2},\ \ldots.$$

If the eccentric angle of P_r is α_r, and that of P_0 is α, we have

$$\tan\frac{\alpha}{2}\tan\frac{\alpha_1}{2} = -k^2,$$

$$\tan\frac{\alpha_1}{2}\tan\frac{\alpha_2}{2} = -\frac{1}{k^2},$$

....

$$\tan\frac{\alpha_{2r}}{2}\tan\frac{\alpha_{2r+1}}{2} = -k^2,$$

$$\tan\frac{\alpha_{2r+1}}{2}\tan\frac{\alpha_{2r+2}}{2} = -\frac{1}{k^2}.$$

Hence
$$\tan\frac{\alpha}{2} = k^4\tan\frac{\alpha_2}{2},$$

$$\tan\frac{\alpha}{2}\tan\frac{\alpha_3}{2} = -k^6;$$

and in general
$$\tan\frac{\alpha_{2r}}{2} = k^{-4r}\tan\frac{\alpha}{2}.$$

$$\tan\frac{\alpha}{2}\tan\frac{\alpha_{2r+1}}{2} = -k^{4r+2}.$$

The last relation shows that, for all values of r, the tangents at P_0, P_{2r+1} will meet on the line

$$\frac{x}{a} = \frac{1+k^{4r+2}}{1-k^{4r+2}},$$

which is independent of the position of P_0.

Again, the chord joining the points t, t' has for its equation

$$\frac{x}{a}(1-tt') + \frac{y}{b}(t+t') = (1+tt').$$

Hence the equation of the chord P_0P_{2r} is

$$\frac{x}{a}(1-k^{-4r}t^2) + \frac{y}{b}(1+k^{-4r})t = 1 + k^{-4r}t^2,$$

i.e.
$$\left(1+\frac{x}{a}\right)k^{-4r}t^2 - \frac{y}{b}(1+k^{-4r})t + \left(1-\frac{x}{a}\right) = 0,$$

t standing for the parameter of P_0.

The envelope of this is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b_1^2}=1, \text{ where } \frac{b_1}{b}=\frac{2k^{2r}}{1+k^{4r}}.$$

Also the coordinates of the intersection of the tangents at P_0, P_{2r} are given by

$$\frac{x}{a}=\frac{1-k^{-4r}t^2}{1+k^{-4r}t^2}, \quad \frac{y}{b}=\frac{(1+k^{-4r})t}{1+k^{-4r}t^2},$$

and hence its locus is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b_2^2}=1, \text{ where } \frac{b}{b_2}=\frac{2k^{2r}}{1+k^{4r}}.$$

Thus for the two chords P_0SP_1, $P_1S'P_2$, the envelope of P_0P_2 is the ellipse with major axis AA′, and minor semi-axis

$$b_1=b\frac{2k^2}{1+k^4}=a\frac{(1-e^2)^{\frac{3}{2}}}{1+e^2},$$

while the locus of the intersection of tangents at P_0, P_2 is the ellipse with minor axis AA′, and major semi-axis

$$b_2=b\frac{1+k^6}{2k^2}=a\frac{1+e^2}{\sqrt{(1-e^2)}}.$$

If we take a third chord P_2SP_3, the locus of intersections of tangents at P_0, P_3 is the straight line

$$\frac{x}{a}=\frac{1+k^6}{1-k^6};$$

while, if the first chord were to pass through S′ instead of S, the locus would be

$$\frac{x}{a}=-\frac{1+k^6}{1-k^6}.$$

§ 2. The general relation, then, has been found between the eccentric angles, measured from CA, of the extremities of successive focal chords, drawn end-to-end. It will now be shown that the focal angles of the extremities form intermediate terms in the series of eccentric angles. For the purposes of this discussion, we shall no longer measure all angles from CA, but those beneath the axes we shall measure from CA′, still in the anti-clockwise sense.

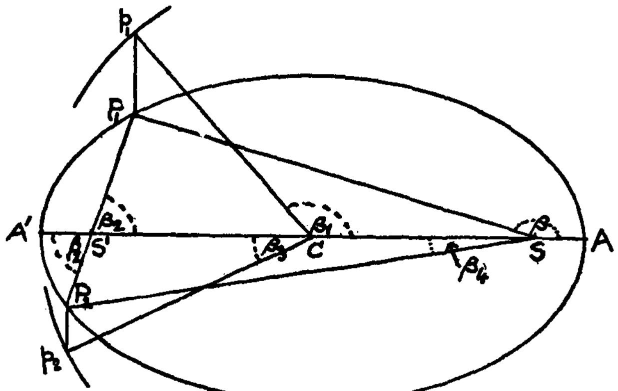

Fig. 2.

Starting from S, draw SP_1, $P_1S'P_2$, P_2SP_3, … ; denote the angles of the figure as follows:

$$\angle\, ASP_1 = \beta, \quad \angle\, A'S'P_2 = \beta_2$$
$$\angle\, ACp_1 = \beta_1, \quad \angle\, A'Cp_2 = \beta_3$$
$$\angle\, AS'P_1 = \beta_2, \quad \angle\, A'SP_2 = \beta_4, \text{ etc.}$$

Then the point P_{2r-1} will be above AA′, and we shall have

$$\angle\, ASP_{2r-1} = \beta_{4r-4}, \quad \angle\, ACp_{2r-1} = \beta_{4r-3}, \quad \angle\, AS'P_{2r-1} = \beta_{4r-2};$$

while P_{2r} will be below AA′, and we shall have

$$\angle\, A'SP_{2r} = \beta_{4r-2}, \quad \angle\, A'Cp_{2r} = \beta_{4r-1}, \quad \angle\, A'SP_{2r} = \beta_{4r}.$$

Then by the well-known formula,

$$\cos\beta_1 = \frac{e + \cos\beta}{1 + e\cos\beta}, \quad i.e. \quad \frac{\tan\frac{\beta_1}{2}}{\tan\frac{\beta}{2}} = k.$$

But from the figure,

$$\cos\beta_2 = \frac{e + \cos\beta_1}{1 + e\cos\beta_1}, \quad i.e. \quad \frac{\tan\frac{\beta_2}{2}}{\tan\frac{\beta_1}{2}} = k.$$

It is now clear that the same relation holds between each consecutive pair of β's, viz.,

$$\cos\beta_n = \frac{e+\cos\beta_{n-1}}{1+e\cos\beta_{n-1}}, \text{ whence } \frac{\tan\frac{\beta_n}{2}}{\tan\frac{\beta_{n-1}}{2}} = k.$$

Hence $\tan\frac{1}{2}\beta_n = k^n \tan\frac{1}{2}\beta$: and we deduce

$$\cos\beta_n = \frac{e_n+\cos\beta}{1+e_n\cos\beta}, \text{ where } e_n = \frac{1-k^{2n}}{1+k^{2n}}.$$

Thus the result of n repetitions of the linear transformation $\left(x, \dfrac{e+x}{1+ex}\right)$ is one of the same type, e being replaced by

$$e_n = \frac{(1+e)^n-(1-e)^n}{(1+e)^n+(1-e)^n};$$

and the process is illustrated by the cosines of the successive focal and eccentric angles in a set of end-to-end focal chords.

The relation of the odd β's in this notation to the a's of the previous section is obvious.

§ 3. Two simple results may now be referred to.

(i) If the two *interior* angles PSS′, PS′S be denoted by γ, γ', then since $\gamma = \pi - \beta$, $\gamma' = \beta_2$, we have $\tan\frac{1}{2}\gamma\tan\frac{1}{2}\gamma' = k^2$, the relation between the dipolar angles for the ellipse.

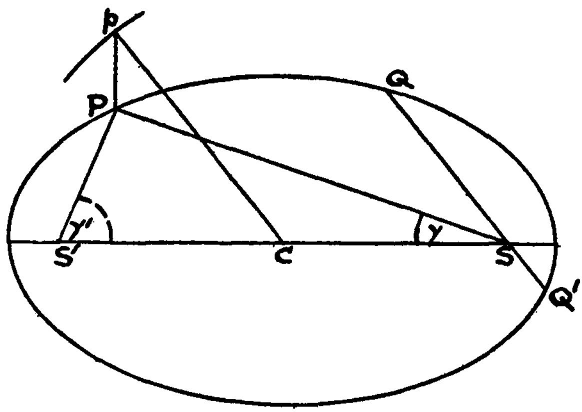

Fig. 3.

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(ii) If Cp, the eccentric ray of P, is parallel to one of the focal distances of Q, then the eccentric ray of Q is parallel to one of the focal distances of P.

Or in another form: If Cp is the eccentric ray parallel to a focal chord QS′Q′, then the eccentric rays of Q, Q′ are parallel respectively to S′P, SP.

This follows at once from the fact that if we take β_1 as a new value of β, β_2 will be the new value of β_1.