THE JULIA SET OF A RANDOM ITERATION SYSTEM

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This paper presents two properties of the Julia set of a random iteration system of rational functions, which are similar to the well-known results in the classical case.

1. Introduction

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_M\}$ be a collection of rational functions. In [7], Zhou and Ren first introduced the random iteration system formed by $\mathcal{R}$. Recently, Hinkkanen and Martin [1] developed the rational semigroup generated by $\mathcal{R}$ with the semigroup operation being functional composition. Some properties of their Julia sets have been investigated.

This paper is devoted to developing two properties of the Julia set of the random iteration system, which are similar to the well-known results in the classical case.

Suppose that $Y = \{1, 2, \ldots, M\}$ and $\Sigma_M = \prod_0^\infty Y$. A metric d' on Σ_M is given by $d'\big(\{j_i\}, \{j_i'\}\big) = \sum_{i=1}^\infty |j_i - j_i'|/2^i$, where $\{j_i\}$ and $\{j_i'\}$ are in Σ_M. For any $\sigma = (j_1, j_2, \ldots, j_n, \ldots) \in \Sigma_M$, we define

$$(1) \qquad W_\sigma^0(z) = z, \quad W_\sigma^n(z) = R_{j_n} \circ R_{j_{n-1}} \circ \cdots \circ R_{j_1}(z),$$

and its inverse $W_\sigma^{-n}(z)$,

$$W_\sigma^{-n}(z) = (W_\sigma^n)^{-1}(z) = R_{j_1}^{-1} \circ R_{j_2}^{-1} \circ \cdots \circ R_{j_n}^{-1}(z). \quad n = 1, 2, \ldots .$$

It is known [6] that if $B \subset \Sigma_M$, the Julia set $J_B(\mathcal{R})$ associated with B is the closure of the union of the sets of non-normality of the sequences $\big\{W_\sigma^n(z)\big\}$ for all orbits σ in B. Specifically, we denote by $J(\mathcal{R})$ the Julia set associated with Σ_M and by $F(\mathcal{R})$ the Fatou set associated with Σ_M, the complement of $J(\mathcal{R})$. (We often call $J(\mathcal{R})$ the Julia set of the random iteration system.) For a random orbit $\sigma \in \Sigma_M$, denote by $J(\sigma)$ the Julia set associated with σ. If R is a rational function, we often denote by $J(R)$ the Julia set of R.

Assume that $z \in J(\mathcal{R})$, and U is a neighbourhood of z. Set

$$E_U = \overline{C} - \bigcup_{\sigma \in \Sigma_M} \bigcup_{n>0} W_\sigma^n(U)$$

By the Montel theorem, E_U contains at most two points. Put $E_z = \bigcup_U E_U$. We call E_z the exceptional set of $\mathcal{R}$ at z and each point in E_z is called the exceptional point of $\mathcal{R}$. From [7], it is seen that E_z is independent of z. Hence we omit the subscript z and denote by $E(\mathcal{R})$ the exceptional set of $\mathcal{R}$. Moreover we let $E(R)$ denote the exceptional set of a rational function R.

It is well-known that if R is a rational function of degree at least two, and U is any non-empty open set which meets the Julia set $J(R)$ of R, then for any compact V which contains no exceptional points of R, there exists an integer l such that $R^n(U) \supset V$, for $n > l$. For the random iteration system we arrive at

THEOREM 1. *Let $J(\mathcal{R})$ denote the Julia set of the random iteration system formed by $\mathcal{R}$ and $E(\mathcal{R})$ be the exceptional set of $\mathcal{R}$. Suppose that U is an open subset of $\overline{C}$ satisfying $U \cap J(\mathcal{R}) \neq \emptyset$. If $V \subset \overline{C}$ is a compact set and $V \cap E(\mathcal{R}) = \emptyset$, then there exist an open set $N \subset \Sigma_M$ and an integer $a > 0$ such that for $n > a$ and $\sigma \in N$, $W_\sigma^n(U) \supset V$.*

Let $\overline{C}$ be the Riemann sphere. By a uniformly perfect set A we mean that $A \subset \overline{C}$ contains at least one point and there is an $m > 0$ such that every annulus that divides A has modulus $\leqslant m$ (see [4]). In [3], Mañé and Da Rocha showed that the Julia set of a rational function is uniformly perfect. Concerning the Julia set of rational semigroup generated by $\mathcal{R}$, a finite set of rational functions, Hinkkanen and Martin [2] also obtained a similar result. Here we consider the Julia set of the random iteration system formed by a finite set of rational functions and obtain the following result using Mañé and Da Rocha's ideas in [3] different from that in [2].

THEOREM 2. *Suppose that $J(\mathcal{R})$ denotes the Julia set of the random iteration system formed by $\mathcal{R}$, then $J(\mathcal{R})$ is uniformly perfect.*

2. LEMMAS

We begin by defining the backward orbit of $\mathcal{R}$ at $z \in \overline{C}$ as follows:

$$O^-(z, \mathcal{R}) = \left\{ W_\sigma^{-n}(z) : \sigma \in \Sigma_M \text{ and } n \in \mathrm{N}^+ \right\},$$

where N^+ is the set of all positive integers.

We can show that if $z \in E(\mathcal{R})$, then the backward orbit $O^-(z, \mathcal{R})$ contains at most two points. Since if $z \in E(\mathcal{R})$, there must be an open set U with $U \cap J(\mathcal{R}) \neq \emptyset$ such that

$$z \in \overline{C} - \bigcup_{\sigma \in \Sigma_M} \bigcup_{n>0} W_\sigma^n(U),$$

or

$$z \notin \bigcup_{\sigma \in \Sigma_M} \bigcup_{n > 0} W_\sigma^n(U).$$

Hence for any $\zeta \in O^-(z, \mathcal{R})$, we have $\zeta \notin \bigcup_{\sigma \in \Sigma_M} \bigcup_{n > 0} W_\sigma^n(U)$. Otherwise there are $\sigma_1, \sigma_2 \in \Sigma_M$ and $n_1, n_2 \in \mathrm{N}^+$ such that $W_{\sigma_1}^{n_1}(\zeta) = z$ and $W_{\sigma_2}^{n_2}(w) = \zeta$ for some $w \in U$, thus there is $\sigma \in \Sigma_M$ such that $W_\sigma^{(n_1 + n_2)}(w) = z$, which is a contradiction. Hence if $z \in E(\mathcal{R})$, we have $\bigcup_{\sigma \in \Sigma_M} \bigcup_{n > 0} W_\sigma^n(U) \cap O^-(z, \mathcal{R}) = \emptyset$. Again since $U \cap J(\mathcal{R}) \neq \emptyset$, the Montel theorem implies that $O^-(z, \mathcal{R})$ contains at most two points.

At the same time, if $O^-(z, \mathcal{R})$ contains at most two points, we have $z \in E(\mathcal{R})$. For if it is not true, then $z \in \bigcup_{\sigma \in \Sigma_M} \bigcup_{n > 0} W_\sigma^n(U)$, for any open set with $U \cap J(\mathcal{R}) \neq \emptyset$. Now take three mutually disjoint open sets U_1, U_2, U_3 with the properties that $U_i \cap J(\mathcal{R}) \neq \emptyset$, $i = 1, 2, 3$. We have $O^-(z, \mathcal{R}) \cap U_i \neq \emptyset$ and $O^-(z, \mathcal{R})$ contains at least three points, which leads to a contradiction.

From the above discussion, we conclude

LEMMA 1. *The following statements are equivalent:*

    (1)   *z is an exceptional point of $\mathcal{R}$;*

    (2)   *the backward orbit $O^-(z, \mathcal{R})$ of $\mathcal{R}$ at z contains at most two points.*

Using the above lemma, we can obtain the following result.

LEMMA 2. *Let $\mathcal{R} = \{R_1, R_2, \ldots, R_M\}$, then*

$$E(\mathcal{R}) = \bigcap_{\sigma \in \Sigma_M} \bigcap_{n > 0} E(W_\sigma^n)$$

where W_σ^n is as defined in (1).

PROOF: If $z \in E(\mathcal{R})$, then $O^-(z, \mathcal{R})$ contains at most two points. Thus for any $\sigma \in \Sigma_M$ and any $n \in \mathrm{N}^+$, $\left\{ (W_\sigma^n)^{-m}(z) \mid m \in \mathrm{N}^+ \right\}$ contains at most two points and $z \in E(W_\sigma^n)$ since $z \in E(W_\sigma^n)$ if and only if $\left\{ (W_\sigma^n)^{-m}(z) \mid m \in \mathrm{N}^+ \right\}$ contains at most two points. Thus we have $z \in \bigcap_{\sigma \in \Sigma_M} \bigcap_{n > 0} E(W_\sigma^n)$.

Conversely, if $z \in \bigcap_{\sigma \in \Sigma_M} \bigcap_{n > 0} E(W_\sigma^n)$, but is not in $E(\mathcal{R})$, then for each $\sigma \in \Sigma_M$ and any $n \in \mathrm{N}^+$, $\left\{ (W_\sigma^n)^{-m}(z) \mid m > 0 \right\}$ contains at most two points and $O^-(z, \mathcal{R})$ contains at least two points. Take arbitrarily two points in $O^-(z, R)$ different from z, say, $z_i \in W_{\sigma_i}^{-n_i}(z)$, $i = 1, 2$, for some $\sigma_1, \sigma_2 \in \Sigma_M$ and some $n_1, n_2 \in \mathrm{N}^+$. Since for any $\sigma \in \Sigma_M$ and any $n \in \mathrm{N}^+$, $z \in E(W_\sigma^n)$, this implies that $z \in \left\{ (W_\sigma^n \circ W_\sigma^n)^{-1}(z) \right\}$. Thus we have $\{z, z_1, z_2\} \subset \left\{ \left(W_{\sigma_1}^{n_1} \circ W_{\sigma_1}^{n_1} \circ W_{\sigma_2}^{n_2} \circ W_{\sigma_2}^{n_2} \right)^{-1}(z) \right\}$ and z is not an exceptional point of $\left(W_{\sigma_2}^{n_2} \circ W_{\sigma_2}^{n_2} \circ W_{\sigma_1}^{n_1} \circ W_{\sigma_1}^{n_1} \right)$, a contradiction. Therefore the lemma follows. ∎

To prove our theorems we need the following results.

LEMMA 3. *Let $E(\mathcal{R})$ denote the exceptional set of $\mathcal{R}$.*

(1) *If $E(\mathcal{R})$ contains only one point, then each $R_i \in \mathcal{R}$ is conjugate to a polynomial by the same Möbius transformation.*

(2) *If $E(\mathcal{R})$ contains two points, then each $R_i \in \mathcal{R}$ is conjugate to the map $z \mapsto c_i z^{\pm d_i}$ by the same Möbius transformation, where $d_i = \deg(R_i)$.*

Moreover, all exceptional points lie in $F(\mathcal{R})$.

This is due to Zhou and Ren [7].

Let

$$\Sigma_{prd} = \left\{ \sigma \in \Sigma_M : \ S^p(\sigma) = \sigma, \text{ for some } p \in \mathrm{N}^+ \right\}$$

where S is the one-sided shift on Σ_M.

Combining [6, Theorem 1] with [5, Theorem 1], we have

LEMMA 4.

(1) *Let $J(\mathcal{R})$ and $J_{\Sigma_{prd}}(\mathcal{R})$ be the Julia set associated with Σ_M and the Julia set associated with Σ_{prd} respectively. Then $J(\mathcal{R}) = J_{\Sigma_{prd}}(\mathcal{R})$.*

(2) *If $\sigma \in \Sigma_M$ and for some integer p, $S^p(\sigma) = \sigma$, then we have $J(\sigma) = J(g)$ where $g(z) = W_\sigma^p(z)$ is the characteristic function of σ.*

3. PROOFS OF THEOREMS

PROOF OF THEOREM 1: By Lemma 3 we can take the neighbourhood $\mathcal{H}$ of $E(\mathcal{R})$ to be small enough such that for any $R_i \in \mathcal{R}$, $R_i(\mathcal{H}) \subset \mathcal{H}$ and $V \subset K = \overline{C} \backslash \mathcal{H}$ (if $E(\mathcal{R}) = \emptyset$, we may assume that $\mathcal{H} = \emptyset$). Hence for each $R_i \in \mathcal{R}, R_i(K) \supset K$. From Lemma 4 it follows that there is an orbit σ_0 and an integer p such that $S^p(\sigma_0) = \sigma_0$ and $J(\sigma_0) \cap U \neq \emptyset$.

Note that for any rational function R of degree more than one, the exceptional set $E(R)$ contains at most two points. Hence by Lemma 2 there must be three orbits $\sigma_i \in \Sigma_M$, $i = 1, 2, 3$, and integers n_1, n_2, n_3 such that $E(\mathcal{R}) = \bigcap_{i=1}^{3} E\left(W_{\sigma_i}^{n_i}\right)$.

From Lemma 4, we have $J(\sigma_0) = J(g)$ where $g(z) = W_{\sigma_0}^p(z)$ is the characteristic function of σ_0. By the classical Fatou-Julia theory, noting that $J\left(W_{\sigma_i}^{n_i}\right)$, $i = 1, 2, 3$ are perfect, we have that $g^l(U)$ for a large enough l covers the Riemann sphere $\overline{C}$ with the possible exception of (arbitrarily) small neighbourhoods of $E(g)$ and $g^l(U) \cap J\left(W_{\sigma_i}^{n_i}\right) \neq \emptyset$, $i = 1, 2, 3$. Hence for l_1 large enough, $\left(W_{\sigma_1}^{n_1}\right)^{l_1} \circ g^l(U)$ covers $\overline{C}$ with the possible exception of (arbitrarily) small neighbourhoods of $E\left(W_{\sigma_1}^{n_1}\right)$ and for l_2 large enough, $\left(W_{\sigma_2}^{n_2}\right)^{l_2} \circ \left(W_{\sigma_1}^{n_1}\right)^{l_1} \circ g^l(U)$ covers $\overline{C}$ with the possible exception of (arbitrarily) small neighbourhoods of $E\left(W_{\sigma_1}^{n_1}\right) \cap E\left(W_{\sigma_2}^{n_2}\right)$. Thus for l_3 large enough, $\left(W_{\sigma_3}^{n_3}\right)^{l_3} \circ \left(W_{\sigma_2}^{n_2}\right)^{l_2} \circ \left(W_{\sigma_1}^{n_1}\right)^{l_1} \circ g^l(U)$ covers $\overline{C}$ with the possible exception of (arbitrarily) small neighbourhoods of $E(\mathcal{R}) = \bigcap_{i=1}^{3} E\left(W_{\sigma_i}^{n_i}\right)$ and $\left(W_{\sigma_3}^{n_3}\right)^{l_3} \circ \left(W_{\sigma_2}^{n_2}\right)^{l_2} \circ \left(W_{\sigma_1}^{n_1}\right)^{l_1} \circ g^l(U) \supset K$ because $E(\mathcal{R}) \cap K = \emptyset$. Note that for any

$R_i \in \mathcal{R}, R_i(K) \supset K$, so we have $R_i \circ \left(W_{\sigma_3}^{n_3}\right)^{l_3} \circ \left(W_{\sigma_2}^{n_2}\right)^{l_2} \circ \left(W_{\sigma_1}^{n_1}\right)^{l_1} \circ g^l(U) \supset K$. Now choose $a = \sum_{i=1}^{3} n_i l_i + pl$ and

$$N = \left\{ \sigma \;\middle|\; \sigma = \left(\sigma_0|_{pl}, \overbrace{\sigma_1|_{n_1}, \ldots, \sigma_1|_{n_1}}^{l_1}, \overbrace{\sigma_2|_{n_2}, \ldots, \sigma_2|_{n_2}}^{l_2}, \overbrace{\sigma_3|_{n_3}, \ldots, \sigma_3|_{n_3}}^{l_3}, j_{a+1}, \ldots \right) \right\}$$

where $\sigma|_k$ denotes the first k elements of σ and $j_n \in \{1, 2, \ldots, M\}$, for $n > a$. Therefore the lemma follows.   □

PROOF OF THEOREM 2: Suppose, by way of contradiction, that there is a sequence of annuli A_n, $n = 1, 2, \ldots$, which divide $J(\mathcal{R})$ and satisfy $\lim_{n\to\infty} \mathrm{mod}(A_n) = \infty$. This implies that for each n, a connected complementary component K_n of A_n can be chosen so that $\lim_{n\to\infty} \mathrm{diam}\, K_n = 0$. Now denote by K'_n the other complementary component of A_n, then $\inf_n \mathrm{diam}\, K'_n > 0$ because otherwise we could take a subsequence $\{K'_{n_j}\}$ with $\lim_{j\to\infty} \mathrm{diam}\, K'_{n_j} = 0$ and points $p'_j \in K'_{n_j} \cap J(\mathcal{R})$ and $p \in K_{n_j} \cap J(\mathcal{R})$ converging to points p' and p in $J(\mathcal{R})$ respectively, and so, by $\lim_{j\to\infty} \mathrm{diam}\, K_{n_j} = \lim_{j\to\infty} \mathrm{diam}\, K'_{n_j} = 0$ and $J(\mathcal{R}) \subset K_{n_j} \cup K'_{n_j}$ for all j, we obtain that $J(\mathcal{R}) = p \cup p'$. This leads to a contradiction.

Let us denote by Δ the unit disk and by $\varphi_n : \Delta \to A_n \cup K_n$ a conformal representation with $\varphi_n(0) \in K_n$. Then $\mathrm{mod}\left(\Delta - \varphi_n^{-1}(K_n)\right) = \mathrm{mod}(A_n)$ and $\lim_{n\to\infty} \mathrm{diam}\, \varphi_n^{-1}(K_n) = 0$ because $\lim_{n\to\infty} \mathrm{mod}(A_n) = \infty$. Take $1 > r_n > \rho_n > 2\, \mathrm{diam}\, \varphi_n^{-1}(K_n)$ satisfying $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} \rho_n/r_n = 0$. Set $D_n = \left\{ z \;\middle|\; |z| < \rho_n \right\}$. The family of functions $\varphi_n : \Delta \to \overline{\mathbf{C}}$ is normal because $\inf_n \mathrm{diam}\left\{\overline{\mathbf{C}} - \varphi_n(\Delta)\right\} = \inf_n \mathrm{diam}\, K'_n > 0$. Hence $\lim_{n\to\infty} \mathrm{diam}\, \varphi_n(D_n) = 0$. But $\varphi_n(D_n)$ is an open set containing points of $J(\mathcal{R})$. Thus, from Theorem 1 it follows that there are an integer $k_n > 0$ and an orbit $\sigma_n \in \Sigma_M$ such that for $r > k_n$, $W_{\sigma_n}^r\left(\varphi_n(D_n)\right)$ covers $J(\mathcal{R})$. Take c with $0 < c < \mathrm{diam}\, J(\mathcal{R})$ and let m_n be the minimal positive integer satisfying $\mathrm{diam}\, W_{\sigma_n}^{m_n}\left(\varphi_n(D_n)\right) \geqslant c$. Since $\lim_{n\to\infty} \mathrm{diam}\, \varphi_n(D_n) = 0$ and $\mathcal{R}$ is a finite set of rational functions, we obtain that $\lim_{n\to\infty} m_n = +\infty$. Moreover, since $\mathrm{diam}\, W_{\sigma_n}^{m_n-1}\left(\varphi_n(D_n)\right) < c$, it follows that $\mathrm{diam}\, W_{\sigma_n}^{m_n}\left(\varphi_n(D_n)\right) < Lc$, where L is the maximum of the Lipschitz constants of all $R_i \in \mathcal{R}$.

Let G be a set of four different points in $J(\mathcal{R})$ and take c so small that that no set of diameter $\leqslant Lc$ contains two of them. Then $W_{\sigma_n}^{m_n}\left(\varphi_n(D_n)\right)$ does not cover three points of G. Define $\psi_n : \Delta \to \overline{\mathbf{C}}$ by $\psi_n(z) = W_{\sigma_n}^{m_n}\left(\varphi_n(r_n z)\right)$. Next we shall show that the family ψ_n is normal. It suffices to show that for all n, $\psi_n(\Delta)$ does not cover three points of G. If $\rho_n/2r_n < |z| < 1$, then $\rho_n/2 < |r_n z| < r_n$ and, since $\mathrm{diam}\, \varphi_n^{-1}(K_n) \leqslant \rho_n/2$ and $0 \in \varphi^{-1}(K_n)$, it follows that $r_n z \notin \varphi_n^{-1}(K_n)$ and $\varphi_n(r_n z) \notin K_n$. Hence $\varphi_n(r_n z) \notin J(\mathcal{R})$ because $K_n \supset J(\mathcal{R}) \cap \varphi_n(\Delta)$. Noting that $F(\mathcal{R})$ is forward invariant under $R_i \in \mathcal{R}$ (see [7]), we obtain $\psi_n(z) = W_{\sigma_n}^{m_n}\left(\varphi_n(r_n z)\right) \in W_{\sigma_n}^{m_n}\left(F(\mathcal{R})\right) \subset F(\mathcal{R})$ and therefore $\psi_n(z) \notin G$ when $\rho_n/2r_n < |z| < 1$. On the other hand, if $|z| \leqslant \rho_n/2r_n$, it follows that $|r_n z| \leqslant \rho_n/2 \leqslant \rho_n$ and then $\psi_n(z) = W_{\sigma_n}^{m_n}\left(\varphi_n(r_n z)\right) \in W_{\sigma_n}^{m_n}\left(\varphi_n(D_n)\right)$ that does not

cover three points of G. This proves the normality of the family $\{\psi_n\}$. Then, given $\varepsilon > 0$, there is a neighbourhood N_0 of 0 such that $\operatorname{diam}\psi_n(N_0) \leqslant \varepsilon$ for all n. But for n sufficiently large, $N_0 \supset \left\{z \mid |z| \leqslant \rho_n/r_n\right\}$ and

$$\varepsilon \geqslant \operatorname{diam}\psi_n(N_0) \geqslant \operatorname{diam}\psi_n\left(\left\{z \mid |z| < \rho_n/r_n\right\}\right) = \operatorname{diam} W_{\sigma_n}^{m_n}\left(\varphi_n(D_n)\right) \geqslant c.$$

Since $\varepsilon > 0$ is arbitrary, this is a contradiction that proves the theorem. □

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