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S-BARRELLED TOPOLOGICAL VECTOR SPACES*

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N. Bourbaki [1] was the first to introduce the class of locally convex topological vector spaces called "espaces tonnelés" or "barrelled spaces." These spaces have some of the important properties of Banach spaces and Fréchet spaces. Indeed, a generalized Banach-Steinhaus theorem is valid for them, although barrelled spaces are not necessarily metrizable. Extensive accounts of the properties of barrelled locally convex topological vector spaces are found in [5] and [8].

In the last few years, many new classes of locally convex spaces having various types of barrelledness properties have been defined (see [6] and [7]).

In this paper, using simple notions of sequential convergence, we introduce several new classes of topological vector spaces which are similar to Bourbaki's barrelled spaces. The most important of these we call S-barrelled topological vector spaces. A topological vector space $(X, \mathcal{T})$ is *S-barrelled* if every sequentially closed, convex, balanced, and absorbing subset of X is a sequential neighborhood of the zero vector 0 in X. Examples are given of spaces which are S-barrelled but not barrelled. Then some of the properties of S-barrelled spaces are given—including permanence properties, and a form of the Banach-Steinhaus theorem valid for them. S-barrelled spaces are useful in the study of sequentially continuous linear transformations and sequentially continuous bilinear functions.

For the most part, we use the notations and definitions of [5]. Proofs which closely follow the standard patterns are omitted.

1. Definitions and Examples. Let $(X, \mathcal{T})$ be a topological vector space (not necessarily locally convex) over the real or complex field $\mathbb{K}$. A subset B of X is said to be a *barrel* if B is closed, convex, balanced, and absorbing. The set B is said to be a *sequential barrel* or an *S-barrel* if B is sequentially closed, convex, balanced, and absorbing. Every barrel is an S-barrel.

The topological vector space $(X, \mathcal{T})$ is said to be *barrelled* if every barrel in X is a neighborhood of the zero vector 0 in X. A subset B of X is a *sequential neighborhood* of the zero vector 0 in X if every sequence in X which converges to 0 is ultimately in B. The space $(X, \mathcal{T})$ is said to be *S-barrelled* if every

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sequential barrel in X is a sequential neighborhood of 0. The space $(X, \mathscr{T})$ is said to be S_1-*barrelled* if every barrel in X is a sequential neighborhood of 0. Finally, $(X, \mathscr{T})$ is said to be S_3-*barrelled* if every sequential barrel in X is a neighborhood of 0.

We would like to call an S-barrelled space a "sequentially barrelled space", but this last name has been used by Webb for a different type of topological vector space([13], p. 353).

For a topological vector space $(X, \mathscr{T})$, the following implications hold:

$$\begin{tikzcd}
 & \text{barrelled} \arrow[dr, Rightarrow] & \\
S_3\text{-barrelled} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & S_1\text{-barrelled} \\
 & S\text{-barrelled} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

A topological vector space $(X, \mathscr{T})$ is said to be *sequential* ([15], pp. 50–54) if every sequentially open subset of X is open, and C-*sequential* ([3], p. 493; or [12], p. 273) if every convex sequentially open subset of X is open. Every sequential, barrelled topological vector space is S-barrelled. Every C-sequential, S-barrelled topological vector space is barrelled. For a topological vector space $(X, \mathscr{T})$, these and some additional implications are given below:

$$\begin{tikzcd}
 & \text{barrelled} \arrow[dl, Rightarrow, "\text{sequential}"'] \arrow[dd, Rightarrow, no head, shift right=3, "\text{C-sequential}"'] \arrow[dd, Rightarrow, no head, shift left=3, "\text{sequential}"] & \\
S_3\text{-barrelled} & & S_1\text{-barrelled} \arrow[ul, Rightarrow, "\text{C-sequential}"'] \arrow[dl, Rightarrow, "\text{sequential}"] \\
 & S\text{-barrelled} \arrow[ul, Rightarrow, "\text{C-sequential}"] &
\end{tikzcd}$$

Let $(X, \mathscr{T})$ be a complete, pseudo-metrizable (1°-countable) topological vector space. Then $(X, \mathscr{T})$ is a Baire space ([5], p. 213); and hence a barrelled topological vector space ([5], p. 214; Proposition 3 on page 214 holds for non-locally convex topological vector spaces.). Furthermore, since $(X, \mathscr{T})$ is 1°-countable, it is sequential. Consequently, $(X, \mathscr{T})$ is S_3-barrelled and, of course, S-barrelled. In particular, Banach spaces and Fréchet spaces are S_3-barrelled and S-barrelled.

Every infinite-dimensional Montel DF-(Hausdorff locally convex topological vector) space is barrelled ([5], p. 231) and sequential ([13], p. 362); and hence both S_3-barrelled and S-barrelled. In particular, the space $(\mathscr{S}', \beta(\mathscr{S}', \mathscr{S}))$ of tempered distributions with the strong topology is S_3-barrelled and S-barrelled ([13], p. 363). Also the space $(\mathscr{E}', \beta(\mathscr{E}', \mathscr{E}))$ of distributions of compact support with the strong topology is S_3-barrelled and S-barrelled.

There exist topological vector spaces which are S-barrelled but not barrelled. One example is the sequence space $\ell^1 = \{(\alpha_n) : \sum_{n=1}^{+\infty} |\alpha_n| < +\infty\}$ with the weak topology $\sigma(\ell^1, \ell^\infty)$. Let $\mathcal{T}$ be the norm topology on ℓ^1. In showing that the topological vector space $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is S-barrelled, we use the fact that weak convergence and norm convergence of sequences in ℓ^1 are the same ([8], p. 281). Let B be a $\sigma(\ell^1, \ell^\infty)$-sequential barrel in ℓ^1. Then B is $\sigma(\ell^1, \ell^\infty)$-sequentially closed, convex, balanced, and absorbing. Since B is $\mathcal{T}$-sequentially closed and hence $\mathcal{T}$-closed, B is a barrel in $(\ell^1, \mathcal{T})$. But since $(\ell^1, \mathcal{T})$ is barrelled, B is a $\mathcal{T}$-neighborhood of the zero vector 0 in ℓ^1 and hence a $\sigma(\ell^1, \ell^\infty)$-sequential neighborhood of 0. This proves that $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is S-barrelled. We must now show that $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is not barrelled. If a locally convex topological vector space $(X, \mathcal{T}_b)$ is barrelled, its topology $\mathcal{T}_b$ is the strongest locally convex vector topology on X among all locally convex vector topologies on X yielding the same topological dual ([14], p. 224). Since $\mathcal{T}$ and $\sigma(\ell^1, \ell^\infty)$ are two locally convex vector topologies on ℓ^1 such that: (i) $\sigma(\ell^1, \ell^\infty)$ is strictly weaker than $\mathcal{T}$ (see [8], p. 235), and (ii) $(\ell^1, \sigma(\ell^1, \ell^\infty))$ and $(\ell^1, \mathcal{T})$ have the same topological duals; the space $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is not barrelled.

Let $(X, \mathcal{T})$ be a sequential Montel (Hausdorff locally convex topological vector) space such that the weak topology $\sigma(X, X')$ is strictly weaker than the original topology $\mathcal{T}$. Then $(X, \mathcal{T})$ is barrelled ([5], p. 231); the original topology $\mathcal{T}$ is the strong topology $\beta(X, X')$, see [5], p. 220; and $\sigma(X, X')$-convergence and $\mathcal{T}$-convergence of sequences in X are the same ([5], p. 232). As in the previous example, $(X, \sigma(X, X'))$ is S-barrelled but not barrelled. Since $(\mathscr{E}', \beta(\mathscr{E}', \mathscr{E}))$ is a sequential Montel space; since $(\mathscr{E}, \mathcal{T})$ is reflexive ($\mathcal{T}$ is the usual topology on $\mathscr{E}$, see [5], pp. 89, 239, and 231) so that $\sigma(\mathscr{E}', \mathscr{E}) = \sigma(\mathscr{E}', (\mathscr{E}'_\beta)')$; and since the weak topology $\sigma(\mathscr{E}', \mathscr{E})$ is strictly weaker than the strong topology $\beta(\mathscr{E}', \mathscr{E})$, because $(\mathscr{E}', \mathcal{T}))$ is a Fréchet space of infinite dimension (see [2], p. 82); the space $(\mathscr{E}', \sigma(\mathscr{E}', \mathscr{E}))$ of distributions of compact support with the weak topology is S-barrelled but not barrelled. Similarly, the space $(\mathscr{S}', \sigma(\mathscr{S}', \mathscr{S}))$ of tempered distributions with the weak topology is S-barrelled but not barrelled.

As a special case of the above, if $(X, \mathcal{T})$ is a Fréchet-Montel (Hausdorff locally convex topological vector) space (an FM-space, see [8], p. 369) such that $\sigma(X, X')$ is strictly weaker than $\mathcal{T}$; then $(X, \sigma(X, X'))$ is S-barrelled but not barrelled. Thus $(\mathscr{S}, \sigma(\mathscr{S}, \mathscr{S}'))$ and $(\mathscr{D}(K), \sigma(\mathscr{D}(K), \mathscr{D}(K)'))$ are S-barrelled but not barrelled (see [5], pp. 238–240).

R. H. Lohman ([9]) has introduced a class of separable Fréchet spaces $(X, \mathcal{T})$, herein called spaces satisfying Lohman's conditions, which have the property that $\mathcal{T}$-convergence and $\sigma(X, X')$-convergence of sequences in X are the same. If $\sigma(X, X')$ is strictly weaker than $\mathcal{T}$; then $(X, \sigma(X, X'))$ is S-barrelled but not barrelled. Since $(\ell^1, \mathcal{T})$, where $\mathcal{T}$ is the norm topology on ℓ^1, is a separable Fréchet space satisfying Lohman's conditions ([9], p. 349); we again see that $(\ell^1, \sigma(\ell^1, \ell^\infty))$ is S-barrelled but not barrelled. Since Grothendieck's non-distinguished separable Fréchet space $(X, \mathcal{T})$ satisfies Lohman's conditions ([10], p. 193; and [9], p. 349); the space $(X, \sigma(X, X'))$ is S-barrelled but not barrelled.

2. Characterizations in Terms of Semi-norms. We shall now give characterizations of S-barrelled, of S_1-barrelled, and of S_3-barrelled spaces in terms of semi-norms. In order to do this, we need the following definitions. Let $(X, \mathcal{T})$ be a topological space. Let $f: X \to \mathbb{R}$ be a real-valued function on X, and let $a \in X$. Then the function f is said to be *sequentially lower semi-continuous at the point a* if given any real number ε where $\varepsilon > 0$, there exists a sequential neighborhood U of a such that: $x \in U \Rightarrow f(x) > f(a) - \varepsilon$. Equivalently, f is sequentially lower semi-continuous at the point a if and only if given any interval $(c, +\infty)$ in $\mathbb{R}$ such that $f(a) \in (c, +\infty)$, the set $f^{-1}[(c, +\infty)]$ is a sequential neighborhood of a. If f is sequentially lower semi-continuous at each point of X, we say that f is *sequentially lower semi-continuous.* Clearly, f is sequentially lower semi-continuous if and only if given any real number λ in $\mathbb{R}$, the set $f^{-1}[(-\infty, \lambda]] = \{x \in X \mid f(x) \le \lambda\}$ is sequentially closed.

THEOREM 1. *Let $(X, \mathcal{T})$ be a topological vector space over the real or complex field $\mathbb{K}$. Then:*

(1) *$(X, \mathcal{T})$ is barrelled if and only if every lower semi-continuous semi-norm on X is continuous.*

(2) *$(X, \mathcal{T})$ is S-barrelled if and only if every sequentially lower semi-continuous semi-norm on X is sequentially continuous.*

(3) *$(X, \mathcal{T})$ is S_1-barrelled if and only if every lower semi-continuous semi-norm on X is sequentially continuous.*

(4) *$(X, \mathcal{T})$ is S_3-barrelled if and only if every sequentially lower semi-continuous semi-norm on X is continuous.*

Proof. Statement (1) is a well-known characterization of barrelled spaces ([5], p. 219). The proofs for (2), (3), and (4) are similar, and they are omitted.

Let X be a linear space over the real or complex field $\mathbb{K}$. Let $\mathcal{T}$ be the strongest locally convex vector topology on X, i.e., let $\mathcal{T}$ be the (locally convex) vector topology on X generated by the family of all semi-norms on X. Since every semi-norm on X is $\mathcal{T}$-continuous, the space $(X, \mathcal{T})$ is S_3-barrelled. In particular, the space $(\mathbb{R}^{(\mathbb{N})}, \mathcal{T})$ of all finite real sequences (or any infinite dimensional linear space X) endowed with the strongest locally convex vector topology is an example of a non-metrizable (not 1°-countable) S_3-barrelled topological vector space.

3. Permanence Properties of S-Barrelled Topological Vector Spaces.

THEOREM 2. *Let the locally convex topological vector space $(X, \mathcal{T})$ over the real or complex field $\mathbb{K}$ be the strict inductive limit* ([5], *pp.* 157–164) *of the sequence of locally convex topological vector spaces $((X_n, \mathcal{T}_n): N \in \mathbb{N})$ over $\mathbb{K}$.*

Assume that given any natural number n in $\mathbb{N}$, the set X_n is $\mathcal{T}_{n+1}$-closed in X_{n+1} and the space $(X_n, \mathcal{T}_n)$ is S-barrelled. Then $(X, \mathcal{T})$ is S-barrelled.

Proof. Let B be a $\mathcal{T}$-sequential barrel in X. We must show that B is a $\mathcal{T}$-sequential neighborhood of the zero vector 0 in X.

We need the following well-known result ([4], pp. 149–150). Let (x_n) be a sequence in X, and let $a \in X$. Then (x_n) converges to a in $(X, \mathcal{T})$ if and only if there exists a natural number k such that (x_n) is a sequence in X_k, the element a of X belongs to the set X_k, and (x_n) converges to a in $(X_k, \mathcal{T}_k)$. Using this fact, we can easily show that a subset U of X is a $\mathcal{T}$-sequential neighborhood of the zero vector 0 in X if and only if given any natural number n, the set $U \cap X_n$ is a $\mathcal{T}_n$-sequential neighborhood of 0 in X_n. We can also show that a subset A of X is $\mathcal{T}$-sequentially closed in X if and only if given any natural number n, the set $A \cap X_n$ is $\mathcal{T}_n$-sequentially closed in X_n.

Thus we need only show that given any natural number n, the set $B \cap X_n$ is a $\mathcal{T}_n$-sequential neighborhood of 0 in X_n. Let $n \in \mathbb{N}$. Clearly, $B \cap X_n$ is a convex, balanced, absorbing subset of X_n. Since B is $\mathcal{T}$-sequentially closed, $B \cap X_n$ is $\mathcal{T}_n$-sequentially closed in X_n. Since $(X_n, \mathcal{T}_n)$ is S-barrelled, $B \cap X_n$ is a $\mathcal{T}_n$-sequential neighborhood of 0 in X_n. This proves that $(X, \mathcal{T})$ is S-barrelled.

We can now give another example of a topological vector space which is S-barrelled but not sequential. Consider the Hausdorff locally convex topological vector space $(\mathcal{D}(\Omega), \mathcal{T})$ of infinitely differentiable functions with compact support equipped with the Schwartz strict inductive limit topology ([5], p. 165). By the above theorem, $(\mathcal{D}(\Omega), \mathcal{T})$ is S-barrelled. However, $(\mathcal{D}(\Omega), \mathcal{T})$ is not sequential ([11], pp. 31–36). Of course, $(\mathcal{D}(\Omega), \mathcal{T})$, being C-sequential, is barrelled—a well-known fact.

Before giving our next permanence property, we must introduce the notion of an almost sequentially open function. Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces. Let $f: X \to Y$ be a function from X into Y. The function f is said to be *almost sequentially open* provided that: given any point x in X and given any $\mathcal{T}_x$-sequential neighborhood U of x; the set $Ad_{ss}(f[U])$, which is the smallest $\mathcal{T}_y$-sequentially closed set containing $f[U]$, is a $\mathcal{T}_y$-sequential neighborhood of the point $f(x)$ in Y. If $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ are topological vector spaces over the real or complex field $\mathbb{K}$, and if $f: X \to Y$ is a linear transformation; then f is almost sequentially open if and only if: given any $\mathcal{T}_x$-sequential neighborhood U of the zero vector 0 in X, the set $Ad_{ss}(f[U])$ is a $\mathcal{T}_y$-sequential neighborhood of the zero vector 0 in Y.

THEOREM 3. *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological vector spaces over the real or complex field $\mathbb{K}$. Let $f: X \to Y$ be a sequentially continuous, almost sequentially open linear transformation from X into Y. If $(X, \mathcal{T}_x)$ is S-barrelled, then $(Y, \mathcal{T}_y)$ is S-barrelled.*

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4. Other Properties of S-Barrelled Topological Vector Spaces. Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological spaces. A function $f: X \to Y$ is said to be *almost sequentially continuous at a point a* in X provided that: given any neighborhood $N(f(a))$ of the point $f(a)$ in Y, the set $Ad_{ss}(f^{-1}[N(f(a))])$ is a sequential neighborhood of a. The function f is said to be *almost sequentially continuous* provided that f is almost sequentially continuous at each point x in X. Every function which is sequentially continuous at a point is almost sequentially continuous at that point; and every function which is sequentially continuous is almost sequentially continuous. If $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ are topological vector spaces over $\mathbb{K}$, and if $f: X \to Y$ is a linear transformation; then f is almost sequentially continuous if and only if f is almost sequentially continuous at the zero vector 0 in X.

THEOREM 4. *Let $(X, \mathcal{T}_x)$ be an S-barrelled topological vector space over the real or complex field $\mathbb{K}$. Let $(Y, \mathcal{T}_y)$ be a locally convex topological vector space over $\mathbb{K}$. If $f: X \to Y$ is a linear transformation, then f is almost sequentially continuous.*

Every sequentially continuous bilinear function is separately sequentially continuous. Although the converse of this does not hold, we have the following result:

THEOREM 5. *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological vector spaces over the real or complex field $\mathbb{K}$; let $(Z, \mathcal{T}_z)$ be a locally convex topological vector space over $\mathbb{K}$; and let $f: X \times Y \to Z$ be a bilinear function. Assume that either $(X, \mathcal{T}_x)$ or $(Y, \mathcal{T}_y)$ is S-barrelled. Then f is sequentially continuous if and only if f is separately sequentially continuous.*

Proof. The proof of this theorem follows the pattern of the proof of Theorem 1 on page 357 of [5].

As a simple application of this last theorem, convolution $(S, T) \to S * T$ is a sequentially continuous bilinear map of $\mathcal{E}' \times \mathcal{D}'$ into $\mathcal{D}'$ when $\mathcal{E}'$ and $\mathcal{D}'$ have the strong topologies. We use the fact that convolution is separately continuous and the fact that the space $(\mathcal{E}', \beta(\mathcal{E}', \mathcal{E}))$ is S-barrelled.

5. Banach-Steinhaus Theorem for S-Barrelled Spaces. Let $(X, \mathcal{T})$ be a topological vector space. A subset B of X is said to be *sequentially precompact* or *S-precompact* if given any sequential neighborhood U of the zero vector 0 in X, there exists a finite subset B_0 of X such that $B \subseteq B_0 + U$. Every subset of X which is S-precompact is precompact.

THEOREM 6 (*Banach-Steinhaus Theorem for S-Barrelled Spaces*). *Let $(X, \mathcal{T}_x)$ be an S-barrelled topological vector space over the real or complex field $\mathbb{K}$. Let $(Y, \mathcal{T}_y)$ be a Hausdorff locally convex topological vector space over $\mathbb{K}$. Let*

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$(\mathscr{F}(X, Y), \mathscr{T}_p)$ be the Hausdorff locally convex topological vector space, over $\mathbb{K}$, of all functions on X into Y with the topology of pointwise convergence. Let $(\mathcal{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T})$ be the Hausdorff locally convex topological vector space, over $\mathbb{K}$, of all sequentially continuous linear transformations on X into Y with the topology of uniform convergence on S-precompact subsets of X. Let $\mathcal{N}$ be a filterbase in $\mathcal{S}\mathscr{C}\mathscr{L}(X, Y)$, and let f be a function in $\mathscr{F}(X, Y)$ such that $\mathcal{N}$ is $\mathscr{T}_p$-convergent to f. Suppose that $\mathcal{N}$ has either one of the following two properties:

(1) $\mathcal{N}$ is $\mathscr{T}_p$-bounded in the sense that there exists a set N in $\mathcal{N}$ such that N is a $\mathscr{T}_p$-bounded subset of $\mathscr{F}(X, Y)$.

(2) $\mathcal{N}$ is countable.

Then $f \in \mathcal{S}\mathscr{C}\mathscr{L}(X, Y)$ and $\mathcal{N}$ is $\mathscr{T}$-convergent to f. In particular, if (f_n) is a sequence in $\mathcal{S}\mathscr{C}\mathscr{L}(X, Y)$ and if f is a function in $\mathscr{F}(X, Y)$ such that (f_n) is $\mathscr{T}_p$-convergent to f; then $f \in \mathcal{S}\mathscr{C}\mathscr{L}(X, Y)$ and (f_n) is $\mathscr{T}$-convergent to f.

Proof. The proof of this theorem closely parallels the proof of the Banach-Steinhaus Theorem for barrelled topological vector spaces as found in the books by Schaefer ([10], pp. 79–81) and Garsoux ([4], pp. 156–177).

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