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Almost Disjointness Preservers

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Abstract. We study the stability of disjointness preservers on Banach lattices. In many cases, we prove that an "almost disjointness preserving" operator is well approximable by a disjointness preserving one. However, this approximation is not always possible, as our examples show.

1 Introduction

Recall that an operator T between Banach lattices E and F is called *disjointness preserving* (DP for short) if $Tx \perp Ty$ whenever $x \perp y$. Such operators have been investigated intensively, and are known to possess many remarkable properties (see [9], [23, Chapter 3], or the survey paper [16]). For instance, it is known that any DP operator on $C(K)$ is a weighted composition [23, Section 3.1]. A similar result was shown for DP maps on Köthe spaces [25]. For many other kinds of spaces, the general form of a DP map is also known [5,17,21]. Compact DP maps on $C(K)$ have been described in [22]. Moreover, the inverse of a DP map is again DP [9].

In this paper, we investigate the "stability" of being disjointness preserving. To be more specific, suppose E and F are Banach lattices. We say that an operator $T: E \to F$ is *ε-disjointness preserving* (ε-DP for short) if for any disjoint $x, y \in E$,

$$\||Tx| \wedge |Ty|\| \le \varepsilon \max\{\|x\|, \|y\|\}.$$

Note that 0-DP operators are precisely the disjointness preserving operators.

Note that if T is ε-DP, then for any scalar λ, λT is $|\lambda|\varepsilon$-DP. Clearly, every operator T is $\|T\|$-DP, so the above notion is only interesting for $\varepsilon < \|T\|$.

The goal of this paper is to investigate the properties of ε-DP operators, and furthermore, to determine whether such operators can be approximated by disjointness preserving ones. More precisely, for what ε-DP operators T does there exist a DP map S with $\|T - S\| \le \phi(\varepsilon, \|T\|)$, where $\lim_{\varepsilon\to 0} \phi(\varepsilon, t) = 0$ for every t?

This question has been considered previously on spaces of continuous functions, namely, G. Dolinar [13] (and later J. Araujo and J. Font [6–8], as well as R. Kantrowitz and M. Neumann [18]) considered a formally different notion of almost disjointness preserving operators between $C(K)$ spaces. More precisely, suppose $E = C(K_E)$ and $F = C(K_F)$. We say that $T: E \to F$ is *Dolinar ε – DP* if

$$\|(Tx)(Ty)\| \le \varepsilon \|x\| \|y\|$$

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for any disjoint x and y. It is easy to see that if $T: C(K_E) \to C(K_F)$ is Dolinar ε-DP, then it is $\sqrt{\varepsilon}-$DP; and, in the converse direction, if $T: C(K_E) \to C(K_F)$ is ε-DP, then it is Dolinar $\|T\|\varepsilon$-DP. Improving the results of [13], Arajo and Font [6]showed that if T is a Dolinar $\varepsilon-$DP contraction $(0 < \varepsilon < 2/17)$, then there exists a (disjointness preserving) weighted composition operator S so that $\|T - S\| < \sqrt{17\varepsilon/2}$. They improved on this for linear functionals [7].

The paper is organized as follows: Section 2 is devoted to collecting basic facts about ε-DP operators. In Section 3, we establish a probabilistic inequality (to be used throughout our work), and list some of its consequences.

In Section 4 we show that positive ε-DP operators from c_0 or c into a Banach lattice with the Fatou property can be nicely approximated by DP operators (Theorem 4.1). Our main technical tool is an inequality from Lemma 3.1, which may be of interest in its own right.

In Section 5, we show that any ε-DP operator from a symmetric sequence space into a σ-Dedekind complete $C(K)$ space can be approximated by DP maps (Theorem 5.1).

Section 6 is devoted to proving that any positive ε-DP operator from ℓ_p into L_p can be approximated by a DP one (Theorems 6.1 and 6.2). In Section 7, we prove similar approximation results for operators from a sequence space with a shrinking basis to L_1.

In Section 8 we show that for $1 \le p < q < \infty$ and any $\varepsilon > 0$, there exists a positive ε-DP contraction $T: \ell_p \to \ell_q$ so that $\|T - S\| \ge 1/2$ for any DP map S (Proposition 8.1). Similar results hold for operators from ℓ_p into a certain class of Banach lattices, including L_q (Proposition 8.3).

Section 9 deals with the connections between the properties of an operator and its modulus. We start by observing that if $T \in B(E, F)$ is regular and $|T|$ is ε-DP, then the same holds for T. Under some conditions on E and F, the converse is true (Proposition 9.1). In general, Proposition 9.4 provides a counterexample.

Finally, in Section 10 we explore notions closely related to ε-DP operators, such as almost-lattice homomorphisms, and operators almost preserving expressions of the form $(|x|^p + |y|^p)^{1/p}$. Further, we explore the connections between ε-DP operators, and operators "almost preserving" order (Proposition 10.1). We also consider a stronger version of ε-DP operators for which approximation results hold in a general setting (see Theorem 10.6).

Throughout this paper, we use standard Banach lattice terminology and notation, as well as some well known facts. For more information we refer the reader to any of the excellent monographs on the topic, such as [3] or [23]. For the peculiarities of complex Banach lattices, one may consult [2].

2 Basic Facts

We start with a few easy observations. First, almost disjointness preservation only needs to be verified on positive elements.

Proposition 2.1 *Suppose E and F are real (complex) Banach lattices. If $T \in B(E,F)$ is such that $\||Tx| \wedge |Ty|\| \leq \varepsilon$ for any positive disjoint $x, y \in \mathbf{B}(E)$, then T is 4ε-DP (16ε-DP in the complex case). Moreover, if T is positive, then it is ε-DP.*

Proof Suppose that T is positive. Then for every $z \in E$, we have $|Tz| \leq T|z|$ (see [2, Lemma 3.22]). If x and y are disjoint, then

$$\big\| |Tx| \wedge |Ty| \big\| \leq \big\| T|x| \wedge T|y| \big\| \leq \varepsilon.$$

For general T in the real case, write $x = x_+ - x_-$, and $y = y_+ - y_-$ (here $x \perp y$). Then

$$\begin{aligned} \big\| |Tx| \wedge |Ty| \big\| &\leq \big\| \big(|Tx_+| + |Tx_-| \big) \wedge \big(|Ty_+| + |Ty_-| \big) \big\| \leq \sum_{\sigma,\delta=\pm} \big\| |Tx_\sigma| \wedge |Ty_\delta| \big\| \\ &\leq \varepsilon \sum_{\sigma,\delta=\pm} \max\{\|x_\sigma\|, \|y_\delta\|\} \leq 4\varepsilon. \end{aligned}$$

The complex case is dealt with similarly. ∎

Furthermore, almost disjointness preserving operators also preserve "almost disjointness".

Proposition 2.2 *Suppose E and F are real Banach lattices, and $T \in B(E,F)$ is ε-DP. Then*

$$\big\| |Tx| \wedge |Ty| \big\| \leq 4\big(\varepsilon \max\{\|x\|, \|y\|\} + \|T\| \big\| |x| \wedge |y| \big\| \big)$$

for any $x, y \in E$. In the complex case, a similar inequality holds, with 16 instead of 4.

Proof We prove the real case. Suppose that x and y are positive. Then $x' = x - x \wedge y$ and $y' = y - x \wedge y$ are disjoint, and therefore,

$$\big\| |Tx'| \wedge |Ty'| \big\| \leq \varepsilon \max\{\|x'\|, \|y'\|\} \leq \varepsilon \max\{\|x\|, \|y\|\}.$$

However,

$$\begin{aligned} \big\| |Tx| \wedge |Ty| \big\| &\leq \Big\| (|Tx'| + |T(x \wedge y)|) \wedge (|Ty'| + |T(x \wedge y)|) \Big\| \\ &= \big\| |Tx'| \wedge |Ty'| + |T(x \wedge y)| \big\| \leq \big\| |Tx'| \wedge |Ty'| \big\| + \|T(x \wedge y)\| \\ &\leq \varepsilon \max\{\|x\|, \|y\|\} + \|T\| \|x \wedge y\|. \end{aligned}$$

For general $x, y \in E$, use the Riesz decompositions $x = x_+ - x_-$ and $y = y_+ - y_-$. For $\sigma, \delta = \pm$, we have $x_\sigma \wedge y_\delta \leq |x| \wedge |y|$. Hence $\|x_\sigma \wedge y_\delta\| \leq \||x| \wedge |y|\|$. By the above,

$$\begin{aligned} \big\| |Tx_\sigma| \wedge |Ty_\delta| \big\| &\leq \varepsilon \max\{\|x_\sigma\|, \|y_\delta\|\} + \|T\| \|x_\sigma \wedge y_\delta\| \\ &\leq \varepsilon \max\{\|x\|, \|y\|\} + \|T\| \||x| \wedge |y|\|. \end{aligned}$$

To finish the proof, recall that $|Tx| \wedge |Ty| \leq \sum_{\sigma,\delta=\pm} |Tx_\sigma| \wedge |Ty_\delta|$. ∎

Finally, we show that if a Banach lattice E is "diffuse enough" and F is "atomic enough", then the norm of an ε-DP operator from E to F cannot exceed 2ε. We say that a Banach lattice E has *Fatou norm with constant* $\mathfrak{f}$ if, for any non-negative increasing net $(x_i) \subset E$, with $\sup_i \|x_i\| < \infty$, we have $\bigvee_i x_i \in E$, and $\|\bigvee_i x_i\| \leq \mathfrak{f} \sup_i \|x_i\|$. Recall that $x \in E_+ \backslash \{0\}$ is called an *atom* of E if it generates a one-dimensional principal ideal E_x. In this case, E_x is actually a projection band [28, Proposition 4.18]. Moreover, x is an atom if and only if whenever $0 \leq x_1, x_2 \leq x$, and $x_1 \perp x_2$, then either $x_1 = 0$

or $x_2 = 0$. A Banach lattice is called *atomic* if it is generated by its atoms as a band [23, §2.5].

Proposition 2.3 *Suppose E and F are Banach lattices, so that E is order continuous and has no atoms, while F is atomic, and has Fatou norm with constant $\mathfrak{f}$. If $T: E \to F$ is ε-DP, then $\|T\| \le 2\varepsilon\mathfrak{f}$.*

The restriction on E being order continuous is essential. For instance, suppose $E = C(K)$ and F is 1-dimensional. Then any scalar multiple of a point evaluation is a DP functional (see [13] for the proof that any ε-DP functional is close to a scalar multiple of a point evaluation).

Proof Denote the atoms of F by $(\delta_i)_{i\in I}$. By the discussion above, for every $i \in I$, $\operatorname{span}[\delta_i]$ is the range of a band projection. We denote this band projection by P_i, and write $P_i x = \langle f_i, x\rangle \delta_i$, where $f_i \in F_+^*$. For a finite set $A \subset I$, define the "basis" projection $Q_A = \sum_{i\in A} P_i$. It is easy to see [26, pp. 142-144] that for any $y \in F$, the net $(Q_A y)$ converges to y in the order topology (here, the net of finite subsets of I is ordered by inclusion).

Fix $c < \|T\|$, and find $x \in E$ so that $\|x\| \le 1$ and $\|Tx\| > c$. Further, find a finite set A so that $\|Q_A Tx\| > c/\mathfrak{f}$. Let P_x be the band projection corresponding to $|x|$, and denote its image by G. Note that G inherits the lack of atoms from E. Indeed, suppose, for the sake of contradiction, that $y \in G_+$ is an atom of G. By [23, Lemma 2.7.12], there exist non-zero disjoint $y_1, y_2 \in E_+$ so that $y = y_1 + y_2$. By the properties of band projections, $y_1, y_2 \in G$.

By [20, Theorem 1.b.14], we can view G as a Köthe function space on (Ω, μ). The proof (in conjunction with the characterization of atoms given above) actually constructs a measure μ without atoms. Moreover, there exist μ-measurable functions ϕ_i so that for every $y \in G$, $\langle f_i, Ty\rangle = \int_\Omega \phi_i y \, d\mu$. By Liapounoff's theorem (see [20, Theorem 2.c.9]), there exists a subset $S \subset \Omega$ so that the equality

$$\langle f_i, T(x\mathbf{1}_S)\rangle = \langle f_i, T(x\mathbf{1}_{S^c})\rangle = \frac{\langle f_i, Tx\rangle}{2}$$

holds for any $i \in A$. As Q_A is a band projection, we have for every $z \in F$,

$$Q_A|z| = |Q_A z| = \sum_{i\in A} |\langle f_i, z\rangle| \delta_i.$$

Consequently, $Q_A|Tx| = \sum_{i\in A} \big|\langle f_i, Tx\rangle\big| \delta_i = 2Q_A|T(x\mathbf{1}_S)| = 2Q_A|T(x\mathbf{1}_{S^c})|$. Hence

$$\big\| |T(x\mathbf{1}_S)| \wedge |T(x\mathbf{1}_{S^c})| \big\| \ge \frac{1}{2}\big\| Q_A|Tx| \big\| > \frac{c}{2\mathfrak{f}}.$$

However, $x\mathbf{1}_S$ and $x\mathbf{1}_{S^c}$ belong to $\mathbf{B}(X)$, hence $\big\| |T(x\mathbf{1}_S)| \wedge |T(x\mathbf{1}_{S^c})| \big\| \le \varepsilon$. To complete the proof, recall that c can be arbitrarily close to $\|T\|$. ■

3 A Probabilistic Inequality

The following lemma may be interesting in its own right.

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Lemma 3.1 *Suppose $(b_i)_{i=0}^n$ is a family of non-negative numbers. Then*

$$\mathbb{E}_S \min\Big\{ \sum_{i\in S} b_i, \sum_{i\in S^c} b_i \Big\} \le \Big(\sum_{i=0}^n b_i - \max_{0\le i\le n} b_i \Big) \le 2^8\, \mathbb{E}_S \min\Big\{ \sum_{i\in S} b_i, \sum_{i\in S^c} b_i \Big\}.$$

Here the expected value is taken over all subsets $S \subset \{0,\dots,n\}$, with equal weight.

Proof Clearly, for every $S \subset \{0,\dots,n\}$ we have

$$\min\Big\{ \sum_{i\in S} b_i, \sum_{i\in S^c} b_i \Big\} \le \sum_{i=0}^n b_i - \max_{0\le i\le n} b_i,$$

and therefore, the first inequality of the claim follows.

For the second one, without loss of generality, we can assume $1 = b_0 \ge b_1 \ge \cdots \ge b_n \ge 0$, and set $b = b_1 + \cdots + b_n$. For $S \subset \{0,\dots,n\}$, let $f(S) = \sum_{i\in S} b_i$ and $g(S) = \min\{f(S), f(S^c)\}$.

Consider two cases.

Case 1: $b \le 2^7$. For $S \subset \{0,\dots,n\}$ set $S' = S$ if $0 \notin S$, and $S' = S^c$ otherwise. Then S' is uniformly distributed over subsets of $\{1,\dots,n\}$. Then

$$2^{-7} \sum_{i\in S'} b_i \le 2^{-7} b \le 1 \le \sum_{i\in\{0,\dots,n\}\setminus S'} b_i.$$

Hence $g(S) \ge 2^{-7} \sum_{i\in S'} b_i$. Note that S' is uniformly distributed over subsets of $\{1,\dots,n\}$. Hence $\mathbb{E}_S g(S) \ge 2^{-7} \mathbb{E}_{S'\subset\{1,\dots,n\}} \sum_{i\in S'} b_i = 2^{-7}\cdot \frac{b}{2} = 2^{-8} b$.

Case 2: $b > 2^7$. Note that $\sum_{i=0}^n b_i^2 \le \sum_{i=0}^n b_i = b+1$. By the large deviation inequality for Bernoulli random variables (see [24, Chapter 7]),

$$\begin{aligned}\mathbb{P}\Big(\big|b+1-2\sum_{i\in S} b_i\big| \ge (b+1)/4 \Big) &\le 2\exp\Big(-((b+1)/4)^2/(4(b+1)) \Big) \\ &= 2e^{-(b+1)/64} < 2e^{-1} < 0.74.\end{aligned}$$

Thus, with probability greater than 0.26, $\sum_{i\in S} b_i \in \big[\frac{b+1}{4}, \frac{3(b+1)}{4} \big]$; hence $g(S) \ge (b+1)/4$. Therefore,

$$\mathbb{E} g(S) \ge 0.26\Big(\frac{b+1}{4} \Big) > 2^{-5} b.$$

Thus, each of the cases gives the desired result. ∎

Now an application of Krivine functional calculus [20, Theorem 1.d.1] yields the following.

Corollary 3.2 *If $f_1,\dots,f_n$ are positive elements in a Banach lattice, then*

$$\mathbb{E}_S \min\Big\{ \sum_{i\in S} f_i, \sum_{i\in S^c} f_i \Big\} \ge 2^{-8}\Big(\sum_{i=1}^n f_i - \bigvee_{1\le i\le n} f_i \Big).$$

Consequently, $\mathbb{E}_S \big\| \min\big\{ \sum_{i\in S} f_i, \sum_{i\in S^c} f_i \big\} \big\| \ge 2^{-8} \big\| \sum_{i=1}^n f_i - \bigvee_{1\le i\le n} f_i \big\|$.

As a consequence, we have the following corollary.

Corollary 3.3 *Suppose $T\colon E \to F$ is a positive operator that is ε-DP. Then for any disjoint $x_1, \dots, x_n \in E$, we have*

$$\Big\| \sum_{i=1}^{n} |Tx_i| - \bigvee_{i=1}^{n} |Tx_i| \Big\| \le 256\varepsilon \Big\| \sum_{i=1}^{n} x_i \Big\| .$$

In particular, for any disjoint $x_1, \dots, x_n \in E$ and every $1 \le p < \infty$, it also holds that

$$\Big\| \Big(\sum_{i=1}^{n} |Tx_i|^p \Big)^{\frac{1}{p}} - T\Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}} \Big\| \le 256\varepsilon \Big\| \sum_{i=1}^{n} x_i \Big\| .$$

Proof For any $S \subset \{1, \dots, n\}$, we have

$$\Big\| \Big(\sum_{i \in S} |Tx_i| \Big) \wedge \Big(\sum_{i \in S^c} |Tx_i| \Big) \Big\| \le \Big\| T\Big| \sum_{i \in S} x_i \Big| \wedge T\Big| \sum_{i \in S^c} x_i \Big| \Big\| \le \varepsilon \Big\| \sum_{i=1}^{n} x_i \Big\| .$$

Now apply Corollary 3.2 with $f_i = |Tx_i|$.

For the second inequality, note that for every $1 \le p < \infty$, we have

$$0 \le \Big(\sum_{i=1}^{n} |Tx_i|^p \Big)^{\frac{1}{p}} - T\Big(\sum_{i=1}^{n} |x_i|^p \Big)^{\frac{1}{p}} \le \sum_{i=1}^{n} |Tx_i| - \bigvee_{i=1}^{n} |Tx_i|. \qquad \blacksquare$$

Corollary 3.4 *Suppose the operator $T \in B(E,F)_+$ is ε-DP, and E is σ-Dedekind complete. Then for any $x_1, \dots, x_n \in E_+$, we have*

$$\max\Big\{ \Big| T\Big(\bigvee_{i=1}^{n} x_i\Big) - \bigvee_{i=1}^{n} (Tx_i) \Big\|, \Big\| \bigwedge_{i=1}^{n} (Tx_i) - T\Big(\bigwedge_{i=1}^{n} x_i\Big) \Big\| \Big\} \le 256\varepsilon \Big\| \bigvee_{i=1}^{n} x_i \Big\|.$$

Proof First prove that

$$\Big\| T\Big(\bigvee_{i=1}^{n} x_i\Big) - \bigvee_{i=1}^{n} (Tx_i) \Big\| \le 256\varepsilon \Big\| \bigvee_{i=1}^{n} x_i \Big\|. \tag{3.1}$$

Fix $c > 0$ and let $x = x_1 + \dots + x_n$. Let $\mathcal{C}$ be the set of *components* of x, *i.e.*, of vectors $y \in E_+$ satisfying $y \wedge (x - y) = 0$. By [3, Theorem 1.49], $\mathcal{C}$ is closed under the operations $\vee$ and $\wedge$. Moreover, if $u, v \in \mathcal{C}$ are such that $u \le v$, then $v - u \in \mathcal{C}$. Finite linear combinations of elements of $\mathcal{C}$ are called *simple functions*.

By [23, Proposition 1.2.20], E has the principal projection property. By the Freudenthal spectral theorem (see [3, Theorem 2.8]), for every i there exists a simple function u_i so that $0 \le x_i - u_i \le c|x|/\|x\|$ (hence $\|u_i - x_i\| \le c$). By considering $u_i \vee 0$ instead of u_i, we can assume that all the u_i's are non-negative. Write $u_i = \sum_{j=1}^{N_i} \alpha_{ij} v_{ij}$, where $\alpha_{ij} > 0$ and $(v_{ij})_{j=1}^{N_i}$ are disjoint components of x. By the discussion above, the elements $\bigwedge_{i=1}^{n} v_{ij_i}$ for any $j_i \le N_i$ are disjoint components of x, and therefore, there exists a family $(w_j)_{j=1}^{M}$ of disjoint components of x, so that for each i, we can write $u_i = \sum_{j=1}^{M} \beta_{ij} w_j$. Note that $\bigvee_i u_i = \sum_j \beta_j w_j$, where $\beta_j = \bigvee_i \beta_{ij}$.

Define the sets (A_i) recursively by setting $A_0 = \varnothing$, and $A_i = \{ j : \beta_{ij} = \beta_j \} \setminus \bigcup_{s<i} A_s$. These sets are clearly disjoint, and their union is $\{1, \dots, M\}$. For $1 \le i \le n$, set $y_i = \sum_{j \in A_i} \beta_j w_j$. Then $0 \le y_i \le u_i$, the y_i's are disjoint, and $\bigvee_i y_i = \bigvee_i u_i \le \bigvee_i x_i$. Thus,

$$T\Big(\bigvee_{i=1}^{n} u_i\Big) - \bigvee_{i=1}^{n} (Tu_i) \le T\Big(\bigvee_{i=1}^{n} y_i\Big) - \bigvee_{i=1}^{n} (Ty_i) = T\Big(\sum_{i=1}^{n} y_i \Big) - \bigvee_{i=1}^{n} (Ty_i).$$

By Corollary 3.3,

$$\begin{aligned}(3.2)\qquad \Big\|T\Big(\bigvee_{i=1}^{n} u_i\Big) - \bigvee_{i=1}^{n}(Tu_i)\Big\| &\le \Big\|T\Big(\sum_{i=1}^{n} y_i\Big) - \bigvee_{i=1}^{n}(Ty_i)\Big\| \\ &\le 256\varepsilon\Big\|\sum_{i=1}^{n} y_i\Big\| \le 256\varepsilon\Big\|\bigvee_{i=1}^{n} x_i\Big\|.\end{aligned}$$

For each i, write $x_i = u_i + z_i$, where $z_i \ge 0$ and $\|z_i\| \le c$. In this notation, $\bigvee_{i=1}^n x_i \le \bigvee_{i=1}^n u_i + \bigvee_{i=1}^n z_i$, and therefore, $\big\|\bigvee_{i=1}^n x_i - \bigvee_{i=1}^n u_i\big\| \le nc$. From this, we conclude that

$$\Big\|T\Big(\bigvee_{i=1}^{n} x_i\Big) - \bigvee_{i=1}^{n}(Tx_i)\Big\| \le \Big\|T\Big(\bigvee_{i=1}^{n} u_i\Big) - \bigvee_{i=1}^{n}(Tu_i)\Big\| + nc\|T\|.$$

To obtain (3.1), invoke (3.2), and recall that c can be arbitrarily small.

To obtain the inequality

$$(3.3)\qquad \Big\|\bigwedge_{i=1}^{n}(Tx_i) - T\Big(\bigwedge_{i=1}^{n} x_i\Big)\Big\| \le 256\varepsilon\Big\|\bigvee_{i=1}^{n} x_i\Big\|,$$

set $x = \bigvee_{i=1}^n x_i$. For each i set $y_i = x - x_i$, then $0 \le y_i \le x$. We have $\bigvee_{i=1}^n y_i = x + \bigvee_{i=1}^n (y_i - x) = x - \bigwedge_{i=1}^n x_i$. Hence $T(\bigwedge_{i=1}^n x_i) = Tx - T(\bigvee_{i=1}^n y_i)$. Similarly, $\bigvee_{i=1}^n Ty_i = Tx + \bigvee_{i=1}^n (T(y_i - x)) = Tx - \bigwedge_{i=1}^n Tx_i$, which yields $\bigwedge_{i=1}^n Tx_i = Tx - \bigvee_{i=1}^n Ty_i$. Therefore, $\bigwedge_{i=1}^n (Tx_i) - T(\bigwedge_{i=1}^n x_i) = T(\bigvee_{i=1}^n y_i) - \bigvee_{i=1}^n (Ty_i)$. To obtain (3.3), combine (3.1) with the fact that $\bigvee_{i=1}^n y_i \le x$. ∎

It was shown [1] that for any rearrangement invariant spaces X, Y over a finite measure such that $X \nsubseteq Y$, there is no non-zero disjointness preserving operator $T\colon X \to Y$. In particular, the only disjointness preserving operator $T\colon L_p[0,1] \to L_q[0,1]$ for $p > q$ is $T = 0$. An application of Corollary 3.3 provides the following version of this fact for positive ε-DP operators.

Proposition 3.5 *Let $1 \le p < q \le \infty$ and E be a q-convex Banach lattice. If*

$$T\colon L_p[0,1] \to E$$

is positive and ε-DP, then $\|T\| \le 256\varepsilon$.

Proof Given a positive $x \in L_p[0,1]$ with $\|x\|_p = 1$, for every $n \in \mathbb{N}$ an application of Liapunov's theorem [20, Theorem 2.c.9] allows us to find a partition of $[0,1]$ in pairwise disjoint measurable sets $(A_i)_{i=1}^n$ such that $\|x\chi_{A_i}\|_p = n^{-1/p}$. Let $x_i = x\chi_{A_i}$, for $i = 1, \dots, n$. We have $(x_i)_{i=1}^n$ are disjoint and $x = \sum_{i=1}^n x_i$.

Since E is q-convex, there is a constant $C > 0$ so that

$$\Big\|\Big(\sum_{i=1}^{n}|Tx_i|^q\Big)^{\frac{1}{q}}\Big\| \le C\Big(\sum_{i=1}^{n}\|Tx_i\|^q\Big)^{\frac{1}{q}} \le C\|T\|n^{\frac{1}{q}-\frac{1}{p}}.$$

Hence, using Corollary 3.3, we have

$$\begin{aligned}\|Tx\| &\leq \Big\| T\Big(\sum_{i=1}^n x_i\Big) - \Big(\sum_{i=1}^n |Tx_i|^q\Big)^{\frac{1}{q}}\Big\| + \Big\|\Big(\sum_{i=1}^n |Tx_i|^q\Big)^{\frac{1}{q}}\Big\| \\ &= \Big\| T\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} - \Big(\sum_{i=1}^n |Tx_i|^q\Big)^{\frac{1}{q}}\Big\| + \Big\|\Big(\sum_{i=1}^n |Tx_i|^q\Big)^{\frac{1}{q}}\Big\| \\ &\leq 256\varepsilon + C\|T\| n^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

Since $p < q$ and n was arbitrary, we get that $\|T\| \leq 256\varepsilon$. ∎

4 Positive Operators on ℓ_∞^n, c_0, and c

Recall that a Banach lattice X has the *Fatou property with constant* $\mathfrak{f}$ if, for any non-negative increasing net $(x_i) \subset X$, with $\sup_i \|x_i\| < \infty$, we have $\bigvee_i x_i \in X$, and $\|\bigvee_i x_i\| \leq \mathfrak{f} \sup_i \|x_i\|$. If $\mathfrak{f} = 1$, we speak simply of the *Fatou property*. Every Banach lattice with the Fatou property is σ-Dedekind complete. Note that if X is a Köthe function space, then it suffices to verify the above inequality for non-negative increasing sequences (x_i). Banach lattices with the Fatou property include dual lattices [23, Proposition 2.4.19] and KB-spaces [3, p. 232].

Theorem 4.1 *Suppose F is a Banach lattice, and let $\varepsilon > 0$.*

(i) *For any positive operator $T\colon \ell_\infty^n \to F$ that is ε-DP, there exists a DP operator $S\colon \ell_\infty^n \to F$, so that $0 \leq S \leq T$ and $\|T - S\| \leq 256\varepsilon$.*
(ii) *Suppose F has the Fatou property with constant $\mathfrak{f}$. Then for any positive operator $T\colon c_0 \to F$, which is ε-DP, there exists a DP operator $S\colon c_0 \to F$, so that $0 \leq S \leq T$ and $\|T - S\| \leq 256\mathfrak{f}\varepsilon$.*
(iii) *Suppose F has the Fatou property with constant $\mathfrak{f}$. Then for any positive operator $T\colon c \to F$ that is ε-DP, there exists a DP operator $S\colon c \to F$, so that $0 \leq S \leq T$ and $\|T - S\| \leq 256\mathfrak{f}^2\varepsilon$.*

The following lemma is needed to prove Theorem 4.1. This result may be known to the experts, but we have not been able to find it in the literature.

Lemma 4.2 *Suppose that for $1 \leq i \leq k$, $(x_n^{(i)})_{n\in\mathbb{N}}$ are increasing positive sequences in a Banach lattice, so that $\bigvee_{n\in\mathbb{N}} x_n^{(i)}$ for $1 \leq i \leq k$ and $\bigvee_{n\in\mathbb{N}}\big(\sum_{i=1}^k x_n^{(i)}\big)$ exist. Then*

$$\bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^k x_n^{(i)}\Big) = \sum_{i=1}^k \bigvee_{n\in\mathbb{N}} x_n^{(i)}.$$

Proof We will proceed by induction on k. For any $m \in \mathbb{N}$, we have

$$\bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^{k+1} x_n^{(i)}\Big) \geq \bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^{k} x_n^{(i)} + x_m^{(k+1)}\Big) = \bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^{k} x_n^{(i)}\Big) + x_m^{(k+1)}.$$

Hence, using the induction hypothesis,

$$\bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^{k+1} x_n^{(i)}\Big) \geq \bigvee_{n\in\mathbb{N}}\Big(\sum_{i=1}^{k} x_n^{(i)}\Big) + \bigvee_{m\in\mathbb{N}} x_m^{(k+1)} = \sum_{i=1}^{k+1} \bigvee_{n\in\mathbb{N}} x_n^{(i)}.$$

The converse inequality follows from the fact that for every m,

$$\bigvee_{n=1}^{m}\Big(\sum_{i=1}^{k} x_n^{(i)}\Big) = \sum_{i=1}^{k} x_m^{(i)} \leq \sum_{i=1}^{k} \bigvee_{n\in\mathbb{N}} x_n^{(i)}. \qquad \blacksquare$$

Proof of Theorem 4.1 Throughout the proof, we denote by (δ_i) the canonical basis of ℓ_∞^n or c_0, and $f_i = T\delta_i$. Furthermore, we assume that $\|T\| \leq 1$. Indeed, if $\|T\| > 1$, then $T' = T/\|T\|$ is $\varepsilon/\|T\|$-DP. If (i) is established for a contractive operator T, then we can find a DP map S' so that $0 \leq S' \leq T'$ and $\|S' - T'\| \leq 256\varepsilon/\|T\|$, and take $S = \|T\|S'$. The same argument works for (ii) and (iii).

For each $n \in \mathbb{N}$ define a 1-homogeneous continuous function $\phi_n\colon \mathbb{R}^n \to \mathbb{R}$:

$$\phi_n\colon (t_1,\dots,t_n) \mapsto \begin{cases} 0 & \text{if } |t_1| \leq \bigvee_{i=2}^n |t_i|,\\ 2\big(|t_1| - \bigvee_{i=2}^n |t_i|\big) & \text{if } \bigvee_{i=2}^n |t_i| \leq |t_1| \leq 2\bigvee_{i=2}^n |t_i|,\\ |t_1| & \text{if } |t_1| > 2\bigvee_{i=2}^n .|t_i|. \end{cases}$$

(i) For $1 \leq i \leq n$, set $g_i = \phi_n(f_i, f_{i+1},\dots,f_n,f_1,\dots,f_{i-1})$. We claim that the operator $S\colon \ell_\infty^n \to F\colon \delta_i \mapsto g_i$ has the desired properties.

Note that $0 \leq \phi_n(t_1,\dots,t_n) \leq t_1$ for any $t_1,\dots,t_n \in \mathbb{R}_+$. Hence $0 \leq g_i \leq f_i$, which shows that $0 \leq S \leq T$.

To show that S is disjointness preserving, consider $i \neq j$. For any $(t_1,\dots,t_n) \in \mathbb{R}^n$, $\phi_n(t_i,t_{i+1},\dots,t_n,t_1,\dots,t_{i-1}) \wedge \phi_n(t_j,t_{j+1},\dots,t_n,t_1,\dots,t_{j-1}) = 0$. Hence g_i and g_j are disjoint.

Finally we estimate $\|T - S\| = \|(T-S)\sum_{i=1}^n \delta_i\| = \|\sum_{i=1}^n (f_i - g_i)\|$. We claim that $\sum_{i=1}^n (f_i - g_i) \leq 2^9 \mathbb{E}_S\big(\sum_{i\in S} f_i\big) \wedge \big(\sum_{i\in S^c} f_i\big)$. Indeed, by functional calculus, we need to show that for any $t_1,\dots,t_n \in \mathbb{R}_+^n$,

$$\sum_{i=1}^n \big(t_i - \phi_n(t_i,t_{i+1},\dots,t_n,t_1,\dots,t_{i-1})\big) \leq 2^9 \mathbb{E}_S\Big(\sum_{i\in S} t_i\Big) \wedge \Big(\sum_{i\in S^c} t_i\Big).$$

By relabeling, we can assume that $t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$. By Lemma 3.1, the right-hand side is at least $2(t_2 + \cdots + t_n)$. In the left-hand side however,

$$t_2 - \phi_n(t_2,t_3,\dots,t_n,t_1) = t_2, \dots, t_n - \phi_n(t_n,t_1,\dots,t_{n-1}) = t_n,$$

while $0 \leq t_1 - \phi_n(t_1,t_2,\dots,t_n) \leq \bigvee_{i\geq 2} t_i = t_2$. Therefore, the right-hand side is at most $2t_2 + t_3 + \cdots + t_n \leq 2(t_2 + \cdots + t_n)$. Finally, since T is ε-DP, the result follows.

(ii) For $T\colon c_0 \to F$, let $f_i = T\delta_i$. For $n \geq i$, set

$$g_i^{(n)} = \phi_n(f_i, f_1,\dots,f_{i-1},f_{i+1},\dots,f_n).$$

Clearly, $0 \leq g_i^{(n)} \leq f_i$. Moreover, it is easy to observe that

$$\phi_n(t_1,\dots,t_n) = \phi_{n+1}(t_1,\dots,t_n,0) \geq \phi_{n+1}(t_1,\dots,t_n,t_{n+1})$$

for any $t_1,\dots,t_{n+1} \in \mathbb{R}_+$. As the Krivine functional calculus preserves lattice operations, we have

$$\begin{aligned} g_i^{(n)} &= \phi_{n+1}(f_i, f_1,\dots,f_{i-1},f_{i+1},\dots,f_n,0)\\ &\geq \phi_{n+1}(f_i,f_1,\dots,f_{i-1},f_{i+1},\dots,f_n,f_{n+1}) = g_i^{(n+1)}. \end{aligned}$$

Hence the sequence $(g_i^{(n)})_n$ is decreasing for every i. Due to the σ-Dedekind completeness of F, $g_i = \bigwedge_n g_i^{(n)}$ exists in F_+.

Define the operator $S: c_0 \to F$ by $S\delta_i = g_i$. Clearly $0 \le S \le T$. Moreover, $g_i^{(n)} \wedge g_j^{(n)} = 0$ whenever $i, j \in \{1, \dots, n\}$ are distinct. Hence $g_i \perp g_j$ for $i \ne j$, and consequently, S is disjointness preserving. Moreover,

$$\|T - S\| = \sup_n \Big\| (T - S) \sum_{i=1}^{n} \delta_i \Big\| = \sup_n \Big\| \sum_{i=1}^{n} (f_i - g_i) \Big\|.$$

Reasoning as in (i), we conclude that for every $k \ge n$,

$$\Big\| \sum_{i=1}^{n} (f_i - g_i^{(k)}) \Big\| \le \Big\| \sum_{i=1}^{k} (f_i - g_i^{(k)}) \Big\| \le 256\varepsilon.$$

By the Fatou property and Lemma 4.2,

$$\Big\| \sum_{i=1}^{n} (f_i - g_i) \Big\| = \Big\| \bigvee_{k=1}^{\infty} \sum_{i=1}^{n} (f_i - g_i^{(k)}) \Big\| \le 256\mathfrak{f}\varepsilon.$$

(iii) As before, let (δ_i) be the canonical basis of $c_0 \subset c$, and denote by $\mathbf{1}$ the constant sequence $(1, 1, \dots) \in c$. Let $f_i = T\delta_i$ and $f_0 = T\mathbf{1} - \bigvee_{n=1}^{\infty} \big(\sum_{i=1}^{n} f_i \big)$. Note that $\sum_{i=1}^{n} f_i = T\big(\sum_{i=1}^{n} \delta_i\big) \le T\mathbf{1}$. Hence the supremum in the centered equation exists due to the σ-Dedekind completeness of F. Note also that for $x = (\alpha_1, \alpha_2, \dots) \in c$,

$$Tx = (\lim_j \alpha_j) T\mathbf{1} + \sum_{i=1}^{\infty} (\alpha_i - \lim_j \alpha_j) f_i.$$

Further observe that for any $S \subset \{0, 1, \dots, n\}$, we have

$$\Big\| \Big(\sum_{i \in S} f_i \Big) \wedge \Big(\sum_{i \in S^c} f_i \Big) \Big\| \le \varepsilon$$

(here $S^c = \{0, 1, \dots, n\} \backslash S$). Indeed, suppose without loss of generality that $0 \in S$. Let $S' = S \backslash \{0\}$, $y = \sum_{i \in S^c} \delta_i$, and $x = \mathbf{1} - y$. As T is ε-DP, $\|Tx \wedge Ty\| \le \varepsilon$. But $Ty = \sum_{i \in S^c} f_i$, while

$$Tx = \sum_{i \in S'} f_i + T\mathbf{1} - \sum_{i=1}^{n} f_i \ge \sum_{i \in S'} f_i + T\mathbf{1} - \bigvee_{m=1}^{\infty} \sum_{i=1}^{m} f_i = \sum_{i \in S'} f_i + f_0 = \sum_{i \in S} f_i.$$

Define $g_i^{(n)} = \phi_{n+1}(f_i, f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$, for $0 \le i \le n$. As in the proof of (ii),

$$\Big\| \sum_{i=0}^{n} (f_i - g_i^{(n)}) \Big\| \le 256\varepsilon.$$

Let $g_i = \lim_k g_i^{(k)}$; then $\big\| \sum_{i=0}^{n} (f_i - g_i) \big\| \le 256\mathfrak{f}\varepsilon$ for every n.

Now observe that $g_i^{(i)} \ge g_i^{(i+1)} \ge \cdots$ and set $\widetilde{g} = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} g_i$. Define $S: c \to F$ by setting $S\delta_i = g_i$, and $S\mathbf{1} = \widetilde{g} + g_0$. This operator is well defined and positive. Moreover, $(T - S)\delta_i = f_i - g_i$ for $i \in \mathbb{N}$, and, by Lemma 4.2, $(T - S)\mathbf{1} = \bigvee_{n=0}^{\infty} \sum_{i=0}^{n} (f_i - g_i)$. Thus, $T \ge S$. Indeed, suppose $x = (\alpha_i)_{i=1}^{\infty} \in c$ is positive. Let $\alpha = \lim_j \alpha_j$. Then

$$(T - S)x = \alpha \Big(\bigvee_{n=0}^{\infty} \sum_{i=0}^{n} (f_i - g_i) \Big) + \sum_{i=1}^{\infty} (\alpha_i - \alpha)(f_i - g_i) \ge \alpha(f_0 - g_0) \ge 0.$$

Consequently,

$$\|T - S\| = \|(T - S)\mathbf{1}\| = \Big\| \bigvee_{n=0}^{\infty} \sum_{i=0}^{n} (f_i - g_i) \Big\| \leq \mathfrak{f} \sup_n \Big\| \sum_{i=0}^{n} (f_i - g_i) \Big\| \leq 256\mathfrak{f}^2 \varepsilon. \qquad \blacksquare$$

5 Operators into $C(K)$ Spaces

In this section we consider operators from sequence spaces into $C(K)$. Throughout the section, K denotes a compact Hausdorff space. First, consider the case when $C(K)$ is σ-Dedekind complete (equivalently, K is a basically disconnected compact Hausdorff set, see [20, Proposition 1.a.4]).

Theorem 5.1 *Suppose X is a Banach lattice with the order structure given by its 1-unconditional basis, and $C(K)$ is σ-Dedekind complete. If $T\colon X \to C(K)$ is ε-DP, then there exists a disjointness preserving $S\colon X \to C(K)$ so that $\|S\| \leq \|T\|$ and $\|S - T\| \leq 257\varepsilon\|T\|$. If T is positive, then S can be chosen so that, in addition, $0 \leq S \leq T$.*

Proof By scaling, we can assume that T is a contraction. Denote the normalized unconditional basis of X by $(\delta_i)_{i=1}^{\infty}$, and let c_{00} be the linear span of $\delta_1, \delta_2, \dots$ in X. For $i \in \mathbb{N}$, set $f_i = T\delta_i$, and note that $|f_i| \leq \mathbf{1}$. Consequently, the sequence (f_i) is order bounded. Hence, by the σ-Dedekind completeness of $C(K)$, $h_i = \bigvee_{j \neq i} |f_j|$ is continuous for every i. Let us define the continuous functions

$$g_i(t) = \begin{cases} 0 & \text{if } |f_i(t)| \leq h_i(t), \\ f_i(t) & \text{if } |f_i(t)| \geq 2h_i(t), \\ 2\big(f_i(t) - \operatorname{sign} f_i(t) \cdot h_i(t)\big) & \text{if } h_i(t) \leq |f_i(t)| \leq 2h_i(t). \end{cases}$$

Now let $S\colon c_{00} \to C(K)\colon \delta_i \mapsto g_i$. Clearly, S is disjointness preserving, since $|g_i| \wedge |g_j| = 0$ for $i \neq j$. It remains to show that $T|_{c_{00}} - S$ is bounded and that its norm does not exceed 257ε (once this is done, we extend S to the whole space X by continuity).

To this end, fix $t \in K$, and pick $\alpha_1, \dots, \alpha_N \in \mathbb{F}$ with $\|\sum_{i=1}^{N} \alpha_i \delta_i\|_X \leq 1$. We must show that for every $t \in K$

$$\sum_{i=1}^{N} |\alpha_i| |f_i(t) - g_i(t)| \leq 257\varepsilon. \tag{5.1}$$

It suffices to consider $\alpha_1, \dots, \alpha_N \geq 0$.

For $S \subset \{1, \dots, N\}$, set $S^c = \{1, \dots, N\} \backslash S$. Consider $x = \sum_{i \in S} \omega_i \alpha_i \delta_i$ and $y = \sum_{i \in S^c} \omega_i \alpha_i \delta_i$, where $\omega_i = |f_i(t)|/f_i(t)$ if $f_i(t) \neq 0$, and $\omega_i = 0$ otherwise. Note that x and y are disjoint elements of $\mathbf{B}(X)$. As T is ε-DP, we have

$$\Big(\sum_{i \in S} \alpha_i |f_i(t)| \Big) \wedge \Big(\sum_{i \in S^c} \alpha_i |f_i(t)| \Big) \leq \big\| |Tx| \wedge |Ty| \big\| \leq \varepsilon.$$

Hence, by Lemma 3.1,

$$\sum_{i=1}^{N} \alpha_i |f_i(t)| - \bigvee_{i=1}^{N} \alpha_i |f_i(t)| \leq 256\varepsilon.$$

Pick k so that $\bigvee_{i=1}^{N} \alpha_i |f_i(t)| = \alpha_k |f_k(t)|$. Note that $|f_k(t) - g_k(t)| \leq \varepsilon$. Indeed, this inequality is evident if $|f_k(t)| \leq \varepsilon$. If $|f_k(t)| > \varepsilon$, note that $|f_j(t)| \leq \varepsilon$ for any $j \neq k$.

Otherwise we would have $\| |T\delta_k| \wedge |T\delta_j| \| > \varepsilon$, contradicting the assumption that T is ε-DP. Thus, if $|f_k(t)| > \varepsilon$, then $h_k(t) \leq \varepsilon$, and we also have $|f_k(t) - g_k(t)| \leq h_k(t)$.

As $\alpha_k \leq 1$, we have

$$\sum_{i=1}^{N} \alpha_i |f_i(t) - g_i(t)| \leq \sum_{i \neq k} \alpha_i |f_i(t)| + |f_k(t) - g_k(t)| \leq 256\varepsilon + \varepsilon,$$

establishing (5.1).

If T is positive, then we have $0 \leq g_i \leq f_i$; hence $0 \leq S \leq T$. ∎

Along the same lines, we prove the following theorem.

Theorem 5.2 *Suppose X is a finite dimensional Banach lattice. If $T: X \to C(K)$ is ε-DP, then there exists a disjointness preserving $S: X \to C(K)$ so that $\|S\| \leq \|T\|$ and $\|S - T\| \leq 256\varepsilon\|T\|$. If T is positive, then S can be chosen so that, in addition, $0 \leq S \leq T$.*

Sketch of a proof It is well known (see [28, Corollary 4.20]) that X has a basis of atoms, which we denote by $(\delta_i)_{i=1}^N$ ($N = \dim X$). Use scaling to assume that T is contractive. Let $f_i = T\delta_i$ and $h_i = \bigvee_{j \neq i} |f_j|$. Define g_i and S as in the proof of Theorem 5.1 and proceed further in the same manner. ∎

For operators from c or c_0 into $C(K)$, the assumption that the range is σ-Dedekind complete is redundant.

Theorem 5.3 *Suppose K is a compact Hausdorff space, and ε is a positive number. Then for any ε-DP operator $T: c_0 \to C(K)$, there exists a DP operator $S: c_0 \to C(K)$ so that $\|S\| \leq \|T\|$ and $\|T - S\| \leq 257\varepsilon$. If T is positive, then S can be selected so that $0 \leq S \leq T$.*

Here and below, we use the notation $(\delta_i)_{i \in \mathbb{N}}$ for the canonical basis of c_0, while c_{00} denotes the set of all finitely supported sequences in c_0. The following straightforward observation will be used throughout the proof.

Lemma 5.4 *A linear map $U: c_{00} \to C(K)$ is bounded if and only if*

$$\sup_{t \in K} \sum_{i=1}^{\infty} \big| [U\delta_i](t) \big|$$

is finite. If this is the case, then the above expression equals $\|U\|$. Moreover, U extends by continuity to an operator from c_0 into $C(K)$ of the same norm.

Proof of Theorem 5.3 We know that if T is ε-DP, then $T/\|T\|$ is $\varepsilon/\|T\|$-DP. We can therefore assume that T is a contraction, and restrict our attention to $\varepsilon < 2^{-8}$. Denote the canonical basis of c_0 by $(\delta_i)_{i=1}^{\infty}$, and set $f_i = T\delta_i$. Note that T is ε-DP if and only if the inequality $\big(\sum_{i \in A} |f_i(t)|\big) \wedge \big(\sum_{i \in B} |f_i(t)|\big) \leq \varepsilon$ holds for any $t \in K$ and for any two disjoint sets A and B. Consequently, for any $t \in K$, there exists at most one $i \in \mathbb{N}$ so that $|f_i(t)| > \varepsilon$.

Consider the function

$$\phi(t) = \begin{cases} 0 & \text{if } |t| \le \varepsilon, \\ 2(|t| - \varepsilon)\operatorname{sign} t & \text{if } \varepsilon \le |t| \le 2\varepsilon, \\ t & \text{if } |t| \ge 2\varepsilon. \end{cases}$$

Let $g_i = \phi \circ f_i$, *i.e.*, $g_i(t) = \phi(f_i(t))$, and define the operator $S: c_{00} \to C(K): \delta_i \mapsto g_i$. As noted above, for any $t \in K$, there exists at most one $i \in \mathbb{N}$ so that $|g_i(t)| \neq 0$. Hence the vectors (g_i) are disjoint, which shows that S is disjointness preserving. Moreover, if T is positive, then for any i, $0 \le S\delta_i = g_i \le f_i = T\delta_i$,

First show that S is, indeed, a well-defined contraction (hence it extends by continuity to a contraction $c_0 \to C(K)$). By Lemma 5.4, $\sum_{i=1}^{\infty} |f_i(t)| \le 1$ for every $t \in K$. By our construction, $|g_i| \le |f_i|$. Hence $\sum_{i=1}^{\infty} |g_i(t)| \le 1$ for every t. Again by Lemma 5.4, $\|S\| \le 1$.

It remains to estimate

$$\|T - S\| = \sup_{t \in K} \sum_{i=1}^{\infty} \big| [(T - S)\delta_i](t) \big| = \sup_{t \in K} \sum_{i=1}^{\infty} |f_i(t) - g_i(t)|.$$

Fix $t \in K$ and $N \in \mathbb{N}$, and show that

$$\sum_{i=1}^{N} |f_i(t) - g_i(t)| \le 257\varepsilon. \tag{5.2}$$

To this end, find $k \in \{1, \dots, N\}$ so that $|f_k(t)| = \max_{1 \le i \le N} |f_i(t)|$. Then $|f_j(t)| \le \varepsilon$ (and consequently, $g_j(t) = 0$) for $j \neq k$. For a set $S \subset \{1, \dots, N\}$, set $S^c = \{1, \dots, N\} \backslash S$. We know that for any such S, $\sum_{i \in S} \big| f_i(t) \big| \wedge \sum_{i \in S^c} \big| f_i(t) \big| \le \varepsilon$. Indeed, consider

$$x = \sum_{i \in S} \overline{\operatorname{sign} f_i(t)} \delta_i \quad \text{and} \quad y = \sum_{i \in S^c} \overline{\operatorname{sign} f_i(t)} \delta_i.$$

The elements x and y belong to the unit ball of c_0 and are disjoint. Thus,

$$\sum_{i \in S} |f_i(t)| \wedge \sum_{i \in S^c} |f_i(t)| \le \big\| |Tx| \wedge |Ty| \big\| \le \varepsilon.$$

Then $\sum_{i=1}^{N} |f_i(t) - g_i(t)| = \sum_{j \neq k} |f_j(t)| + |f_k(t) - g_k(t)|$. By Lemma 3.1, $\sum_{j \neq k} |f_j(t)| \le 256\varepsilon$. Moreover, $\sup_s |\phi(s) - s| = \varepsilon$. Hence, $|f_k(t) - g_k(t)| \le \varepsilon$. This yields (5.2). ■

Theorem 5.5 *Suppose K is a compact Hausdorff space, and ε is a positive number. For any ε-DP operator $T: c \to C(K)$, there exists a DP operator $S: c \to C(K)$ so that $\|T - S\| \le 536\varepsilon$. If T is positive, then S can be chosen to be positive as well.*

Throughout the proof, we identify c_0 with its canonical image in c. Then $c = \operatorname{span}[c_0, \mathbf{1}]$. As before, we denote the canonical basis of c_0 by $(\delta_i)_{i \in \mathbb{N}}$. The following lemma can be easily verified.

Lemma 5.6 *For any operator $V: c \to X$ (X is an arbitrary Banach space), we have $\|V\| \le 2\|V|_{c_0}\| + \|V\mathbf{1}\|$.*

Proof Consider the projection Q from c to $\mathbb{F}\mathbf{1}$, defined by

$$Q((\alpha_1, \alpha_2, \dots)) = \lim_i \alpha_i \mathbf{1}.$$

Clearly $\|Q\| = 1$. Hence $\|I_c - Q\| \le 2$, and $\ker Q = \operatorname{ran}(I - Q) = c_0$. We complete the proof by writing $V = VQ + V(I - Q)$. ∎

We also need a simple fact about complex numbers. Fix $c > 0$. For a complex number $z = |z|e^{\imath \arg z}$, define $\phi_c(z) = (|z| - c)_+ e^{\imath \arg z}$.

Lemma 5.7 *Given $c > 0$, for any $z, w \in \mathbb{C}$, we have $|\phi_c(z) - \phi_c(w)| \le |z - w|$.*

Proof By scaling, we may assume $c = 1$. Without loss of generality, $|z| \ge |w|$.

The case of $|w| \le 1$ is easy: $\phi_c(w) = 0$ and by the triangle inequality,

$$|z - w| \ge |z| - |w| \ge (|z| - 1)_+ = |\phi_c(z) - \phi_c(w)|.$$

Now if $|z| \ge |w| > 1$, use the law of cosines: $|z - w|^2 = a^2 + b^2 - \kappa ab$, where $a = |z|$, $b = |w|$, and $\kappa = 2\cos(\arg z - \arg w)$ (note that $-2 \le \kappa \le 2$). Similarly, $|\phi(z) - \phi(w)|^2 = (a-1)^2 + (b-1)^2 - \kappa(a-1)(b-1)$. Thus,

$$|z - w|^2 - |\phi(z) - \phi(w)|^2 = (2 - \kappa)(a + b - 1) \ge 0. \qquad ∎$$

Lemma 5.8 *Suppose K is a compact Hausdorff space and a contraction $U\colon c \to C(K)$ is σ-DP. Suppose, moreover, that $U|_{c_0}$ is disjointness preserving and the functions $f = U\mathbf{1}$ and $f_i = U\delta_i$ are such that*

$$\text{(5.3)} \qquad \textit{if } i \in \mathbb{N} \textit{ and } t \in K \textit{ with } |f_i(t)| > \sigma, \textit{ then } |f(t) - f_i(t)| \le \sigma.$$

Then there exists a DP operator $S\colon c \to C(K)$ such that $\|U - S\| \le 11\sigma$. If U is positive, then S can be chosen positive as well.

Proof We shall construct $g, g_1, g_2, \ldots \in C(K)$ so that:

- For any i, $\|g_i - f_i\| \le 4\sigma$.
- $\|g - f\| \le 3\sigma$.
- The functions $g_1, g_2, \ldots$ are disjoint; if i and t are such that $g_i(t) \ne 0$, then $g_i(t) = g(t)$.
- If the functions $f, f_1, f_2, \ldots$ are positive, then the same holds for $g, g_1, g_2, \ldots$.

Once these functions are selected, we define $S\colon c \to C(K)$ by setting $S\delta_i = g_i$ ($i \in \mathbb{N}$) and $S\mathbf{1} = g$. Then $\|(S - U)|_{c_0}\| \le 4\sigma$ and $\|(S - U)\mathbf{1}\| \le 3\sigma$. Hence, by Lemma 5.6, $\|S - U\| \le 11\sigma$.

Moreover, S is disjointness preserving. Indeed, consider two disjoint elements of c: $x = (\alpha_i)_{i \in A}$ and $y = (\beta_i)_{i \in B}$, where the sets A and B are disjoint. If the sets $\{i \in A : \alpha_i \ne 0\}$ and $\{i \in B : \beta_i \ne 0\}$ are both infinite, then x and y belong to c_0, and we finish the proof invoking the disjointness of the functions g_i. Otherwise, suppose A is finite. Then we can assume that $B = \mathbb{N} \backslash A$. Let $\beta = \lim_i \beta_i$, and write $y = \beta\mathbf{1} + \sum_{i=1}^{\infty} \gamma_i \delta_i$, where

$$\gamma_i = \begin{cases} \beta_i - \beta & \text{if } i \in B, \\ -1 & \text{if } i \in A. \end{cases}$$

Then $Sx = \sum_{i \in A} \alpha_i g_i$ and $Sy = g - \sum_{i \in A} g_i + \sum_{i \in B} \gamma_i g_i$. If $[Sx](t) \ne 0$, then there exists $i \in A$ so that $g_i(t) \ne 0$, and therefore, $[Sy](t) = g(t) - g_i(t) = 0$. Thus, Sx and Sy are disjoint.

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Finally, suppose $g, g_1, g_2, \dots$ are positive. For $x = (\alpha_1, \alpha_2, \dots) \in c_+$, let $\alpha = \lim_i \alpha_i$. Then $Sx = \alpha g + \sum_{i=1}^\infty (\alpha_i - \alpha) g_i \geq 0$. Indeed, suppose $t \in K$ is such that there exists i with $g_i(t) > 0$. Such an i is unique. Hence

$$[Sx](t) = \alpha g(t) - (\alpha_i - \alpha) g_i(t) = \alpha_i g(t) \geq 0.$$

If there is no such i, then $[Sx](t) = \alpha g(t) \geq 0$.

To construct $g, g_1, g_2, \dots$, let $h = \phi_\sigma(f)$ (that is, $h(t) = (|f(t)| - \sigma)_+ e^{\iota \arg f(t)}$). For $i \in \mathbb{N}$, set $h_i = \phi_\sigma(f_i)$. Clearly $\|f - h\| \leq \sigma$, and $\|f_i - h_i\| \leq \sigma$ for any i. Also, if i and t are such that $h_i(t) \neq 0$, then $|h(t) - h_i(t)| \leq \sigma$, by Lemma 5.7 and (5.3).

Now define $\rho: \mathbb{R} \to [0,1]$ via

$$\rho(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t/\sigma & \text{if } 0 \leq t \leq \sigma, \\ 1 & \text{if } t \geq \sigma, \end{cases}$$

and let $k_i(t) = (1 - \rho(|h_i(t)|)) h_i(t) + \rho(|h_i(t)|) h(t)$. Clearly the function k_i is continuous, and $k_i(t) = 0$ whenever $h_i(t) = 0$. If $h_i(t) \neq 0$, then

$$|k_i(t) - h_i(t)| = \rho(|h_i(t)|) |h(t) - h_i(t)| < \sigma.$$

Hence $\|h_i - k_i\| \leq \sigma$. Finally, if $|k_i(t)| > 2\sigma$, then $k_i(t) = h(t)$. Indeed, if $|k_i(t)| > 2\sigma$, then $|h_i(t)| > \sigma$. Hence $\rho(|h_i(t)|) = 1$, yielding $k_i(t) = h(t)$.

Now set $g_i = \phi_{2\sigma}(k_i)$, and $g = \phi_{2\sigma}(h)$. From the above, if $g_i(t) \neq 0$, then $g_i(t) = g(t)$. Clearly the functions g_i are disjoint. Furthermore,

$$\|f_i - g_i\| \leq \|f_i - h_i\| + \|h_i - k_i\| + \|k_i - g_i\| \leq 4\sigma,$$
$$\|f - g\| \leq \|f - h\| + \|h - g\| \leq 3\sigma.$$

Thus, $g, g_1, g_2, \dots$ have the desired properties. ∎

Corollary 5.9 *Suppose K is a compact Hausdorff space and a contraction $U: c \to C(K)$ is σ-DP. Suppose, moreover, that $U|_{c_0}$ is disjointness preserving. Then there exists a DP operator $S: c \to C(K)$ so that $\|U - S\| \leq 11\sigma$. If U is positive, then S can be chosen positive as well.*

Proof Let $f_i = U\delta_i$ and $f = U\mathbf{1}$. The functions f_i are disjoint. Now fix i and t, and set $x = \delta_i$ and $y = \mathbf{1} - \delta_i$. Both x and y belong to the unit ball of c. Hence

$$|f_i(t)| \wedge |f(t) - f_i(t)| \leq \big\| |Tx| \wedge |Ty| \big\| \leq \sigma.$$

Thus, (5.3) holds. To complete the proof, apply Lemma 5.8. ∎

Proof of Theorem 5.5 By Theorem 5.3, there exists a disjointness preserving map $V: c_0 \to C(K)$ so that $\|V\| \leq \|T\|$, and $\|V - T|_{c_0}\| \leq 257\varepsilon$ (if T is positive, then $0 \leq V \leq T$). Define $U: c \to C(K)$ by setting $U|_{c_0} = V$ and $U\mathbf{1} = T\mathbf{1}$. By Lemma 5.6, $\|T - U\| \leq 514\varepsilon$.

Set $f = T\mathbf{1} = U\mathbf{1}$, $f_i = U\delta_i$, and $F_i = T\delta_i$. Note that if T is positive, then so is V. Indeed, by the construction in the proof of Theorem 5.3, $0 \leq f_i \leq F_i$ for every i. Note that $T(\mathbf{1} - \delta_i) = f - F_i \geq 0$ for every i. Hence $f \geq f_i$. For $x = (\alpha_1, \alpha_2, \dots) \in c_+$ set

$\alpha = \lim_i \alpha_i$. Then $Ux = \alpha f + \sum_{i=1}^n (\alpha_i - \alpha) f_i$. Fix $t \in K$. If $f_i(t) = 0$ for every i, then $[Ux](t) = \alpha f(t) \ge 0$. Otherwise, there is a unique i so that $f_i(t) > 0$. Then

$$[Ux](t) = \alpha f(t) + (\alpha_i - \alpha) f_i(t) = \alpha_i f_i(t) + \alpha(f(t) - f_i(t)) \ge 0.$$

We shall show that (5.3) holds with $\sigma = 2\varepsilon$, that is, if i and t satisfy $f_i(t) \ne 0$, then $|f_i(t) - f(t)| \le 2\varepsilon$. Once this is done, we can apply the proof of Lemma 5.8 to obtain S with the desired properties.

Let $x = \delta_i$ and $y = \mathbf{1} - \delta_i$. In the above notation, $Tx = F_i$ and $Ty = f - F_i$. Hence, for any $t \in K$, $\min\{|F_i(t)|, |f(t) - F_i(t)|\} \le \varepsilon$. By the proof of Theorem 5.3, $|F_i(t) - f_i(t)| \le \varepsilon$ (we use the fact that $|\phi(s) - s| \le \varepsilon$ for every s).

Now suppose $|f_i(t)| \ge 2\varepsilon$. Then $|F_i(t)| \ge 2\varepsilon$ as well, hence $|f(t) - F_i(t)| \le \varepsilon$. The triangle inequality implies

$$|f(t) - f_i(t)| \le |f(t) - F_i(t)| + |f_i(t) - F_i(t)| \le 2\varepsilon.$$

By the proof of Lemma 5.8, there exists a "good" S with $\|U - S\| \le 22\varepsilon$. By the triangle inequality, $\|T - S\| \le 536\varepsilon$. ∎

6 Positive Operators From ℓ_p to L_p

We start this section exploring the case of ε-DP operators defined on the space ℓ_1. We use the following classical result of Dor [14, Corollary 3.2]. Suppose (Ω, μ) is a measure space, $(f_n)_{n\in\mathbb{N}}$ are functions in $L_1(\Omega, \mu)$, and there exists $\theta \in (0,1]$ so that the inequality $\|\sum_{i=1}^n a_i f_i\| \ge \theta \sum_{i=1}^n |a_i|$ holds for any finite sequence $(a_i)_{i=1}^n$. Then there are disjoint measurable sets $(A_n)_{n\in\mathbb{N}}$ in Ω so that

$$\inf_n \int_{A_n} |f_n|\, d\lambda \ge 1 - \frac{4}{3}(1 - \theta).$$

Dor proved this theorem for the Lebesgue measure on $[0,1]$. However (as noted [4]) an inspection shows that the proof works for an arbitrary measure space. Moreover, one can select the sets A_i from the σ-algebra generated by the functions $(f_n)_{n\in\mathbb{N}}$.

Theorem 6.1 *Suppose (Ω, μ) is a measure space and $T\colon \ell_1 \to L_1(\mu)$ is a positive ε-DP operator with $\varepsilon \in (0, \|T\|/16)$. Then there exists a positive disjointness preserving operator $S\colon \ell_1 \to L_1(\mu)$ such that $0 \le S \le T$ and $\|T - S\| \le 2\sqrt{2\varepsilon\|T\|/3}$.*

Proof As usual, we can assume $\|T\| = 1$. Then we need to prove the existence of a disjointness preserving $S\colon \ell_1 \to L_1(\mu)$ such that $0 \le S \le T$ and $\|T - S\| \le 2\sqrt{2\varepsilon/3}$.

For $n \in \mathbb{N}$, let $f_n = T\delta_n$. Since $\|T\| \le 1$, we have $\|f_n\| \le 1$. By positivity, $f_n \ge 0$. Let $c = 2\sqrt{2\varepsilon/3}$ and $M = \{n \in \mathbb{N} : \|f_n\| \ge c\}$.

Now, for $n \in M$, let $g_n = f_n/\|f_n\|$. These form a normalized sequence in $L_1(\mu)$, which is equivalent to the unit vector basis of ℓ_1. In fact, given real scalars $(a_n)_{n\in M}$, let $P = \{n \in M : a_n > 0\}$, $N = \{n \in M : a_n < 0\}$ and $x = \sum_{n\in P} |a_n| g_n$, $y = \sum_{n\in N} |a_n| g_n$. We have

$$\begin{aligned}\Big\|\sum_{n\in M} a_n g_n\Big\| &= \Big\|\sum_{n\in P} |a_n| g_n - \sum_{n\in N} |a_n| g_n\Big\| = \|x - x\wedge y + x\wedge y - y\| \\ &= \|x - x\wedge y\| + \|x\wedge y - y\| \ge \|x\| + \|y\| - 2\|x\wedge y\|.\end{aligned}$$

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Since $g_n \geq 0$ and $\|g_n\| = 1$, we have $\|x\| = \sum_{n\in P} |a_n|$ and $\|y\| = \sum_{n\in N} |a_n|$. Now, since $P \cap N = \varnothing$ and $P, N \subset M$, we have

$$\begin{aligned}\|x \wedge y\| &= \Big\| \Big(\sum_{n\in P} \frac{|a_n|}{\|f_n\|} f_n \Big) \wedge \Big(\sum_{n\in N} \frac{|a_n|}{\|f_n\|} f_n \Big) \Big\| \\ &= \Big\| T\Big(\sum_{n\in P} \frac{|a_n|}{\|f_n\|} \delta_n \Big) \wedge T\Big(\sum_{n\in N} \frac{|a_n|}{\|f_n\|} \delta_n \Big) \Big\| \\ &\leq \varepsilon \max\Big\{ \sum_{n\in P} \frac{|a_n|}{\|f_n\|}, \sum_{n\in N} \frac{|a_n|}{\|f_n\|} \Big\} \\ &\leq \frac{\varepsilon}{c} (\|x\| + \|y\|).\end{aligned}$$

Hence, we get that $\| \sum_{n\in M} a_n g_n \| \geq (1 - \frac{2\varepsilon}{c}) \sum_{n\in M} |a_n|$. Now by Dor's theorem quoted above, there exist pairwise disjoint measurable sets $(A_n) \subset \Omega$ such that $\|g_n|_{A_n}\| \geq 1 - \frac{8\varepsilon}{3c} = 1 - c$.

Let us define the operator $S: \ell_1 \to L_1(\mu)$ given by

$$S\delta_n = \begin{cases} f_n|_{A_n} & \text{if } n \in M \\ 0 & \text{elsewhere.} \end{cases}$$

Since the (A_n) are pairwise disjoint, S is disjointness preserving. We have $\|T - S\| = \sup_n \|(T - S)\delta_n\|$. Now for $n \in M$, we have

$$\|(T - S)\delta_n\| = \|f_n|_{A_n^c}\| = \|f_n\| - \|f_n|_{A_n}\| = \|f_n\|(1 - \|g_n|_{A_n}\|) \leq c,$$

while for $n \notin M$, we get $\|(T - S)\delta_n\| = \|f_n\| \leq c$. Thus, $\|T - S\| \leq c$. ∎

Theorem 6.2 *Suppose* $1 < q < \infty$, $\varepsilon \in (0, 1/8^{\frac{1}{q}})$, *and* (Ω, μ) *is a measure space. If* $T: \ell_q \to L_q(\mu)$ *is positive and* ε*-DP, then there exists* $S: \ell_q \to L_q(\mu)$ *so that* $0 \leq S \leq T$, *and*

$$\|T - S\| \leq 2^8 \varepsilon + 2\sqrt{\frac{2\varepsilon\|T\|}{3}}.$$

To deduce this theorem from Theorem 6.1, we need an auxiliary result.

Lemma 6.3 *Suppose* $1 \leq q \leq \infty$, (Ω, μ) *is a measure space, and the positive operator* $R: \ell_q \to L_q(\mu)$ *satisfies the following.*

(i) *If* $x, y \in \mathbf{B}(\ell_q)_+$ *are disjoint, then* $\|Rx \wedge Ry\| \leq \varepsilon_1$.
(ii) $\sup_i \|R\delta_i\| \leq \varepsilon_2$, *where* $(\delta_i)_{i=1}^\infty$ *is the canonical basis of* ℓ_q.

Then $\|R\| \leq 2^8 \varepsilon_1 + \varepsilon_2$.

Proof Write $R\delta_i = f_i$. Then $\sup_i \|f_i\| \leq \varepsilon_2$. It suffices to show that $\| \sum_{i=1}^n \alpha_i f_i \| \leq 2^8 \varepsilon_1 + \varepsilon_2$ whenever $\alpha_1, \dots, \alpha_n \geq 0$ satisfy $\sum_i \alpha_i^q \leq 1$. By the triangle inequality,

$$\Big\| \sum_{i=1}^n \alpha_i f_i \Big\| \leq \Big\| \sum_{i=1}^n \alpha_i f_i - \bigvee_{i=1}^n \alpha_i f_i \Big\| + \Big\| \bigvee_{i=1}^n \alpha_i f_i \Big\|. \tag{6.1}$$

However,

$$\Big\|\bigvee_{i=1}^{n}\alpha_i f_i\Big\|^q \le \Big\|\Big(\sum_{i=1}^{n}(\alpha_i f_i)^q\Big)^{1/q}\Big\|^q$$
$$= \int \sum_{i=1}^{n}\alpha_i^q f_i(t)^q\,d\mu(t) \le \sup_{1\le i\le n}\|f_i\|^q\cdot\sum_{i=1}^{n}\alpha_i^q \le \varepsilon_2^q.$$

Furthermore, by Corollary 3.2,

$$\Big\|\sum_{i=1}^{n}\alpha_i f_i - \bigvee_{i=1}^{n}\alpha_i f_i\Big\| \le 2^8\mathbb{E}_S\Big\|\Big(\sum_{i\in S}\alpha_i f_i\Big)\wedge\Big(\sum_{i\in S^c}\alpha_i f_i\Big)\Big\|$$
$$= 2^8\mathbb{E}_S\Big\|R\Big(\sum_{i\in S}\alpha_i\delta_i\Big)\wedge R\Big(\sum_{i\in S^c}\alpha_i\delta_i\Big)\Big\| \le 2^8\varepsilon_1$$

(we average over all $S\subset\{1,\dots,n\}$). Plugging this into (6.1), we finish the proof. ∎

Proof of Theorem 6.2 By scaling, we can assume $\|T\|\le 1$. We denote the canonical basis on ℓ_p by $(\delta_i^{[p]})_{i=1}^{\infty}$ (below, we consider $p=q$ and $p=1$). Let $f_i = T\delta_i^{[q]}\in L_q(\mu)$ and $g_i = f_i^q\in L_1$. Define $T'\colon\ell_1\to L_1(\mu)$ by setting $T'\delta_i^{[1]} = g_i$, for every i. Clearly,

$$\|T'\| = \sup_i\|T\delta_i^{[1]}\|_1 = \sup_i\|g_i\|_1 = \sup_i\|f_i\|_q^q = \sup_i\|T\delta_i^{[q]}\|_q^q \le \|T\|^q \le 1.$$

We show that T' is ε^q-DP. It suffices to prove that for disjoint $x,y\in\ell_1$ with finite support, we have $\||T'x|\wedge|T'y|\|_1 \le \varepsilon^q\max\{\|x\|_1,\|y\|_1\}$. Write $x=\sum_{i\in A}\alpha_i\delta_i^{[1]}$ and $y=\sum_{i\in B}\beta_i\delta_i^{[1]}\in\mathbf{B}(\ell_1)$, where A and B are disjoint finite sets. Define $\widetilde{x} = \sum_{i\in A}|\alpha_i|^{1/q}\delta_i^{[q]}$, $\widetilde{y}=\sum_{i\in B}|\beta_i|^{1/q}\delta_i^{[q]}\in\ell_q$. Then

$$\big\||T'x|\wedge|T'y|\big\|_1 \le \Big\|\Big(\sum_{i\in A}|\alpha_i|g_i\Big)\wedge\Big(\sum_{i\in B}|\beta_i|g_i\Big)\Big\|_1$$
$$= \int\Big(\sum_{i\in A}|\alpha_i|g_i(t)\Big)\wedge\Big(\sum_{i\in B}|\beta_i|g_i(t)\Big)\,d\mu(t).$$

However, it is easy to see that for any positive $\gamma_1,\dots,\gamma_m$, we have $\sum_i\gamma_i\le(\sum_i\gamma_i^{1/q})^q$. Hence

$$\big\||T'x|\wedge|T'y|\big\|_1 \le \int\Big(\Big(\sum_{i\in A}|\alpha_i|^{1/q}f_i(t)\Big)\wedge\Big(\sum_{i\in B}|\beta_i|^{1/q}f_i(t)\Big)\Big)^q\,d\mu(t)$$
$$= \big\|(T\widetilde{x})\wedge(T\widetilde{y})\big\|_q^q \le \varepsilon^q\max\big\{\|\widetilde{x}\|_q^q,\|\widetilde{y}\|_q^q\big\} = \varepsilon^q\max\big\{\|x\|_1,\|y\|_1\big\}.$$

Use Theorem 6.1 to find an operator $S'\colon\ell_1\to L_1(\mu)$ so that $0\le S'\le T'$, and $\|T'-S'\|\le(8/3)^{1/2}\varepsilon^{q/2}$. Define $S\colon\ell_q\to L_q$ by setting $S(\sum_i\alpha_i\delta_i^{[q]}) = \sum_i\alpha_i(S'\delta_i^{[1]})^{1/q}$. We clearly have $0\le S\le T$. Hence S is a bounded operator. It remains to estimate $\|T-S\|$ from the above.

As $0\le T-S\le T$, $T-S$ must be ε-DP. Furthermore, for any i,

$$\|(T-S)\delta_i^{[q]}\|_q^q = \|T\delta_i^{[q]}-S\delta_i^{[q]}\|_q^q = \int\big((T\delta_i^{[q]})(t)-(S\delta_i^{[q]})(t)\big)^q\,d\mu(t).$$

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Note that for $0 \le \alpha \le \beta$, we have $(\beta-\alpha)^q \le \beta^q - \alpha^q$. Recall that $(T\delta_i^{[q]})(t) = f_i(t) = g_i(t)^{1/q} = (T'\delta_i^{[1]})(t)^{1/q}$, and $(S\delta_i^{[q]})(t) = (S'\delta_i^{[1]})(t)^{1/q}$. Thus,

$$\|(T-S)\delta_i^{[q]}\|_q^q \le \int \big((T'\delta_i^{[1]})(t) - (S'\delta_i^{[1]})(t) \big)\, d\mu(t) \le \|T'-S'\| \le \sqrt{\frac{8}{3}}\varepsilon^{q/2}.$$

Lemma 6.3 gives the desired estimate for $\|T-S\|$. ∎

Remark 6.4 It is well known that for $p \neq 2$, every linear isometry $T: L_p(\mu) \to L_p(\nu)$ is disjointness preserving [11, p. 77]. Along the same lines, it can be shown that for $p \neq 2$, there is a constant C_p such that every linear ε-isometry $T: L_p(\mu) \to L_p(\nu)$ (that is, such that $(1+\varepsilon)^{-1}\|x\| \le \|Tx\| \le (1+\varepsilon)\|x\|$) is also $C_p\varepsilon$-DP.

7 Positive Operators From Sequence Spaces to L_1

Throughout this section, the Banach lattice structure on E is assumed to be given by its 1-unconditional basis (δ_i).

Denote by $\mathbf{S}(Z)$ the unit sphere of a normed space Z. We define the set-valued *duality mapping* $\mathfrak{D}$ by letting $\mathfrak{D}(x) = \{f \in \mathbf{S}(E^*) : f(x) = \|x\|\}$ for $x \in E\backslash\{0\}$. The map $\mathfrak{D}$ is said to be *lower semicontinuous* if for any $x \in E\backslash\{0\}$ and any open set U with $U \cap \mathfrak{D}(x) \neq \varnothing$, there exists $\varepsilon \in (0, \|x\|)$ so that $U \cap \mathfrak{D}(y) \neq \varnothing$ whenever $\|x-y\| < \varepsilon$.

We call the space E *smooth* if $\mathfrak{D}(x)$ is a singleton for very x. In this case, we can define $\mathfrak{D}^o: E\backslash\{0\} \to E^*$ so that $\mathfrak{D}(x) = \{\mathfrak{D}^o(x)\}$ for every x. It is known [12, §2.2] that $\mathfrak{D}^o$ is continuous (with respect to the norm topology) if and only if the norm of E is Fréchet differentiable away from 0. Clearly, for smooth spaces $\mathfrak{D}^o$ is continuous if and only if $\mathfrak{D}$ is lower semi-continuous.

Theorem 7.1 *Suppose the order in a reflexive Banach lattice E is determined by its 1-unconditional basis and the duality map is lower semi-continuous on $E\backslash\{0\}$. Suppose, furthermore, that the operator $T \in B(E,\ell_1)_+$ is ε-DP. Then there exists a disjointness preserving operator $S \in B(E,\ell_1)_+$ such that $S \le T$ and $\|T-S\| \le 256\varepsilon$.*

Let us begin with some auxiliary results. The first one is straightforward.

Lemma 7.2 *If E is a space with a 1-unconditional basis δ_i and δ_i^* denote the corresponding biorthogonal functionals, then for any $T \in B(E, L_1(\mu))_+$, $\|T\| = \big\|\sum_i \|T\delta_i\|\delta_i^*\big\|_{E^*}$.*

Proof For the sake of brevity, set $f_i = T\delta_i$. Suppose $(\alpha_i) \in c_{00}$ is a finite sequence of non-negative numbers. Then $\|T(\sum_i \alpha_i\delta_i)\| = \int(\sum_i \alpha_i f_i) = \sum_i \alpha_i\|f_i\|$. Therefore,

$$\begin{aligned}\|T\| &= \sup\Big\{\Big\|T\Big(\sum_i \alpha_i\delta_i\Big)\Big\| : \Big\|\sum_i \alpha_i\delta_i\Big\| \le 1\Big\}\\ &= \sup\Big\{\sum_i \alpha_i\|f_i\| : \Big\|\sum_i \alpha_i\delta_i\Big\| \le 1\Big\} = \Big\|\sum_i \|f_i\|\delta_i^*\Big\|_{E^*}.\end{aligned}$$ ∎

The next lemma may be known to the experts in Banach space geometry.

Lemma 7.3 *Suppose Z is a real Banach space whose duality mapping $\mathfrak{D}$ is lower semi-continuous. Suppose, furthermore, that there exist $z, z_1, z_2, \ldots \in Z$ so that $z \neq 0$,*

$\lim_n \|z - z_n\| = 0$, *and for each* n, *there exists* $z_n^* \in \mathfrak{D}(z)$ *so that*

$$\limsup_n \frac{\|z\| - \langle z_n^*, z_n\rangle}{\|z - z_n\|} > 0.$$

Then $\|z_n\| < \|z\|$ *for some value of* n.

Proof By rescaling, we can assume that $\|z\| = 1$. Furthermore, by passing to a subsequence, we can assume that for every n, $\langle z_n^*, z_n\rangle < 1 - c\|z - z_n\|$, where $c > 0$ is a constant. By the lower semi-continuity of the duality map, we can find a sequence $\widetilde{z}_n^* \in \mathfrak{D}(z_n)$ so that $\lim_n \|z_n^* - \widetilde{z}_n^*\| = 0$. We then have

$$\|z_n\| = \langle \widetilde{z}_n^*, z_n\rangle = \langle \widetilde{z}_n^*, z\rangle - \langle z_n^*, z\rangle + \langle \widetilde{z}_n^* - z_n^*, z_n - z\rangle + \langle z_n^*, z_n\rangle. \tag{7.1}$$

As $z_n^* \in \mathfrak{D}(z)$, and $\|\widetilde{z}_n^*\| = 1$, we have $\langle \widetilde{z}_n^*, z\rangle - \langle z_n^*, z\rangle \le 0$. Furthermore, $\langle z_n^*, z_n\rangle \le 1 - c\|z - z_n\|$, and $\langle \widetilde{z}_n^* - z_n^*, z_n - z\rangle \le \|\widetilde{z}_n^* - z_n^*\|\|z_n - z\| = o(\|z - z_n\|)$. Now (7.1) shows that $\|z_n\| \le 1 - c\|z - z_n\| + o(\|z - z_n\|)$. ∎

Proof of Theorem 7.1 We assume that the basis (δ_i) is normalized. Let $f_i = T\delta_i$. By Corollary 3.3, for every sequence $(\alpha_i) \in c_{00}$, we have

$$\Big\|\sum_i \alpha_i f_i - \bigvee_i \alpha_i f_i\Big\| \le 256\varepsilon \Big\|\sum_i \alpha_i \delta_i\Big\|.$$

We will find mutually disjoint sets $A_i \subset \mathbb{N}$ with the property that

$$\Big\|\sum_i \|\mathbf{1}_{A_i^c} f_i\|\delta_i^*\Big\| \le 256\varepsilon. \tag{7.2}$$

Once this is done, we define $S: E \to \ell_1: \delta_i \mapsto \mathbf{1}_{A_i} f_i$. Then clearly $0 \le S \le T$, and by Lemma 7.2,

$$\|T - S\| = \Big\|\sum_i \|f_i - \mathbf{1}_{A_i} f_i\|\delta_i^*\Big\| = \Big\|\sum_i \|\mathbf{1}_{A_i^c} f_i\|\delta_i^*\Big\| \le 256\varepsilon.$$

For the purpose of finding (A_i), we use some ideas of [14]. Consider the space

$$\mathcal{H} = \Big\{(h_1, h_2, \dots) \in \prod_i \mathbf{B}(\ell_\infty)_+ : \sum_i h_i \le \mathbf{1}\Big\}.$$

Here $\prod_i \mathbf{B}(\ell_\infty)_+$ is equipped with the topology of the product of infinitely many copies of (ℓ_∞, w^*). It is easy to see that $\mathcal{H}$ is compact. Now define

$$F: \mathcal{H} \to \mathbb{R}: (h_i)_{i\in\mathbb{N}} \mapsto \Big\|\sum_i \|(\mathbf{1} - h_i) f_i\|\delta_i^*\Big\|.$$

Note that the function F is convex. Indeed, suppose $h_i = t h_i^{(0)} + (1-t) h_i^{(1)}$ for every i. For convenience, set $\phi_i = f_i(\mathbf{1} - h_i)$ and $\phi_i^{(j)} = f_i(\mathbf{1} - h_i^{(j)})$ for $j = 0, 1$. Then $\phi_i = t\phi_i^{(0)} + (1-t)\phi_i^{(1)}$, and as all the functions are non-negative, $\|\phi_i\| = t\|\phi_i^{(0)}\| + (1-t)\|\phi_i^{(1)}\|$.

$$\begin{aligned} F((h_i)_i) &= \Big\|\sum_i \|\phi_i\|\delta_i^*\Big\| = \Big\|\sum_i (t\|\phi_i^{(0)}\| + (1-t)\|\phi_i^{(1)}\|)\delta_i^*\Big\| \\ &\le t\Big\|\sum_i \|\phi_i^{(0)}\|\delta_i^*\Big\| + (1-t)\Big\|\sum_i \|\phi_i^{(1)}\|\delta_i^*\Big\| \\ &= tF((h_i^{(0)})_i) + (1-t)F((h_i^{(0)})_i). \end{aligned}$$

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Moreover, F is continuous. Indeed, fix $\varepsilon' > 0$ and $(h_i) \in \mathcal{H}$. Find N so that

$$\Big\| \sum_{i=N+1}^{\infty} \|f_i\| \delta_i^* \Big\| < \varepsilon'/2.$$

Then $|F((h_i)) - F(h_i'))| < \varepsilon'$ whenever, for $1 \le i \le N$,

$$\Big| \|(1-h_i)f_i\| - \|(1-h_i')f_i\| \Big| = \|(h_i'-h_i)f_i\| = \Big|\langle h_i - h_i', f_i\rangle\Big| < \frac{\varepsilon'}{2N}$$

($\langle\cdot,\cdot\rangle$ denotes the duality bracket between ℓ_∞ and ℓ_1). The centered equation above clearly defines a relatively open subset of $\mathcal{H}$.

By the above, for any $n \in \mathbb{N}$, there exists an extreme point $(h_i^{(n)})_i \in \mathcal{H}$ so that $F((h_i^{(n)})_i) < \inf F + 1/n$. As noted in [14], (h_i) is an extreme point of $\mathcal{H}$ if and only if there exist disjoint sets A_i so that $h_i = \mathbf{1}_{A_i}$, for every i. Moreover, the set of the extreme points of $\mathcal{H}$ is closed. Indeed, one can observe that $\mathcal{H}$ is metrizable. Suppose $((h_i^{(n)})_{i\in\mathbb{N}})_{n\in\mathbb{N}}$ is a sequence of extreme points, converging to some $(h_i)_{i\in\mathbb{N}} \in \mathcal{H}$. Write $h_i^{(n)} = \mathbf{1}_{A_i^{(n)}}$. Then for any i, $h_i^{(n)} \underset{n}{\to} h_i$ pointwise, hence $h_i = \mathbf{1}_{A_i}$. Moreover, for each $i, t \in \mathbb{N}$, only two situations are possible.

(i) For n large enough, $t \in A_i^{(n)}$ (that is, $h_i^{(n)}(t) = 1$), and consequently, $t \in A_i$.

(ii) For n large enough, $t \notin A_i^{(n)}$, and then, $t \notin A_i$.

This shows that the sets (A_i) are disjoint.

We therefore conclude that F attains its minimum on an extreme point $(\mathbf{1}_{A_i})$. By enlarging the sets A_i if necessary, we can assume that $\bigcup_i A_i = \mathbb{N}$. It remains to show that these sets satisfy (7.2).

For the sake of brevity, write $\beta_i = \|\mathbf{1}_{A_i^c} f_i\|$, and $x = \sum_i \beta_i \delta_i^*$. Find $z = \sum_i \alpha_i \delta_i \in \mathbf{S}(E)_+$ so that $\sum_i \alpha_i \beta_i = \|\sum_i \beta_i \delta_i^*\|$. We will show that for any $t \in A_i$, $\alpha_i f_i(t) = \bigvee_j \alpha_j f_j(t)$. Indeed, suppose, by way of contradiction, that there exist $t \in A_i$, and $j \neq i$, so that $\alpha_i f_i(t) < \alpha_j f_j(t)$. For $k \in \mathbb{N}$, let $h_k = \mathbf{1}_{A_k}$. Furthermore, for any $\varepsilon \in (0, (\alpha_j f_j(t) - \alpha_i f_i(t))/2)$, define $h_k^{(\varepsilon)}$ by setting $h_k^{(\varepsilon)} = h_k$ for $k \notin \{i,j\}$, $h_i^{(\varepsilon)} = h_i - \varepsilon \mathbf{1}_{\{t\}}$, and $h_j^{(\varepsilon)} = h_j + \varepsilon \mathbf{1}_{\{t\}}$. Let $\beta_k^{(\varepsilon)} = \|(\mathbf{1} - h_k^{(\varepsilon)}) f_k\|$, Then $\beta_k^{(\varepsilon)} = \beta_k$ for $k \notin \{i,j\}$, $\beta_i^{(\varepsilon)} = \beta_i + \varepsilon f_i(t)$ and $\beta_j^{(\varepsilon)} = \beta_j - \varepsilon f_j(t)$. Write $x = \sum_k \beta_k \delta_k^*$ and $x^{(\varepsilon)} = \sum_k \beta_k^{(\varepsilon)} \delta_k^*$. Then $\|x - x^{(\varepsilon)}\| = \|\varepsilon f_i(t)\delta_i - \varepsilon f_j(t)\delta_j\| \le (|f_i(t)| + |f_j(t)|)\varepsilon$. Moreover,

$$\begin{aligned}\langle z, x^{(\varepsilon)}\rangle &= \sum_k \alpha_k \beta_k^{(\varepsilon)} = \sum_k \alpha_k \beta_k + \varepsilon\big(\alpha_i f_i(t) - \alpha_j f_j(t)\big) \\ &= 1 - \varepsilon\big(\alpha_j f_j(t) - \alpha_i f_i(t)\big).\end{aligned}$$

An application of Lemma 7.3 shows that for some ε,

$$F((h_i^{(\varepsilon)})_i) = \|x^{(\varepsilon)}\| < \|x\| = F((h_i)_i),$$

contradicting our assumption that F attains its minimum at (h_i).

For $N \in \mathbb{N}$, let $B_N = \bigcup_{k=1}^N A_k$ and $\phi_N = \sum_{i=1}^N \alpha_i \mathbf{1}_{A_i^c} f_i$. By the above, $\phi_N(t) = \sum_{i=1}^N \alpha_i f_i(t) - \bigvee_i \alpha_i f_i(t)$ for $t \in B_N$. Consequently,

$$\big\| \phi_N \mathbf{1}_{B_N} \big\| \le \Big\| \sum_{i=1}^N \alpha_i f_i - \bigvee_i \alpha_i f_i \Big\| \le 256\varepsilon.$$

Now consider a finite set $B \subset \mathbb{N}$. Then $B \subset B_N$ for N large enough. Hence

$$\Big\| \Big(\sum_{i=1}^{N} \alpha_i \mathbf{1}_{A_i^c} f_i \Big) \mathbf{1}_B \Big\| \leq 256\varepsilon$$

for every N. By the Fatou Property of ℓ_1,

$$\Big\| \Big(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i^c} f_i \Big) \mathbf{1}_B \Big\| \leq 256\varepsilon,$$

and as B can be arbitrarily large, $\| \sum_i \alpha_i \mathbf{1}_{A_i^c} f_i \| \leq 256\varepsilon$. Now, since

$$\Big\| \sum_i \|\mathbf{1}_{A_i^c} f_i\| \delta_i^* \Big\| = \sum_i \alpha_i \|\mathbf{1}_{A_i^c} f_i\| = \Big\| \sum_i \alpha_i \mathbf{1}_{A_i^c} f_i \Big\| \leq 256\varepsilon,$$

we get (7.2) as claimed. ∎

Theorem 7.4 *Suppose the order in a reflexive Banach lattice E is determined by its 1-unconditional basis, and the operator $T \in B(E, \ell_1)_+$ is ε-DP. Then for every $c > 1$, there exists a disjointness preserving operator $S \in B(E, \ell_1)_+$ so that $S \leq T$, and $\|T - S\| \leq 256c\varepsilon$.*

For the proof we need a renorming result similar to [15, Proposition 1.4]. Recall that a Banach space Z is called *locally uniformly rotund* (LUR, for short) if, for any $z, z_1, z_2, \ldots \in Z$, $\lim \|z_n - z\| = 0$ whenever $\lim_n (2(\|z\|^2 + \|z_n\|^2) - \|z + z_n\|^2) = 0$. We say that a basis in a Banach space Z is *shrinking* if its biorthogonal functionals form a basis of the dual space Z^*. For unconditional bases, this condition holds precisely when the space contains no subspace isomorphic to ℓ_1 [19, Theorem 1.c.9].

Lemma 7.5 *Suppose $(E, \|\cdot\|)$ is a space with a shrinking 1-unconditional basis (δ_i). Then for every $c > 1$, E admits an equivalent norm $\|\cdot\|_0$ such that*

(i) *for any $x \in E$, $\|x\|_0 \leq \|x\| \leq c\|x\|_0$;*
(ii) *$(E, \|\cdot\|_0)^*$ is LUR;*
(iii) *the basis (δ_i) is 1-unconditional in $(E, \|\cdot\|_0)$.*

Sketch of the proof We follow the reasoning of [15, Proposition 1.4]. The minor changes that are required are indicated below. As before, we assume that the basis (δ_i) is normalized, and we denote the corresponding biorthogonal functionals by δ_i^*. To distinguish between the (originally given) norms on E and E^*, we denote them by $\|\cdot\|$ and $\|\cdot\|^*$, respectively.

Find $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots > 0$ so that $\sum_{i=0}^{\infty} \varepsilon_i < c$. For $f = \sum_i f_i \delta_i^* \in E^*$, set

$$\|f\|_1^* = \Big(\|f\|^{*2} + \sum_i \varepsilon_i |f_i|^2 \Big)^{1/2}.$$

Then $(E^*, \|\cdot\|_1^*)$ is smooth, and for any f, $\|f\|^* \leq \|f\|_1^* \leq \sqrt{c}\|f\|^*$. Moreover, $\|\cdot\|_1^*$ is a dual norm, and we can define the predual norm $\|\cdot\|_1$ on E. Finally, the basis (δ_i^*) is 1-unconditional in $(E^*, \|\cdot\|_1^*)$. Hence (δ_i) is 1-unconditional in $(E, \|\cdot\|_1)$.

Now set

$$\|f\|_0^* = \Big(\sum_{i=0}^{\infty} \varepsilon_i \Big\| \sum_{k=i+1}^{\infty} f_k \delta_k^* \Big\|_1^{*2} \Big)^{1/2}.$$

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This is a dual LUR norm, and $\|f\|_1^* \le \|f\|_0^* \le \sqrt{c}\|f\|_1^*$. Finally, the 1-unconditionality is once again preserved. ∎

Proof of Theorem 7.4 By Lemma 7.5, we can equip E with an equivalent norm $\|\cdot\|_0$, with the properties that $\|\cdot\|_0 \le \|\cdot\| \le c\|\cdot\|_0$, the basis $(\delta_i)_{i=1}^\infty$ is 1-unconditional, and $(E, \|\cdot\|_0)^*$ is LUR. By [15, Corollary 1.16], $\|\cdot\|_0$ is Fréchet differentiable on $E\backslash\{0\}$.

Now consider T as a map from $(E, \|\cdot\|_0)$ into ℓ_1. As $\mathbf{B}(E, \|\cdot\|_0) \subset c\mathbf{B}(E)$, we conclude that T is $c\varepsilon$-DP with respect to $\|\cdot\|_0$. By Theorem 7.1, we can find a disjointness preserving mapping $S: (E, \|\cdot\|_0) \to \ell_1$ so that $0 \le S \le T$, and $\|T - S\| \le 256c\varepsilon$. To finish the proof, recall that $\|\cdot\|_0 \le \|\cdot\|$. ∎

In the case of operators with values in $L_1(\Omega, \mu)$ (for an arbitrary measure space (Ω, μ)), we obtain the following.

Theorem 7.6 *Suppose the order in a Banach lattice E is determined by its 1-unconditional shrinking basis, and the operator $T \in B(E, L_1(\Omega, \mu))_+$ is ε-DP. Then for every $\sigma > 0$ there exists a disjointness preserving finite rank operator $S \in B(E, L_1(\Omega, \mu))_+$ so that $\|T - S\| \le 256\varepsilon + \sigma$.*

Remark 7.7 Note that if the order on a Banach lattice E is determined by a shrinking unconditional basis, then any $T \in B(E, L_1(\Omega, \mu))_+$ is necessarily compact. Indeed, if T as above is not compact, then a standard small perturbation argument produces a disjoint normalized positive sequence (x_i) so that $\inf_i \|Tx_i\| > 0$. By [19, Theorem 1.c.9], E contains no isomorphic copies of ℓ_1. Hence (x_i) is weakly null. Then (Tx_i) is weakly null. However, any positive weakly null sequence in L_1 must also be norm null, yielding a contradiction.

Proof As before denote the normalized 1-unconditional basis of E by (δ_i), and set $f_i = T\delta_i$. Then E^* is spanned by $(\delta_i^*)_{i\in\mathbb{N}}$, and by Lemma 7.2, $\|T\| = \|\sum_{i=1}^\infty \|f_i\|\delta_i^*\|$. Given $\sigma > 0$, find N so that $\|\sum_{i=N+1}^\infty \|f_i\|\delta_i^*\| < \sigma/4$. Let $E_N = \operatorname{span}[\delta_1, \dots, \delta_N] \subset E$. Find a finite σ-algebra $\mathcal{A}$ in (Ω, μ), so that, for every $x \in \mathbf{B}(E_N)$,

$$\|Tx - PTx\| < 2^{-11}\sigma$$

(here P denotes the conditional expectation onto $L_1(\mathcal{A}, \mu)$). Then $T' = PT|_{E_N}$ is $(\varepsilon + 2^{-10}\sigma)$-DP. Indeed, for every disjoint $x_1, x_2 \in \mathbf{B}(E_N)$,

$$\begin{aligned}
\big\| |T'x_1| \wedge |T'x_2| \big\| &\le \big\| |T'x_1| \wedge |(T' - T)x_2| \big\| + \big\| |T'x_1| \wedge |Tx_2| \big\| \\
&\le \|(T' - T)x_2\| + \big\| |(T' - T)x_1| \wedge |Tx_2| \big\| + \big\| |Tx_1| \wedge |Tx_2| \big\| \\
&\le \|(T' - T)x_2\| + \|(T' - T)x_1\| + \big\| |Tx_1| \wedge |Tx_2| \big\| \le 2^{-10}\sigma + \varepsilon.
\end{aligned}$$

Fix $c \in (1, (256\varepsilon + \sigma/4)^{-1}(256\varepsilon + 3\sigma/4))$. As in the proof of Theorem 7.4, we can find $S': E_N \to L_1(\mathcal{A}, \mu)$ so that $0 \le S' \le T'$ and $\|S - T\| \le (256\varepsilon + \sigma/4)c$. Now define $S: E \to L_1(\Omega, \mu)$ by setting $S\delta_i = S'\delta_i$ for $1 \le i \le N$, $S\delta_i = 0$ otherwise. Clearly S is positive and disjointness preserving, and

$$\|T - S\| \le \|T' - S'\| + \big\| T|_{\operatorname{span}[\delta_i : i > N]} \big\| \le 256c(2^{-10}\sigma + \varepsilon) + \frac{\sigma}{4} < 256\varepsilon + \sigma,$$

due to the choice of c. ∎

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8 Counterexamples

In this section we show that, in general, not every positive almost DP operator can be approximated by a disjointness preserving one. Actually, our examples produce positive operators T which are not merely ε-DP, but have a stronger property:

$$\||Tx| \wedge |Ty|\| \le \varepsilon\sqrt{\|x\|\|y\|} \quad \text{for any } x \perp y.$$

Proposition 8.1 *Suppose $1 \le p < q < \infty$. Then for any $\varepsilon > 0$, there exists a finite rank positive ε – DP operator $T: \ell_p \to \ell_q$, so that $\|T\| \le 2^{1-1/q}$ and $\|T - S\| \ge 2^{-1/q} \ge \|T\|/2$ whenever S is disjointness preserving.*

Start with a combinatorial lemma.

Lemma 8.2 *For $N \in \mathbb{N}$, let $M = N(N+1)/2$. Then $\{1, \dots, M\}$ contains sets $F_1, \dots, F_{N+1}$ of cardinality N each, so that*

(i) *each number $s \in \{1, \dots, M\}$ belongs to exactly two of the sets F_i;*
(ii) *$|F_i \cap F_j| = 1$ if $i \ne j$.*

Proof Consider the complete graph on $N+1$ vertices, and denote its sets of vertices and edges by V and E, respectively. Write $V = \{v_1, \dots, v_{N+1}\}$ and $E = \{e_1, \dots, e_M\}$. Let F_i be the set of all s so that e_s is adjacent to v_i. ■

Proof of Proposition 8.1 Pick $N \in \mathbb{N}$ so that

$$\varepsilon \ge \begin{cases} N^{-1/q} & \text{if } \infty > q \ge 2p, \\ (N^{-1}(N+1)^{2-q/p})^{1/q} & \text{if } 2p > q > p. \end{cases}$$

Define the operator $T: \ell_p^{N+1} \to \ell_q^M$ by setting $T\delta_i = N^{-1/q}\mathbf{1}_{F_i}$, where (δ_i) is the canonical basis for ℓ_p^{N+1}. Clearly, T is positive. Moreover,

$$\|T: \ell_1^{N+1} \to \ell_1^M\| = \max_i \|T\delta_i\|_1 = N^{1/q'},$$

where $1/q + 1/q' = 1$. Furthermore,

$$\|T: \ell_\infty^{N+1} \to \ell_\infty^M\| = \|T\mathbf{1}\|_\infty = N^{-1/q}\Big\|\sum_i \mathbf{1}_{F_i}\Big\| = 2N^{-1/q}$$

(for $1 \le s \le M$, $(\sum_i \mathbf{1}_{F_i})(s) = 2$, since $s \in F_i$ for exactly two indices i). By interpolation,

$$\|T: \ell_q^{N+1} \to \ell_q^M\| \le \|T: \ell_1^{N+1} \to \ell_1^M\|^{1/q}\, \|T: \ell_\infty^{N+1} \to \ell_\infty^M\|^{1/q'} \le 2^{1/q'}.$$

As the formal identity from ℓ_p^{N+1} to ℓ_q^{N+1} is contractive, the desired estimate for $\|T\|$ follows.

Next we show that T is ε – DP. Consider disjoint elements $x = \sum_{i \in P_x} \alpha_i \delta_i$ and $y = \sum_{j \in P_y} \beta_j \delta_j$, where $P_x \cap P_y = \varnothing$ and $P_x \cup P_y = \{1, \dots, N+1\}$. For $s \in \{1, \dots, M\}$, let Q_s be the set of i's for which $s \in F_i$ (we have $|Q_s| = 2$). If $Q_s \subset P_x$ or $Q_s \subset P_y$, then $(|Tx| \wedge |Ty|)(s) = 0$. If $Q_s = \{i, j\}$ with $i \in P_x$ and $j \in P_y$, then

$$N^{1/q}(|Tx| \wedge |Ty|)(s) = |\alpha_i| \wedge |\beta_j| \le |\alpha_i|^{1/2}|\beta_j|^{1/2}.$$

Note that any pair (i,j) appears in the right-hand side of the centered inequality at most once (when $Q_s = (i,j)$). Therefore,

$$N\big\| |Tx| \wedge |Ty| \big\|_q^q = N\sum_s \big| (|Tx| \wedge |Ty|)(s) \big|^q \le \sum_{i,j} (|\alpha_i|^{1/2}|\beta_j|^{1/2})^q$$
$$= \sum_i |\alpha_i|^{q/2} \sum_j |\beta_j|^{q/2}.$$

For $q \ge 2p$, $\big(\sum_i |\alpha_i|^{q/2}\big)^{2/q} \le \big(\sum_i |\alpha_i|^p\big)^{1/p} = \|x\|_p$, and therefore, $\sum_i |\alpha_i|^{q/2} \le \|x\|_p^{q/2}$. Similarly, $\sum_j |\beta_j|^{q/2} \le \|y\|_p^{q/2}$. Thus,

$$\big\| |Tx| \wedge |Ty| \big\|_q^2 \le N^{-2/q}\|x\|_p\|y\|_p \le \varepsilon^2\|x\|_p\|y\|_p,$$

due to our definition of ε.

For $p < q < 2p$,

$$\Big(\sum_i |\alpha_i|^{q/2}\Big)^{2/q} \le (N+1)^{2/q-1/p}\Big(\sum_i |\alpha_i|^p\Big)^{1/p} = (N+1)^{2/q-1/p}\|x\|_p.$$

Hence $\sum_i |\alpha_i|^{q/2} \le (N+1)^{1-q/(2p)}\|x\|_p^{q/2}$. Handling $\sum_j |\beta_j|^{q/2}$ similarly, we conclude that $N\big\| |Tx| \wedge |Ty| \big\|_q^q \le (N+1)^{2-q/p}\|x\|_p^{q/2}\|y\|_p^{q/2}$. Hence

$$\big\| |Tx| \wedge |Ty| \big\|_q \le (N^{-1}(N+1)^{2-q/p})^{1/q}\sqrt{\|x\|_p\|y\|_p} \le \varepsilon\sqrt{\|x\|_p\|y\|_p}.$$

Finally, we show that T is poorly approximated by disjointness preserving operators. Suppose $S\colon \ell_p^{N+1} \to \ell_q^M$ is disjointness preserving. Let $G_i = \operatorname{supp}(S\delta_i)$ and $H_i = F_i \backslash G_i$. The sets G_i are disjoint, and $\sum_{i=1}^{N+1} |G_i| \le M = N(N+1)/2$. Hence $|G_i| \le N/2$ for some i. Then $|H_i| \ge N/2$. Hence

$$\|T - S\| \ge \|(T-S)\delta_i\| \ge N^{-1/q}|H_i|^{1/q} \ge 2^{-1/q}.$$

Thus T has all the desired properties. ∎

The above results can be generalized somewhat by extending the range space. Recall that a Banach lattice X *satisfies a lower q-estimate with constant* $\mathfrak{C}_q$ if, for any disjoint $x_1,\dots,x_n \in X$, $\|\sum_i x_i\| \ge \mathfrak{C}_q\big(\sum_i \|x_i\|^q\big)^{1/q}$.

Proposition 8.3 *Suppose $1 \le p < q < \infty$, and X is an infinite dimensional Banach lattice satisfying a lower q-estimate with constant $\mathfrak{C}_q$. Suppose, moreover, that X does not satisfy a lower r-estimate for any $r < q$. Then for any $\varepsilon > 0$ there exists a finite rank positive ε – DP operator $T\colon \ell_p \to X$ such that $\|T\| \le 2^{1-1/q}(1+\varepsilon)$ and whenever S is disjointness preserving, $\|T-S\| \ge \mathfrak{C}_q/(2^{-1/q}3^{-(q-1)/q})$. In the particular case of $X = L_q$, we can have $\|T\| \le 2^{1-1/q}$ and $\|T-S\| \ge 2^{-1/q}$.*

Remark 8.4 Recall that there are no non-zero disjointness preserving operators from $L_p(0,1)$ to $L_q(0,1)$, when $p < q$ (see [1], and also Proposition 3.5.)

Proof Follow the proof of Proposition 8.1. Pick $N \in \mathbb{N}$ so that

$$\frac{\varepsilon}{2} > \begin{cases} N^{-1/q} & \text{if } \infty > q \geq 2p, \\ (N^{-1}(N+1)^{2-q/p})^{1/q} & \text{if } 2p > q > p. \end{cases}$$

Let $M = N(N+1)/2$. Fix $\delta \in (0, 1/4)$. By Krivine's theorem for lattices (see [27]), there exist disjoint positive norm one $x_1, \dots, x_M \in X$ so that for any $\alpha_1, \dots, \alpha_M \in \mathbb{C}$,

$$\frac{1}{1+\delta} \Big\| \sum_i \alpha_i x_i \Big\| \leq \Big(\sum_i |\alpha_i|^q \Big)^{1/q} \leq (1+\delta) \Big\| \sum_i \alpha_i x_i \Big\|.$$

Define the operator $T: \ell_p^{N+1} \to X$ by setting $T\delta_i = N^{-1/q} \sum_{j \in F_i} x_j$, where (δ_i) is the canonical basis for ℓ_p^{N+1}. Clearly, T is positive. From the proof of Proposition 8.1, $\|T\| \leq (1+\delta)2^{1/q'}$ and T is $(1+\delta)\varepsilon/2 - \mathrm{DP}$.

It remains to show that if $S: \ell_p^{N+1} \to X$ is disjointness preserving, then

$$\max_{1 \leq i \leq N} \|(T-S)\delta_i\| \geq \mathfrak{C}_q / (3 \cdot 2^{-1/q}).$$

It is easy to see that any disjoint order bounded sequence in X is norm null. Hence X is order continuous (see [23, §2.4]). This, in turn, implies that any ideal in X is a projection band. For $x \in X$, we shall denote by $\mathbf{P}_x$ the band projection corresponding to x. Let $P_i = \mathbf{P}_{S\delta_i} \mathbf{P}_{T\delta_i}$. If P is a projection, we use the shorthand $P^\perp = I - P$. By the basic properties of band projections (see [23, §1.2]), P_i's are band projections and $P_i P_j = 0$ if $i \neq j$.

Recall that for $1 \leq s \leq M$, $Q_s = \{1 \leq i \leq N+1 : s \in F_i\}$ and $|Q_s| = 2$. Let $y_{is} = P_i x_s$, and note that $y_{is} = 0$ unless $s \in F_i$, or equivalently, $i \in Q_s$. Also let $y_{0s} = x_s - \sum_{i \in Q_s} P_i x_s = (\sum_{i \in Q_s} P_i)^\perp x_s$. The elements y_{is} are disjoint. We have

$$\begin{aligned} N^{1/q} \|(T-S)\delta_i\| &\geq N^{1/q} \|\mathbf{P}_{S\delta_i}^\perp \mathbf{P}_{T\delta_i}(T\delta_i)\| \\ &= \Big\| \sum_{s \in F_i} (x_s - y_{is}) \Big\| = \Big\| \sum_{s \in F_i} (y_{0s} + y_{i's}) \Big\|, \end{aligned}$$

where i' is such that $Q_s = \{i, i'\}$. By the lower q-estimate,

$$N \|(T-S)\delta_i\|^q \geq \mathfrak{C}_q^q \sum_{s \in F_i} (\|y_{0s}\|^q + \|y_{i's}\|^q).$$

Consequently,

$$\begin{aligned} \mathfrak{C}_q^{-q} N \sum_{i=1}^{N+1} \|(T-S)\delta_i\|^q &\geq \sum_{i=1}^{N+1} \sum_{s \in F_i} (\|y_{0s}\|^q + \|y_{i's}\|^q) \\ &= \sum_{s=1}^{M} \sum_{i \in Q_s} (\|y_{0s}\|^q + \|y_{i's}\|^q) \\ &= \sum_{s=1}^{M} \Big(2\|y_{0s}\|^q + \sum_{i \in Q_s} \|y_{is}\|^q \Big). \end{aligned}$$

An easy computation shows that the inequality

$$2a^q + b^q + c^q \geq a^q + b^q + c^q \geq 3^{1-q}(a+b+c)^q$$

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holds for any non-negative reals a, b, c. Hence

$$2\|y_{0s}\|^q + \sum_{i\in Q_s} \|y_{is}\|^q \geq 3^{1-q}\Big(\|y_{0s}\| + \sum_{i\in Q_s}\|y_{is}\|\Big)^q \geq 3^{1-q}\|y_{0s} + \sum_{i\in Q_s} y_{is}\|^q = 3^{1-q}\|x_s\|^q.$$

Therefore,

$$\mathfrak{C}_q^{-q} N \sum_{i=1}^{N+1} \|(T-S)\delta_i\|^q \geq \frac{1}{3^{q-1}} \sum_{s=1}^{M} \|x_s\|^q = \frac{M}{3^{q-1}}.$$

Thus, for some i,

$$\|(T-S)\delta_i\|^q \geq \frac{\mathfrak{C}_q^q M}{3^{q-1}N(N+1)} = \frac{\mathfrak{C}_q^q}{2\cdot 3^{q-1}}.$$

The particular case of $X = L_q(\mu)$ is more straightforward. In this case, $\mathfrak{C}_q = 1$ and the x_is satisfy $\|\sum_i \alpha_i x_i\| = \big(\sum_i |\alpha_i|^q\big)^{1/q}$, that is, we can take $\delta = 0$. Keeping the same notation as before, we obtain:

$$\begin{aligned} N\sum_{i=1}^{N+1}\|(T-S)\delta_i\|^q &\geq \sum_{s=1}^{M}\sum_{i\in\{0\}\cup Q_s}\|y_{is}\|^q \\ &= \sum_{s=1}^{M}\Big\|\sum_{i\in\{0\}\cup Q_s} y_{is}\Big\|^q \\ &= \sum_{i=1}^{M}\|x_s\|^q = M. \end{aligned}$$

Hence, for some i, $\|(T-S)\delta_i\|^q \geq M/(N(N+1)) = 1/2$. ∎

9 Modulus of an ε-DP Operator

By [23, §3.1], the modulus of a disjointness preserving operator T exists, and for any $x \geq 0$, $|T|x = |Tx|$. It is easy to see that $\||T|\| = \|T\|$, and that $|T|$ preserves disjointness. Conversely, if $|T|$ exists and is disjointness preserving, then the same is true for T. More generally, if $|T|$ is ε-DP, then T is ε-DP. Indeed, suppose $|T|$ is ε-DP, and pick disjoint x and y:

$$\big\||Tx| \wedge |Ty|\big\| \leq \big\||T||x| \wedge |T||y|\big\| \leq \varepsilon \max\big\{\|x\|, \|y\|\big\}.$$

For operators into Dedekind complete $C(K)$ spaces we have a converse.

Proposition 9.1 *Consider $T \in B(E, F)$, where E and F are Banach lattices, and F is an M-space. If $T \in B(E, F)$ is ε-DP and the modulus $|T|$ exists, then $|T|$ is ε-DP.*

Remark 9.2 Suppose, in Proposition 9.1, F is a Dedekind complete M-space, with a strong order unit (equivalently, $F = C(K)$, where K is a Stonian compact Hausdorff space [20, §1.a–b]). Then any operator $T \in B(E, F)$ has modulus $|T|$ and $\||T|\| = \|T\|$ [29].

Proof Recall that for any $x \in E$ we have $|T||x| = \bigvee_{|y| \le |x|} |Ty|$. Now given disjoint x_1, x_2, we have

$$\begin{aligned}\Big\| \big||T|x_1\big| \wedge \big||T|x_2\big| \Big\| &\le \Big\| |T|\,|x_1| \wedge |T|\,|x_2| \Big\| \\ &= \Big\| \bigvee_{|y_1| \le |x_1|} |Ty_1| \wedge \bigvee_{|y_2| \le |x_2|} |Ty_2| \Big\| \\ &= \Big\| \bigvee_{\substack{|y_1| \le |x_1| \\ |y_2| \le |x_2|}} |Ty_1| \wedge |Ty_2| \Big\|.\end{aligned}$$

As F is an M-space,

$$\Big\| \bigvee_{\substack{|y_1| \le |x_1| \\ |y_2| \le |x_2|}} |Ty_1| \wedge |Ty_2| \Big\| = \sup_{\substack{|y_1| \le |x_1| \\ |y_2| \le |x_2|}} \big\| |Ty_1| \wedge |Ty_2| \big\|.$$

Recall that T is ε-DP, hence

$$\big\| |Ty_1| \wedge |Ty_2| \big\| \le \varepsilon \max\{\|y_1\|, \|y_2\|\} \le \varepsilon \max\{\|x_1\|, \|x_2\|\},$$

and therefore, $\Big\| \big||T|x_1\big| \wedge \big||T|x_2\big| \Big\| \le \varepsilon \max\{\|x_1\|, \|x_2\|\}$. ∎

Incidentally, in the non-locally convex setting, we have some stability for the modulus of an ε-DP operator.

Proposition 9.3 *Let $0 < p \le 1/2$ be a Banach lattice E and $T\colon \ell_p \to E$ be an ε-DP operator. The modulus $|T|$ (which is also bounded) is $\sqrt{\varepsilon\|T\|}$-DP.*

Proof Let $f_n = T\delta_n$, where $(\delta_n)_{n=1}^\infty$ form the canonical basis of ℓ_p. We have that $|T|\delta_n = |f_n|$. Indeed, since δ_n is an atom, we have

$$|T|\delta_n = \sup\{|Ty| : |y| \le \delta_n\} = \sup\{|T\lambda\delta_n| : |\lambda| \le 1\} = |T\delta_n|.$$

Therefore, $|T|\colon \ell_p \to E$ is given by $|T|\big(\sum_n a_n\delta_n\big) = \sum_n a_n|f_n|$ (which defines a bounded operator). We claim that for $n \ne m$,

$$\big\| |a_n f_n| \wedge |b_m f_m| \big\| \le \sqrt{\varepsilon\|T\|}\sqrt{|a_n||b_m|}. \tag{9.1}$$

Indeed, as T is ε-DP, we have $\big\| |a_n f_n| \wedge |b_m f_m| \big\| \le \varepsilon(|a_n| \vee |b_m|)$. Also,

$$\big\| |a_n f_n| \wedge |b_m f_m| \big\| \le \|a_n f_n\| \wedge \|b_m f_m\| \le \|T\|\,|a_n| \wedge |b_m|.$$

Assume without loss of generality that $|a_n| \le |b_m|$. Then

$$\big\| |a_n f_n| \wedge |b_m f_m| \big\| \le \varepsilon|b_m| \wedge \|T\|\,|a_n| \le \sqrt{\varepsilon|b_m|\|T\||a_n|},$$

establishing (9.1).

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Now let $x, y \in \ell_p$ be disjoint elements. We can write $x = \sum_{i\in A} a_i\delta_i$, $y = \sum_{j\in B} b_j\delta_j$ with $A \cap B = \varnothing$. Taking (9.1) into account, we obtain

$$\begin{aligned}\big\| \big||T|x\big| \wedge \big||T|y\big| \big\| &\le \big\| |T|\big(\sum_{i\in A}|a_i|\delta_i\big) \wedge |T|\big(\sum_{j\in B}|b_j|\delta_j\big) \big\| \\ &\le \Big\| \sum_{i\in A}\sum_{j\in B} |a_i f_i| \wedge |b_j f_j| \Big\| \le \sum_{i\in A}\sum_{j\in B} \big\| |a_i f_i| \wedge |b_j f_j| \big\| \\ &\le \sum_{i\in A}\sum_{j\in B} \sqrt{\varepsilon\|T\|}\sqrt{|a_i||b_j|} = \sqrt{\varepsilon\|T\|}\sqrt{\|x\|_{\frac12}\|y\|_{\frac12}} \\ &\le \sqrt{\varepsilon\|T\|}\sqrt{\|x\|_p\|y\|_p} \le \sqrt{\varepsilon\|T\|}\max\big\{\|x\|_p, \|y\|_p\big\}. \qquad \blacksquare\end{aligned}$$

The result below shows that, in general, the ε-disjointness preserving properties of T do not allow us to conclude anything about the ε-disjointness properties of $|T|$, even if the latter exists.

Proposition 9.4 *For every $\varepsilon > 0$, there exists an operator $T \in B(\ell_2)$ such that $\|T\| \ge 1$, $\big\||T|\big\| \le 2$, and T is ε-DP, yet $|T|$ is not c-DP whenever $c \le 1/2$. Moreover, $\|T - I_{\ell_2}\| < \varepsilon$, while $\||T| - U\| \ge 1/(3\sqrt{2})$ whenever U is disjointness preserving.*

Start by observing that the property of being ε-DP is preserved by direct sums.

Lemma 9.5 *Suppose $(E_i)_{i\in\mathbb{N}}$, $(F_i)_{i\in\mathbb{N}}$ are Banach lattices, U is a Banach space with a 1-unconditional basis, and the operators $T_i \in B(E_i, F_i)$ are such that $\sup_i \|T_i\| < \infty$. Define the Banach lattices $E = (\bigoplus_i E_i)_U$ and $F = (\bigoplus_i F_i)_U$, and the operator $T = \bigoplus_i T_i \in B(E, F)$. If T_i is ε-DP for every $i \in \mathbb{N}$, then T is 2ε-DP.*

Proof Consider disjoint $x = (x_i)_{i\in\mathbb{N}}, y = (y_i)_{i\in\mathbb{N}} \in E$ (here $x_i, y_i \in E_i$, for every $i \in \mathbb{N}$). By [20, Proposition 1.d.2], we have

$$\begin{aligned}\big\| |Tx| \wedge |Ty| \big\| &= \big\| \big(\||T_i x_i| \wedge |T_i y_i|\| \big)_i \big\|_U \\ &\le \varepsilon \|(\max\{\|x_i\|, \|y_i\|\})_i\|_U \\ &\le \varepsilon \|(\|x_i\| + \|y_i\|)_i\|_U \\ &\le 2\varepsilon \max\big\{\|x\|, \|y\|\big\}. \qquad \blacksquare\end{aligned}$$

Proof of Proposition 9.4 Consider the operators $S_i \in B(\ell_2^{2^i})$, given by unitary Walsh matrices. It is known that $|S_i| = 2^{i/2}\xi_i \otimes \xi_i$, where ξ_i is the unit vector $2^{-i/2}\sum_{j=1}^{2^i} e_j$ ($e_1, \dots, e_{2^i}$ is the canonical basis of $\ell_2^{2^i}$). Let $T_i = I_{\ell_2^{2^i}} + 2^{-i/2}S_i$. Pick $k \in \mathbb{N}$ so that $2^{-k/6} < \varepsilon/6$. Identify $(\bigoplus_{i\ge k}\ell_2^{2^i})_2$ with ℓ_2. Then we can view $T = \bigoplus_{i\ge k} T_i$ as an operator on ℓ_2. We show that T has the required properties.

Indeed, for any i, $\|T_i\| \ge 1 - 2^{-i/2}$. Hence $\|T\| = \sup_i \|T_i\| \ge 1$. Furthermore, $\|T - I_{\ell^2}\| = \sup_i 2^{-i/2}\|S_i\| \le \varepsilon$. The operator $|T| = \bigoplus_i (I_{\ell_2^{2^i}} + \xi_i \otimes \xi_i)$ has norm 2.

Now fix $i > k$, and consider disjoint vectors

$$x = 2^{-(i-1)/2}\sum_{j=1}^{2^{i-1}} e_j \quad \text{and} \quad y = 2^{-(i-1)/2}\sum_{j=2^{i-1}+1}^{2^i} e_j$$

in the unit ball of $\ell_2^{2^i}$. Then $|T|x = |T|y = 2^{-1/2}\xi_i$. Hence $\big\| \big||T|x\big| \wedge \big||T|y\big| \big\| = 2^{-1/2}$. Thus, $|T|$ cannot be c-DP for $c < 1/2$.

To prove that T is ε-DP, it suffices to prove (in light of Lemma 9.5) that for any $i > k$, $I + 2^{-i/2}S_i$ is $\varepsilon/2$-DP. If $x, y \in \mathbf{B}(\ell_2^{2^i})$ are disjoint, then

$$\begin{aligned}\big|\big(I + 2^{-i/2}S_i\big)x\big| \wedge \big|\big(I + 2^{-i/2}S_i\big)y\big| &\le \big(|x| + 2^{-i/2}|S_i x|\big) \wedge \big(|y| + 2^{-i/2}|S_i y|\big)\\ &\le |x| \wedge 2^{-i/2}|S_i y| + 2^{-i/2}|S_i x| \wedge |y| + 2^{-i/2}|S_i x| \wedge 2^{-i}|S_i y|.\end{aligned}$$

Hence

$$\begin{aligned}\big\|\big|(I + 2^{-i/2}S_i)x\big| \wedge \big|(I + 2^{-i/2}S_i)y\big|\big\| &\le \min\big\{2^{-i/2}\|x\|, \|y\|\big\} + \min\big\{2^{-i/2}\|y\|, \|x\|\big\}\\ &\quad + \min\big\{2^{-i/2}\|x\|, 2^{-i/2}\|y\|\big\}\\ &\le 3 \cdot 2^{-i/2} \le \varepsilon/2,\end{aligned}$$

by our choice of k.

Finally, suppose $U \in B(\ell_2)$ is a disjointness preserving operator. Let $V = |T| - U$, and suppose, for the sake of contradiction, that $\|V\| < 1/(3\sqrt{2})$. As before, take $x = 2^{-(i-1)/2}\sum_{j=1}^{2^{i-1}} e_j$ and $y = 2^{-(i-1)/2}\sum_{j=2^{i-1}+1}^{2^i} e_j$. Then $\big\|\big||T|x\big| \wedge \big||T|y\big|\big\| = 2^{-1/2}$. On the other hand,

$$\begin{aligned}|T|x \wedge |T|y = (Ux + Vx) \wedge (Uy + Vy) &\le (|Ux| + |Vx|) \wedge (|Uy| + |Vy|)\\ &\le |Ux| \wedge |Vy| + |Vx| \wedge |Uy| + |Vx| \wedge |Vy|.\end{aligned}$$

Hence $\frac{1}{\sqrt{2}} = \big\|\big||T|x\big| \wedge \big||T|y\big|\big\| \le \|Vy\| + 2\|Vx\| \le 3\|V\| < \frac{1}{\sqrt{2}}$, yielding a contradiction. ∎

10 Lattice Homomorphisms and Operators Preserving p-estimates

Let us now consider positive operators being "almost lattice homomorphisms." We say that an operator $T \in B(E,F)$ is an *ε-lattice homomorphism* (*ε-LH* for short) if, for any $x \in E$, $\big\|\big|T|x|\big| - |Tx|\big\| \le \varepsilon\|x\|$. A positive operator $T \in B(E,F)$ is said to be *ε-minimum preserving* (*ε-MP*) if, for any positive $x, y \in \mathbf{B}(E)$,

$$\|(Tx) \wedge (Ty) - T(x \wedge y)\| \le \varepsilon.$$

It is known [23, §3.1] that a positive operator is disjointness preserving if and only if it is 0-LH, if and only if it is 0-MP; in this case, it is a lattice homomorphism. In the "approximate" case, the notions introduced above are connected to being ε'-DP as well (for some ε' depending on ε).

Proposition 10.1 *For Banach lattices E and F, and $T \in B(E,F)$, the following hold.*

(i) *If T is positive, then T is ε-MP if and only if it is ε-DP.*
(ii) *Any ε-DP operator between real Banach lattices is a 2ε-LH.*
(iii) *If T is ε-LH, then T is 4ε-DP in the real case, or 16ε-DP in the complex case. If, in addition, T is positive, then it is ε-DP.*

Proof (i) If T is ε-MP, then it is ε-DP, by Proposition 2.1. To prove the converse, consider $x, y \in \mathbf{B}(E)_+$. Then $x_0 = x - x \wedge y$ and $y_0 = x - x \wedge y$ are positive and

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disjoint, and

$$Tx \wedge Ty - T(x \wedge y) = (Tx_0 + T(x \wedge y)) \wedge (Ty_0 + T(x \wedge y)) - T(x \wedge y) = Tx_0 \wedge Ty_0.$$

If T is ε-DP, then $\|Tx \wedge Ty - T(x \wedge y)\| = \|Tx_0 \wedge Ty_0\| \leq \varepsilon$.

(ii) Suppose T is a ε-DP map between real Banach lattices. Then, for any $x \in E$,

$$\Big| |Tx| - \big|T|x|\big| \Big| = \big| |Tx_+ - Tx_-| - |Tx_+ + Tx_-| \big| = 2\big(|Tx_+| \wedge |Tx_-|\big).$$

As $\max\{\|x_+\|, \|x_-\|\} \leq \|x\|$, and $x_+ \perp x_-$ we have $\| |Tx| - |T|x|| \| \leq 2\varepsilon\|x\|$.

(iii) Suppose T is ε-LH, and pick disjoint positive $y, z \in \mathbf{B}(E)$. Let $x = y - z$. As in part (ii), we obtain

$$\big\| |Ty| \wedge |Tz| \big\| = \frac{1}{2} \Big\| |Tx| - \big|T|x|\big| \Big\| \leq \frac{\varepsilon}{2}\|x\| \leq \frac{\varepsilon}{2}(\|y\| + \|z\|) \leq \varepsilon.$$

To finish the proof, apply Proposition 2.1. ∎

In the rest of the section we consider operators that almost preserve estimates of the form $(|x|^p + |y|^p)^{1/p}$ and their connections with ε-DP operators and lattice homomorphisms. This approach is in part motivated by Corollary 3.3. In particular, this will allow us to extend some of the previous results to the complex setting (see Proposition 10.5.)

Given $1 \leq p \leq \infty$, a positive operator between Banach lattices $T: E \to F$ is said to be ε *preserving p-estimates* if for every $x, y \in E$ we have

$$\Big\| T(|x|^p + |y|^p)^{\frac{1}{p}} - (|Tx|^p + |Ty|^p)^{\frac{1}{p}} \Big\| \leq \varepsilon(\|x\| + \|y\|),$$

while for $p = \infty$, we would have

$$\big\| T(|x| \vee |y|) - (|Tx| \vee |Ty|) \big\| \leq \varepsilon(\|x\| + \|y\|).$$

It is easy to see that an operator is ε preserving 1-estimates if and only if it is an ε-lattice homomorphism. More generally, we have the following proposition.

Proposition 10.2 *Let E and F be real Banach lattices. If $T \in B(E, F)$ is a positive ε-DP operator, then for every $1 < p < \infty$, T is $K \log_2(\varepsilon(\|T\| + 1))^{-1}(\varepsilon(\|T\| + 1))^{1/2}$ preserving p-estimates (where K is a universal constant).*

Recall that according to Proposition 10.1 (i), a positive operator is ε-MP if and only if it is ε-DP. Before giving the proof, we need a preliminary lemma.

Lemma 10.3 *If $T \in B(E, F)$ is a positive ε-MP operator, then for any $x_1, \dots, x_n \in \mathbf{B}(E_+)$ we have $\|T(\bigvee_{i=1}^n x_i) - \bigvee_{i=1}^n Tx_i\| \leq \varepsilon\lceil \log_2 n \rceil n$.*

Proof It suffices to show that for any $m \in \mathbb{N}$, $\|T(\bigvee_{i=1}^{2^m} x_i) - \bigvee_{i=1}^{2^m} Tx_i\| \leq \varepsilon m 2^{m-1}$. Proceed by induction on m. The case of $m = 1$ is contained in the definition of T being ε-MP. To deal with the induction step, suppose the statement holds for m, and prove it for $m + 1$. For $j = 0, 1$ let

$$y_j = \bigvee_{i=2^m j+1}^{2^m j+2^m} x_i \quad \text{and} \quad z_j = Ty_j - \bigvee_{i=2^m j+1}^{2^m j+2^m} Tx_i.$$

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By the induction hypothesis, $\|z_j\| \le \varepsilon m 2^{m-1}$ (and it is easy to see that $z_j \ge 0$). Also,

$$\|T(y_0 \vee y_1) - (Ty_0) \vee (Ty_1)\| \le \varepsilon \max\{\|y_0\|, \|y_1\|\} \le 2^m \varepsilon.$$

We clearly have

$$\begin{aligned} T\Big(\bigvee_{i=1}^{2^m} x_i\Big) - \bigvee_{i=1}^{2^m} Tx_i &= T(y_0 \vee y_1) - (Ty_0 - z_0) \vee (Ty_1 - z_1) \\ &\le \big(T(y_0 \vee y_1) - (Ty_0) \vee (Ty_1) \big) + z_0 + z_1. \end{aligned}$$

Hence,

$$\Big\| T\Big(\bigvee_{i=1}^{2^m} x_i\Big) - \bigvee_{i=1}^{2^m} Tx_i \Big\| \le \big\| T(y_0 \vee y_1) - (Ty_0) \vee (Ty_1) \big\| + (\|z_0\| + \|z_1\|).$$

From the above, $\| \bigvee_{i=1}^n Tx_i - T(\bigvee_{i=1}^n x_i)\| \le 2^m \varepsilon + 2 \cdot m 2^{m-1} \varepsilon = (m+1) 2^m \varepsilon$. ■

We also need a simple calculus result.

Lemma 10.4 *Suppose ϕ is a monotone continuous function on an interval $[a,b]$, continuously differentiable on (a,b). Then the arclength of the graph of ϕ does not exceed $b - a + |\phi(b) - \phi(a)|$.*

Proof For the arclength in question we have

$$L = \int_a^b \sqrt{1 + (\phi'(t))^2}\, dt \le \int_a^b \big(1 + |\phi'(t)|\big)\, dt.$$

The monotonicty of ϕ implies $\int_a^b |\phi'(t)|\, dt = |\phi(b) - \phi(a)|$. ■

Proof of Proposition 10.2 For any u and v in a Banach lattice, if $1/p + 1/q = 1$, then (see [20, 1.d])

$$\big(|u|^p + |v|^p \big)^{1/p} = \bigvee \big\{ \alpha|u| + \beta|v| : \alpha, \beta \in [0,1], \alpha^q + \beta^q = 1 \big\}.$$

For any $N \in \mathbb{N}$, let $\{(x_j, y_j) : j = 0, 1, \dots N\}$ be a collection of points satisfying $x_j, y_j \in [0,1]$, $x_j^q + y_j^q = 1$ and such that for any (α, β) with $\alpha, \beta \in [0,1]$ and $\alpha^q + \beta^q = 1$, there exists $0 \le j \le N$ for which $\max\{|\alpha - x_j|, |\beta - y_j|\} \le C_q/N$, where C_q is the length of the curve $\{(x,y) : x, y \in [0,1],\ x^q + y^q = 1\}$. By Lemma 10.4, $C_q \le 2$. Thus, for any (α, β) with $\alpha, \beta \in [0,1]$ and $\alpha^q + \beta^q = 1$, there exists j so that

$$\alpha|u| + \beta|v| \le (x_j|u| + y_j|v|) + \frac{2}{N}(|u| + |v|).$$

Taking the supremum, we obtain

$$\bigvee \big\{ \alpha|u| + \beta|v| : \alpha, \beta \in [0,1], \alpha^q + \beta^q = 1 \big\} \le \bigvee_{j=0}^{N} (x_j|u| + y_j|v|) + \frac{2}{N}(|u| + |v|),$$

and by the triangle inequality we get

$$\Big\| (|u|^p + |v|^p)^{1/p} - \bigvee_{j=0}^{N} (x_j|u| + y_j|v|) \Big\| \le \frac{2}{N}(\|u\| + \|v\|).$$

By Lemma 10.3,

$$\Big\| T \bigvee_{j=0}^{N} (x_j|x| + y_j|y|) - \bigvee_{j=0}^{N} (x_j T|x| + y_j T|y|) \Big\| \le \varepsilon 2^{\frac{1}{p}} \lceil \log_2(N+1) \rceil (N+1).$$

By Proposition 10.1 (ii), T is ε-LH, hence $\|T|x| - |Tx|\|, \|T|y| - |Ty|\| \le \varepsilon$. Hence

$$\Big\| \bigvee_{j=0}^{N} (x_j T|x| + y_j T|y|) - \bigvee_{j=0}^{N} (x_j |Tx| + y_j |Ty|) \Big\| \le 2(N+1)\varepsilon.$$

Thus, by the triangle inequality,

$$\begin{aligned}
\Big\| T(|x|^p + |y|^p)^{1/p} - \big(|Tx|^p + |Ty|^p\big)^{1/p} \Big\| & \\
\le \Big\| T(|x|^p + |y|^p)^{1/p} - T \bigvee_{j=0}^{N} (x_j|x| + y_j|y|) \Big\| & \\
+ \Big\| T \bigvee_{j=0}^{N} (x_j|x| + y_j|y|) - \bigvee_{j=0}^{N} (x_j T|x| + y_j T|y|) \Big\| & \\
+ \Big\| \bigvee_{j=0}^{N} (x_j T|x| + y_j T|y|) - \bigvee_{j=0}^{N} (x_j |Tx| + y_j |Ty|) \Big\| & \\
+ \Big\| \bigvee_{j=0}^{N} (x_j |Tx| + y_j |Ty|) - (|Tx|^p + |Ty|^p)^{1/p} \Big\| & \\
\le \frac{4\|T\|}{N} + \varepsilon(N+1)\Big(2^{\frac{1}{p}} \lceil \log_2(N+1) \rceil + 2\Big). &
\end{aligned}$$

To finish the proof, select $N \sim (\varepsilon(\|T\|+1))^{-1/2}$. ∎

As a consequence of this result, we can now give the complex version of Proposition 10.1 (ii). We follow [2] in representing a complex Banach lattice X as a complexification of its real part $X_{\mathbb{R}}$. More precisely, any $x \in X$ can be represented as $x = a + \imath b$, with $a, b \in X_{\mathbb{R}}$. Then $|x| = (a^2 + b^2)^{1/2}$.

Proposition 10.5 *Suppose E and F are complex Banach lattices, and $T \in B(E,F)$ is a positive ε-DP operator. Then T is a $C \log_2(\varepsilon(\|T\|+1))^{-1}(\varepsilon(\|T\|+1))^{1/2}$-LH (with C a universal constant).*

Proof Consider $T \in B(E,F)$ as in the statement; we show that, for any $x \in \mathbf{B}(E)$,

$$\|T|x| - |Tx|\| \le C \log_2(\varepsilon(\|T\|+1))^{-1}(\varepsilon(\|T\|+1))^{1/2}.$$

By Proposition 10.1 (i, iii), $T|_{E_{\mathbb{R}}}$ is 2ε-LH. Hence by Proposition 10.2, it follows that $T|_{E_{\mathbb{R}}}$ is $K \log_2(2\varepsilon(\|T\|+1))^{-1}(2\varepsilon(\|T\|+1))^{1/2}$ preserving 2-estimates.

Now write $x = a + \imath b$, where a and b belong to $E_{\mathbb{R}}$. We have that

$$\begin{aligned}
\|T|x| - |Tx|\| &= \big\| T(a^2+b^2)^{1/2} - ((Ta)^2 + (Tb)^2)^{1/2} \big\| \\
&\le 2K \log_2(2\varepsilon(\|T\|+1))^{-1}(2\varepsilon(\|T\|+1))^{1/2}.
\end{aligned}$$ ∎

Motivated by Lemma 10.3 we will consider next a strengthening of operators that are ε preserving ∞-estimates. For $\varepsilon > 0$, we say that a positive operator $T \in B(E,F)$

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(E and F are Banach lattices) is ε-*strongly maximum preserving* (ε-*SMP* for short) if, for any $x_1, \dots, x_n \in \mathbf{B}(E)_+$, we have $\|T(\bigvee_{i=1}^n x_i) - \bigvee_{i=1}^n Tx_i\| \le \varepsilon$.

We say that $T \in B(E, F)$ is an ε-*strongly disjointness preserving* (ε-*SDP*) if, for any mutually disjoint $x_1, \dots, x_n \in \mathbf{B}(E)$, we have $\|\sum_{i=1}^n |Tx_i| - \bigvee_{i=1}^n |Tx_i|\| \le \varepsilon$. Clearly, any ε-SMP positive operator is also ε-SDP.

Note that these properties are much harder to satisfy. For instance, it is easy to see that any operator T is $\|T\|$-DP. On the other hand, for a pair of Banach lattices (E, F), the following are equivalent.

(1) E is lattice isomorphic to an M-space.
(2) There exists $C > 0$ so that any $T \in B(E, F)_+$ is $C\|T\|$-SDP.

To prove (1) $\Rightarrow$ (2), suppose E is an M-space. Fix a positive operator $T: E \to F$, and consider mutually disjoint $x_1, \dots, x_n \in \mathbf{B}(E)$. Then

$$\Big\| \sum_i |Tx_i| \Big\| \le \Big\| T \sum_i |x_i| \Big\| \le \|T\| \Big\| \sum_i |x_i| \Big\| = \|T\| \max_i \|x_i\|,$$

which implies (2).

For (2) $\Rightarrow$ (1), recall that, by [23, §2.1, 2.8], the following are equivalent:

- E is lattice isomorphic to an M-space;
- there exists a constant K so that the inequality $\|\sum_i x_i\| \le K \max_i \|x_i\|$ holds whenever $x_1, \dots, x_n \in E$ are mutually disjoint;
- there exists a constant K so that the inequality $\|\sum_i x_i^*\| \ge K^{-1} \sum_i \|x_i^*\|$ holds whenever $x_1^*, \dots, x_n^* \in E^*$ are mutually disjoint.

Suppose now that (1) fails; we show that (2) fails as well.

If (1) fails, then for every $C > 1$ there exist mutually disjoint non-zero $x_1^*, \dots, x_n^* \in E_+^*$, satisfying $\|\sum_i x_i^*\| < (C+2)^{-1} \sum_i \|x_i^*\|$. Without loss of generality, we can assume $1 = \max_i \|x_i^*\|$. Applying [23, Proposition 1.4.13] to $x_i^*/\|x_i^*\|$, we see that, for any $\sigma > 0$, there exist mutually disjoint $x_1, \dots, x_n \in \mathbf{B}(E)_+$ so that $\langle x_i^*, x_i \rangle > \|x_i^*\| - n^{-1}$ for any i.

Now let $x^* = \sum_i x_i^*$; pick a norm one positive $y \in F$, and define $T: E \to \operatorname{span}[y] \subset F: x \mapsto \langle x^*, x \rangle y$. Clearly $\|T\| = \|x^*\|$. On the other hand, $\max_i \|x_i\| = 1$, $\bigvee_i Tx_i \le y$, and $\sum_i Tx_i = \big(\sum_i \sum_j \langle x_i^*, x_j \rangle\big) y \ge \big(\sum_i \|x_i^*\| - 1\big) y$. Consequently, if T is $\gamma\|T\|$-SDP, then

$$\gamma \ge \frac{\sum_i \|x_i^*\| - 2}{\|\sum_i x_i^*\|} > C.$$

As C can be arbitrarily large, we are done.

Theorem 10.6 *Suppose E and F are Banach lattices, and $T \in B(E, F)$ is a positive ε-SDP operator.*

(i) *Suppose E is finite dimensional. Then there exists a disjointness preserving $S \in B(E, F)$ so that $0 \le S \le T$, and $\|T - S\| \le 2\varepsilon$.*
(ii) *Suppose the order on E is determined by its* 1*-unconditional basis, while F has the Fatou property with constant $\mathfrak{f}$. Then there exists a disjointness preserving $S \in B(E, F)$ so that $0 \le S \le T$, and $\|T - S\| \le 2\mathfrak{f}\varepsilon$.*

Remark 10.7 By Corollary 3.3, if a positive operator T is ε-DP, then for any mutually disjoint $x_1, \dots, x_n \in \mathbf{B}(E)$, we have $\|\sum_{i=1}^n |Tx_i| - \bigvee_{i=1}^n |Tx_i|\| \le 256\varepsilon \|\sum_i x_i\|$. In particular, this holds for the operator T from Proposition 8.1. However, in light of Theorem 10.6, if T is σ-SDP, then $\sigma > 1/4$. Thus, there is no function $f:(0,\infty)\to(0,\infty)$, with $\lim_{t\to 0} f(t) = 0$, so that being ε-DP implies being $f(\varepsilon)$-SDP.

Proof (i) It is well known [28, Corollary 4.20] that X has a basis of atoms, which we denote by $(\delta_i)_{i=1}^n$ ($n = \dim X$), and they form a 1-unconditional basis. Use scaling to assume that T is contractive. Let $f_i = T\delta_i$. As in the proof of Theorem 4.1, define the function $\phi_n:\mathbb{R}^n \to \mathbb{R}$ by setting

$$\phi_n:(t_1,\dots,t_n)\mapsto \begin{cases} 0 & \text{if } |t_1| \le \bigvee_{i=2}^n |t_i|, \\ 2(|t_1| - \bigvee_{i=2}^n |t_i|) & \text{if } \bigvee_{i=2}^n |t_i| \le |t_1| \le 2\bigvee_{i=2}^n |t_i|, \\ |t_1| & \text{if } |t_1| \le \bigvee_{i=2}^n |t_i|. \end{cases}$$

For $1 \le i \le n$, set $g_i = \phi_n(f_i, f_{i+1},\dots,f_n,f_1,\dots,f_{i-1})$. We claim that the operator $S: E \to F: \delta_i \mapsto g_i$ has the desired properties.

Note that $0 \le \phi_n(t_1,\dots,t_n) \le t_1$. Hence $0 \le g_i \le f_i$, which shows that $0 \le S \le T$.

To show that S is disjointness preserving, consider $i \ne j$. Note that

$$\phi_n(t_i, t_{i+1},\dots,t_n,t_1,\dots,t_{i-1}) \wedge \phi_n(t_j, t_{j+1},\dots,t_n,t_1,\dots,t_{j-1}) = 0,$$

for any $(t_1,\dots,t_n)\in\mathbb{R}^n$. Hence g_i and g_j are disjoint.

Now note that $\|T - S\| \le \|(T-S)\sum_{i=1}^n \delta_i\| = \|\sum_{i=1}^n (f_i - g_i)\|$. It therefore suffices to show that

$$\sum_{i=1}^n (f_i - g_i) \le 2\Big(\sum_{i=1}^n f_i - \bigvee_{i=1}^n f_i\Big). \tag{10.1}$$

Indeed, applying the definition of ε-SDP to $x_i = \delta_i$, we obtain $\big\|\sum_{i=1}^n f_i - \bigvee_{i=1}^n f_i\big\| \le \varepsilon$. To establish (10.1), by functional calculus it suffices to show that

$$(t_1 - \phi_n(t_1,t_2,\dots,t_n)) + (t_2 - \phi_n(t_2,t_3,\dots,t_n,t_1)) + \cdots \\ + (t_n - \phi_n(t_n,t_1,\dots,t_{n-1})) \le 2\Big(\sum_{i=1}^n t_i - \bigvee_{i=1}^n t_i\Big),$$

for any $t_1,\dots,t_n \in \mathbb{R}_+^n$. By relabeling, we can assume that $t_1 \ge t_2 \ge \cdots \ge t_n$. In the left-hand side, the i-th term equals t_i, while the first term does not exceed t_2. Thus, the left-hand side does not exceed $2t_2 + t_3 + \cdots + t_n$ On the other hand, the right-hand side equals $2\sum_{i=2}^n t_i$.

(ii) Now denote the basis of X by $(\delta_i)_{i=1}^\infty$, and set $f_i = T\delta_i$. With the notation of (i), set $g_i^{(n)} = \phi_n(f_i, f_{i+1},\dots,f_n,f_1,\dots,f_{i-1})$. Note that, for any $t_1,\dots,t_{n+1}\in\mathbb{R}_+$, we have $\phi_n(t_1,t_2,\dots,t_n) = \phi_{n+1}(t_1,t_2,\dots,t_n,0) \ge \phi_{n+1}(t_1,t_2,\dots,t_n,t_{n+1})$. Hence we have $f_i \ge g_i^{(i)} \ge g_i^{(i+1)} \ge g_i^{(i+2)} \ge \cdots \ge 0$. By the σ-Dedekind completeness of F, $g_i = \lim_n g_i^{(n)}$ exists for every i. Define $S: E \to F$ by setting $S\delta_i = g_i$. Clearly $0 \le S \le T$. Furthermore, S is disjointness preserving. Indeed, if $i \ne j$, and $n \ge i \vee j$, then for any $t_1,\dots,t_n \in \mathbb{R}_+$,

$$\phi_n(t_i, t_{i+1},\dots,t_n,t_1,\dots,t_{i-1}) \wedge \phi_n(t_j, t_{j+1},\dots,t_n,t_1,\dots,t_{j-1}) = 0.$$

Hence $g_i^{(n)} \wedge g_j^{(n)} = 0$.

To estimate $\|T-S\|$, note that

$$\|T-S\| \leq \sup_m \Big\| (T-S)\sum_{i=1}^{m} \delta_i \Big\| = \sup_m \Big\| \sum_{i=1}^{m} (f_i - g_i) \Big\|.$$

For each m, $\| \sum_{i=1}^{m}(f_i - g_i)\| \leq \mathfrak{f}\sup_n \| \sum_{i=1}^{m}(f_i - g_i^{(n)})\|$. By the proof of part (i),

$$\Big\| \sum_{i=1}^{m}(f_i - g_i^{(n)}) \Big\| \leq 2\varepsilon,$$

and the proof is complete. ∎

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