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STABLE PROPERTIES OF HYPERRELEXIVITY

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Abstract. Recently, a new equivalence relation between weak* closed operator spaces acting on Hilbert spaces has appeared. Two weak* closed operator spaces $\mathcal{U}, \mathcal{V}$ are called weak TRO equivalent if there exist ternary rings of operators $\mathcal{M}_i, i = 1, 2$ such that $\mathcal{U} = [\mathcal{M}_2\mathcal{V}\mathcal{M}_1^*]^{-w^*}$, $\mathcal{V} = [\mathcal{M}_2^*\mathcal{U}\mathcal{M}_1]^{-w^*}$. Weak TRO equivalent spaces are stably isomorphic, and conversely, stably isomorphic dual operator spaces have normal completely isometric representations with weak TRO equivalent images. In this paper, we prove that if $\mathcal{U}$ and $\mathcal{V}$ are weak TRO equivalent operator spaces and the space of $I \times I$ matrices with entries in $\mathcal{U}$, $M_I^w(\mathcal{U})$, is hyperreflexive for suitable infinite I, then so is $M_I^w(\mathcal{V})$. We describe situations where if $\mathcal{L}_1, \mathcal{L}_2$ are isomorphic lattices, then the corresponding algebras $\mathrm{Alg}(\mathcal{L}_1)$, $\mathrm{Alg}(\mathcal{L}_2)$ have the same complete hyperreflexivity constant.

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1. Introduction. Recently, a new equivalence relation between weak* closed operator spaces acting on Hilbert spaces has appeared:

DEFINITION 1.1 ([**7**]). Let $H_i, K_i, i = 1, 2$ be Hilbert spaces, and $\mathcal{U} \subset B(K_1, K_2)$, $\mathcal{V} \subset B(H_1, H_2)$ be weak* closed spaces. We call them weak TRO equivalent if there exist ternary rings of operators (TRO's) $\mathcal{M}_i \subset B(H_i, K_i)$, $i = 1, 2$, i.e., spaces satisfying $\mathcal{M}_i\mathcal{M}_i^*\mathcal{M}_i \subset \mathcal{M}_i$, $i = 1, 2$, such that

$$\mathcal{U} = [\mathcal{M}_2\mathcal{V}\mathcal{M}_1^*]^{-w^*}, \quad \mathcal{V} = [\mathcal{M}_2^*\mathcal{U}\mathcal{M}_1]^{-w^*}.$$

Weak TRO equivalence is an equivalence relation. In [**7**] we called this relation, simply, TRO equivalence. Recently in [**5**] appeared the notion of strong TRO equivalence. Thus, throughout this paper we decided to call the notion in Definition 1.1 weak TRO equivalence in order to distinguish the two equivalences.

Weak TRO equivalence is related to the very important notion of stable isomorphism of operator spaces:

THEOREM 1.1 ([**7**]). *If $\mathcal{U}$ and $\mathcal{V}$ are weak TRO equivalent operator spaces then $\mathcal{U}$ and $\mathcal{V}$ are weakly stably isomorphic. This means that there exists a cardinal I such that the spaces $\mathcal{U}\bar{\otimes}B(l^2(I))$, $\mathcal{V}\bar{\otimes}B(l^2(I))$ are completely isometrically isomorphic through a weak* continuous map. Here, $\bar{\otimes}$ is the normal spatial tensor product. Conversely, if $\mathcal{U}$ and $\mathcal{V}$ are weakly stably isomorphic, then they have completely isometric weak* continuous representations ϕ and ψ such that $\phi(\mathcal{U})$ and $\psi(\mathcal{V})$ are weak TRO equivalent.*

In Section 3 of this paper, we prove that if $\mathcal{U}$ and $\mathcal{V}$ are weak TRO equivalent operator spaces, and if the weak* closed space of $I \times I$ matrices with entries in $\mathcal{U}$, $M_I^w(\mathcal{U})$ is hyperreflexive for suitable infinite I, then so is $M_I^w(\mathcal{V})$. In the case of separably acting $\mathcal{U}$ and $\mathcal{V}$, we have $k(M_\infty^w(\mathcal{U})) = k(M_\infty^w(\mathcal{U}))$ where $k(\mathcal{X})$ is the hyperreflexivity constant of $\mathcal{X}$ and ∞ is aleph 0. As a consequence, in Section 4 we prove that if $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic commutative subspace lattice (CSL) algebras acting on separable Hilbert spaces and if $\mathcal{A}$ is completely hyperreflexive, then $\mathcal{B}$ is also a completely hyperreflexive space with the same complete hyperreflexivity constant. We also prove that if $\mathcal{L}_i \subset B(H_i)$, $i = 1, 2$ are separably acting reflexive lattices and there exists a *-isomorphism $\theta : \mathcal{L}_1'' \to \mathcal{L}_2''$ such that $\theta(\mathcal{L}_1) = \mathcal{L}_2$, then $\mathrm{Alg}(\mathcal{L}_1)$ is completely hyperreflexive iff $\mathrm{Alg}(\mathcal{L}_2)$ is completely hyperreflexive. We also prove that if $\mathcal{L}_i \subset B(H_i)$, $i = 1, 2$ are totally atomic separably acting isomorphic CSL's, then $\mathrm{Alg}(\mathcal{L}_1)$ is completely hyperreflexive iff $\mathrm{Alg}(\mathcal{L}_2)$ is completely hyperreflexive. Finally, we prove that separably acting von Neumann algebras with isomorphic commutants have the same complete hyperreflexive constant.

In what follows, the symbol $[\mathcal{S}]$ denotes the linear span of $\mathcal{S}$. If $\mathcal{L} \subset B(H)$, we denote by $\mathcal{L}'$ the set of operators which commute with the elements of $\mathcal{L}$. The set of projections in $\mathcal{L}$ is written as $pr(\mathcal{L})$. If T is an operator and I is a cardinal, T^I denotes the $I \times I$ diagonal matrix with diagonal entries T. If $\mathcal{X}$ is a space of operators, we define $\mathcal{X}^I$ to be the space containing all operators of the form T^I where $T \in \mathcal{X}$.

A set of projections of a Hilbert space is called a lattice if it contains the zero and identity projections and is closed under arbitrary suprema and infima. If $\mathcal{A}$ is a subalgebra of $B(H)$ for some Hilbert space H, the set

$$\mathrm{Lat}(\mathcal{A}) = \{L \in pr(B(H)) : L^\perp \mathcal{A} L = 0\}$$

is a lattice. Dually, if $\mathcal{L}$ is a lattice, the space

$$\mathrm{Alg}(\mathcal{L}) = \{A \in B(H) : L^\perp A L = 0 \ \ \forall \ \ L \in \mathcal{L}\}$$

is an algebra. A lattice $\mathcal{L}$ is called reflexive if $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) = \mathcal{L}$.

A CSLis a projection lattice $\mathcal{L}$ whose elements commute; the algebra $\mathrm{Alg}(\mathcal{L})$ is called a CSL algebra. Two CSL's $\mathcal{L}_1, \mathcal{L}_2$ are called isomorphic if there exists an order preserving 1-1 and onto map from $\mathcal{L}_1$ onto $\mathcal{L}_2$.

Let H_1, H_2 be Hilbert spaces and $\mathcal{U}$ a subset of $B(H_1, H_2)$. The reflexive hull of $\mathcal{U}$ is defined to be the space

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : Tx \in \overline{[\mathcal{U}x]} \text{ for each } x \in H_1\}.$$

A subspace $\mathcal{U}$ is called reflexive if $\mathcal{U} = \mathrm{Ref}(\mathcal{U})$.

Let $\mathcal{U}$ be a subspace of $\mathcal{B}(H, K)$. If $T \in \mathcal{B}(H, K)$, we call

$$d(T, \mathcal{U}) = \inf_{X \in \mathcal{U}} \|T - X\|$$

the distance from T to $\mathcal{U}$. We also set

$$r_{\mathcal{U}}(T) = \sup_{\|\xi\| = \|\eta\| = 1} \{|\langle T\xi, \eta\rangle| : \langle U\xi, \eta\rangle = 0 \ \ \forall \ U \in \mathcal{U}\}.$$

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Trivially, $r_{\mathcal{U}}(T) \leq d(T, \mathcal{U})$. We can see that $\mathcal{U}$ is reflexive if $r_{\mathcal{U}}(T) = 0$ implies that $T \in \mathcal{U}$, for $T \in \mathcal{B}(H, K)$. If there exists $k > 0$ such that

$$d(T, \mathcal{U}) \leq k r_{\mathcal{U}}(T), \quad T \in B(H, K),$$

we say that the space $\mathcal{U}$ is hypereflexive. The space $\mathcal{U}$ is called completely hyperreflexive if $\mathcal{U}\bar{\otimes}B(\mathcal{H})$ is hyperreflexive, where H is a separable infinite dimensional Hilbert space. It is not known if hyperreflexivity implies complete hyperreflexivity.

If $\mathcal{U}$ is a reflexive space, let

$$k(\mathcal{U}) = \sup_{T \notin \mathcal{U}} \frac{d(T, \mathcal{U})}{r_{\mathcal{U}}(T)}$$

be the hyperreflexivity constant of $\mathcal{U}$. Clearly, $\mathcal{U}$ is hyperreflexive if and only if $k(\mathcal{U}) < \infty$.

Throughout this paper, we shall use the following Lemma.

LEMMA 1.2. *Let $\mathcal{U} \subset B(K_1, K_2)$ be a weak* closed space, K_1, K_2 Hilbert spaces, and $\mathcal{B}$ and $\mathcal{A}$ von Neumann algebras such that $\mathcal{B}\mathcal{U}\mathcal{A} \subset \mathcal{U}$. Then for every $T \in B(K_1, K_2)$,*

$$r_{\mathcal{U}}(T) = \sup\{\|QTP\| : Q \in pr(\mathcal{B}'), P \in pr(\mathcal{A}'), Q\mathcal{U}P = 0\}.$$

Proof. Choose $T \in B(K_1, K_2)$ and $Q \in pr(\mathcal{B}')$, $P \in pr(\mathcal{A}')$ such that $Q\mathcal{U}P = 0$. We have

$$\|QTP\| = \sup_{\|\xi\|=\|\eta\|=1} |\langle QTP\xi, \eta\rangle| = \sup_{\|\xi\|=\|\eta\|=1} |\langle TP\xi, Q\eta\rangle|.$$

Since, $\langle UP\xi, Q\eta\rangle = 0, \forall U \in \mathcal{U}, \xi, \eta$ we have

$$|\langle TP\xi, Q\eta\rangle| \leq r_{\mathcal{U}}(T)\|P\xi\|\|Q\eta\| \leq r_{\mathcal{U}}(T).$$

For the converse inequality, suppose $\epsilon > 0$. Then, there exist unit vectors ξ, η such that $\langle U\xi, \eta\rangle = 0, \quad \forall \quad U \in \mathcal{U}$ and

$$r_{\mathcal{U}}(T) - \epsilon < |\langle T\xi, \eta\rangle|.$$

Since $\langle UA\xi, B\eta\rangle = 0, \quad \forall A \in \mathcal{A}, \quad \forall B \in \mathcal{B}$ if P is the projection onto the space generated by $\mathcal{A}\xi$ and Q is the projection onto the space generated by $\mathcal{B}\eta$, we have $Q\mathcal{U}P = 0$ and $Q \in pr(\mathcal{B}')$, $P \in pr(\mathcal{A}')$. But

$$|\langle T\xi, \eta\rangle| = |\langle QTP\xi, \eta\rangle| \leq \|QTP\|.$$

Since ϵ is arbitrary, the proof is complete. □

Now we present some concepts introduced in [**8**].

Let $\mathcal{P}_i = pr(B(H_i))$, $i = 1, 2$. Let $\phi = \mathrm{Map}(\mathcal{U})$ be the map $\phi : \mathcal{P}_1 \to \mathcal{P}_2$, which to each $P \in \mathcal{P}_1$ associates the projection onto the subspace $[TPy : T \in \mathcal{U}, y \in H_1]^-$. The map ϕ is $\vee-$continuous (that is, it preserves arbitrary suprema) and is 0 preserving.

Let $\phi^* = \mathrm{Map}(\mathcal{U}^*)$, $\mathcal{S}_{1,\phi} = \{\phi^*(P)^\perp : P \in \mathcal{P}_2\}$, $\mathcal{S}_{2,\phi} = \{\phi(P) : P \in \mathcal{P}_1\}$ and observe that $\mathcal{S}_{1,\phi} = \mathcal{S}_{2,\phi^*}^\perp$. Erdos proved that $\mathcal{S}_{1,\phi}$ is $\wedge$-complete and contains the identity projection, $\mathcal{S}_{2,\phi}$ is $\vee$-complete and contains the zero projection, while $\phi|_{\mathcal{S}_{1,\phi}} : \mathcal{S}_{1,\phi} \to \mathcal{S}_{2,\phi}$ is a bijection.

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In fact,

$$\mathrm{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : \phi(P)^{\perp} TP = 0 \text{ for each } P \in \mathcal{S}_{1,\phi}\}.$$

When $\phi(I) = I$ and $\phi^*(I) = I$, we call the space $\mathcal{U}$ essential.

In [9] it is proved that a TRO $\mathcal{M}$ is weak* closed if and only if it is *wot* closed if and only if it is reflexive. In this case, if $\chi = \mathrm{Map}(\mathcal{M})$,

$$\mathcal{M} = \{T \in B(H_1, H_2) : TP = \chi(P)T \text{ for all } P \in \mathcal{S}_{1,\chi}\}.$$

In the following theorem, we isolate some consequences of [9, Theorem 2.10].

THEOREM 1.3. *(i) A TRO $\mathcal{M}$ is essential if and only if the algebras $[\mathcal{M}^*\mathcal{M}]^{-w^*}$ $[\mathcal{M}\mathcal{M}^*]^{-w^*}$ contain the identity operators.*

(ii) If $\mathcal{M}$ is an essential TRO and $\chi = \mathrm{Map}(\mathcal{M})$, then $\mathcal{S}_{1,\chi} = pr((\mathcal{M}^*\mathcal{M})')$, $\mathcal{S}_{2,\chi} = pr((\mathcal{M}\mathcal{M}^*)')$ and the map $\chi|_{\mathcal{S}_{1,\chi}} : \mathcal{S}_{1,\chi} \to \mathcal{S}_{2,\chi}$ is an ortholattice isomorphism with inverse $\chi^*|_{\mathcal{S}_{2,\chi}}$.

If K_1, K_2 are Hilbert spaces, $\mathcal{U} \subset B(K_1, K_2)$ is a weak* closed operator space, and I is a cardinal, then $M_I^w(\mathcal{U})$ is the set of $I \times I$ matrices with entries in $\mathcal{U}$ whose finite submatrices have uniformly bounded norm, [1]. We consider $M_I^w(\mathcal{U})$ as a subspace of the set of bounded operators from K_1^I to K_2^I. We can see that the space $M_I^w(\mathcal{U})$ is unitarily equivalent with $\mathcal{U}\bar{\otimes}B(l^2(I))$. Therefore, if $\mathcal{U}$ is a completely hyperreflexive space, then $k(\mathcal{U}\bar{\otimes}B(l^2(\mathbb{N}))) = k(M_\infty^w(\mathcal{U}))$. Also, $C_I^w(\mathcal{U})$ is the subspace of $I \times 1$ columns with entries in $\mathcal{U}$, or, equivalently, the space of bounded operators from K_1 to K_2^I of the form $(U_i)_{i\in I}$, where every U_i belongs to $\mathcal{U}$.

LEMMA 1.4.

$$k(C_I^w(\mathcal{U})) \leq k(M_I^w(\mathcal{U})).$$

Proof. We denote by $E = (E_{i,j})_{i,j\in I}$ the $I \times I$ matrix where $E_{i_0,i_0} = I_{K_1}$ and $E_{i,j} = 0$ for $(i,j) \neq (i_0, i_0)$. Observe that $M_I^w(\mathcal{U})E$ contains elements of the form $(C_i)_{i\in I}$, where $C_{i_0} = C_I^w(\mathcal{U})$ and C_i is a zero column for $i \neq i_0$.

Lemma 6.2 in [2] implies that $k(M_I^w(\mathcal{U})E) \leq k(M_I^w(\mathcal{U}))$. Obviously

$$k(C_I^w(\mathcal{U})) \leq k(M_I^w(\mathcal{U})E).$$

□

In this paper, we shall use the following lemma from 8.5.23 in [1].

LEMMA 1.5. *If $\mathcal{M} \subset B(K_1, K_2)$ is an essential weak* closed TRO, and K_1, K_2 are Hilbert spaces, and I is the cardinal of an orthonormal basis of K_1, there exists a column $M = (M_i)_{i\in I} \in C_I^w(\mathcal{M})$ where every M_i is a partial isometry such that $M_i^*M_i$ is orthogonal to $M_j^*M_j$ for every $i \neq j$ and such that $M^*M = I_{K_1}$.*

2. Weak TRO equivalence of operator spaces. In this section, we fix Hilbert spaces H_1, H_2, K_1, K_2 and essential reflexive operator spaces

$$\mathcal{U} \subset B(K_1, K_2), \quad \mathcal{V} \subset B(H_1, H_2)$$

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which are weak TRO equivalent: i.e., there exist TRO's $\mathcal{M}_i \subset B(H_i, K_i)$, $i = 1, 2$ such that

$$\mathcal{U} = [\mathcal{M}_2\mathcal{V}\mathcal{M}_1^*]^{-w^*}, \quad \mathcal{V} = [\mathcal{M}_2^*\mathcal{U}\mathcal{M}_1]^{-w^*}.$$

Observe that since $\mathcal{U}, \mathcal{V}$ are essential spaces, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are essential too. We assume that

$$\phi = \mathrm{Map}(\mathcal{U}), \psi = \mathrm{Map}(\mathcal{V}), \chi_i = \mathrm{Map}(\mathcal{M}_i), \quad i = 1, 2,$$

$$\mathcal{B}_i = \mathcal{S}'_{i,\phi} \subset B(K_i), \mathcal{A}_i = \mathcal{S}'_{i,\psi} \subset B(H_i), i = 1, 2.$$

In this section, we are going to find *-isomorphisms $\zeta_i : \mathcal{B}'_i \to \mathcal{A}'_i$, $i = 1, 2$ such that if

$$\mathcal{N}_i = \{T \in B(H_i, K_i) : T\zeta_i(P) = PT \ \ \forall\, P \in pr(\mathcal{B}'_i)\}, \quad i = 1, 2,$$

then

$$\mathcal{U} = [\mathcal{N}_2\mathcal{V}\mathcal{N}_1^*]^{-w^*}, \mathcal{V} = [\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1]^{-w^*}.$$

LEMMA 2.1.

$$\mathcal{A}_i = [\mathcal{M}_i^*\mathcal{B}_i\mathcal{M}_i]^{-w^*}, \mathcal{B}_i = [\mathcal{M}_i\mathcal{A}_i\mathcal{M}_i^*]^{-w^*}, \quad i = 1, 2.$$

Proof. Choose

$$Q \in pr(B(H_1)) \Rightarrow \psi(Q) \in \mathcal{S}_{2,\psi}.$$

Since $\mathcal{M}_2\mathcal{V} \subset \mathcal{U}\mathcal{M}_1$, we have $\mathcal{M}_2\mathcal{V}Q \subset \mathcal{U}\mathcal{M}_1Q$. The projection onto the space generated by $\mathcal{V}Q(H_1)$ is $\psi(Q)$ and the projection onto the space generated by $\mathcal{U}\mathcal{M}_1Q(H_1)$ is $\phi(\chi_1(Q))$. Thus,

$$\phi(\chi_1(Q))^\perp\mathcal{M}_2\psi(Q) = 0. \tag{1}$$

Since

$$\mathcal{M}_2^*\mathcal{U}\mathcal{M}_1Q \subset \mathcal{V}Q,$$

we have

$$\psi(Q)^\perp\mathcal{M}_2^*\phi(\chi_1(Q)) = 0. \tag{2}$$

If $B \in \mathcal{B}_2$, $M, N \in \mathcal{M}_2$, then by using (1) we have

$$M^*BN\psi(Q) = M^*B\phi(\chi_1(Q))N\psi(Q) = M^*\phi(\chi_1(Q))BN\psi(Q).$$

Using (2), the last operator is equal to $\psi(Q)M^*BN\psi(Q)$. Therefore,

$$\psi(Q)^\perp\mathcal{M}_2^*\mathcal{B}_2\mathcal{M}_2\psi(Q) = 0.$$

Since, $\mathcal{B}_2$ is a self-adjoint algebra, we also have

$$\psi(Q)\mathcal{M}_2^*\mathcal{B}_2\mathcal{M}_2\psi(Q)^\perp = 0.$$

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Therefore,

$$\mathcal{M}_2^*\mathcal{B}_2\mathcal{M}_2 \subset \mathcal{S}'_{2,\psi} = \mathcal{A}_2.$$

Similarly, we can prove that

$$\mathcal{M}_2\mathcal{A}_2\mathcal{M}_2^* \subset \mathcal{B}_2.$$

Proposition 2.1 in [**3**] implies that

$$\mathcal{A}_2 = [\mathcal{M}_2^*\mathcal{B}_2\mathcal{M}_2]^{-w^*}, \mathcal{B}_2 = [\mathcal{M}_2\mathcal{A}_2\mathcal{M}_2^*]^{-w^*}.$$

Similarly, we can prove

$$\mathcal{A}_1 = [\mathcal{M}_1^*\mathcal{B}_1\mathcal{M}_1]^{-w^*}, \mathcal{B}_1 = [\mathcal{M}_1\mathcal{A}_1\mathcal{M}_1^*]^{-w^*}.$$

□

By Proposition 2.8 in [**3**], and its proof we have that χ_i^* is an ortholattice isomorphism from $pr(\mathcal{B}_i')$ onto $pr(\mathcal{A}_i')$, $i = 1, 2$ and if we denote

$$\mathcal{N}_i = \{T \in B(H_i, K_i) : T\chi_i^*(P) = PT \ \ \forall\ P \in pr(\mathcal{B}_i')\}, \quad i = 1, 2,$$

then,

$$\mathcal{A}_i = [\mathcal{N}_i^*\mathcal{N}_i]^{-w^*}, \quad \mathcal{B}_i = [\mathcal{N}_i\mathcal{N}_i^*]^{-w^*}.$$

If $\gamma_i = \mathrm{Map}(\mathcal{N}_i)$ by Theorem 1.3, γ_i^* is also an ortholattice isomorphism from $pr(\mathcal{B}_i')$ onto $pr(\mathcal{A}_i')$, $i = 1, 2$. If $P \in pr(\mathcal{B}_i')$ then, since $TP = \chi_i^*(P)T$, $\forall T \in \mathcal{N}_i^*$ the projection onto the space generated by $\mathcal{N}_i^*P(K_i)$ is dominated by $\chi_i^*(P)$. Thus, $\gamma_i^*(P) \le \chi_i^*(P)$. Similarly, $\gamma_i^*(P^\perp) \le \chi_i^*(P^\perp)$. But $\gamma_i^*|_{pr(\mathcal{B}_i')}$ and $\chi_i^*|_{pr(\mathcal{B}_i')}$ are ortholattice homomorphisms, thus $\gamma_i^*|_{pr(\mathcal{B}_i')} = \chi_i^*|_{pr(\mathcal{B}_i')}$, $i = 1, 2$.

By Theorem 3.2 in [**3**] and its proof, there exist *-isomorphisms $\zeta_i : \mathcal{B}_i' \to \mathcal{A}_i'$, $i = 1, 2$ which extend $\gamma_i^*|_{pr(\mathcal{B}_i')} = \chi_i^*|_{pr(\mathcal{B}_i')}$, such that

$$\mathcal{N}_i = \{T \in B(H_i, K_i) : T\zeta_i(X) = XT \ \ \forall\ X \in \mathcal{B}_i'\}, \quad i = 1, 2, .$$

Since $\mathcal{B}_2\mathcal{U}\mathcal{B}_1 \subset \mathcal{U}$ and $\mathcal{A}_2\mathcal{V}\mathcal{A}_1 \subset \mathcal{V}$, the spaces

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_2 & \mathcal{U} \\ 0 & \mathcal{B}_1 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} \mathcal{A}_2 & \mathcal{V} \\ 0 & \mathcal{A}_1 \end{pmatrix}$$

are algebras. These algebras are reflexive because $\mathcal{U}, \mathcal{V}, \mathcal{A}_i, \mathcal{B}_i, i = 1, 2$ are reflexive spaces. Their lattices are the following:

$$\mathrm{Lat}(\mathcal{B}) = \{P_2 \oplus P_1 : P_i \in pr(\mathcal{B}_i'), P_2^\perp\mathcal{U}P_1 = 0\},$$

and

$$\mathrm{Lat}(\mathcal{A}) = \{Q_2 \oplus Q_1 : Q_i \in pr(\mathcal{A}_i'), Q_2^\perp\mathcal{V}Q_1 = 0\}.$$

LEMMA 2.2. *The algebras $\mathcal{A}, \mathcal{B}$ are weak TRO equivalent.*

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Proof. We have

$$\begin{pmatrix} \mathcal{M}_2^* & 0 \\ 0 & \mathcal{M}_1^* \end{pmatrix}\begin{pmatrix} \mathcal{B}_2 & \mathcal{U} \\ 0 & \mathcal{B}_1 \end{pmatrix}\begin{pmatrix} \mathcal{M}_2 & 0 \\ 0 & \mathcal{M}_1 \end{pmatrix} =$$

$$\begin{pmatrix} \mathcal{M}_2^*\mathcal{B}_2\mathcal{M}_2 & \mathcal{M}_2^*\mathcal{U}\mathcal{M}_1 \\ 0 & \mathcal{M}_1^*\mathcal{B}_1\mathcal{M}_1 \end{pmatrix} \subset \begin{pmatrix} \mathcal{A}_2 & \mathcal{V} \\ 0 & \mathcal{A}_1 \end{pmatrix} = \mathcal{A}.$$

Similarly, we can prove that

$$(\mathcal{M}_2 \oplus \mathcal{M}_1)\mathcal{A}(\mathcal{M}_2 \oplus \mathcal{M}_1)^* \subset \mathcal{B}.$$

Proposition 2.1 in [**3**] implies that

$$[(\mathcal{M}_2 \oplus \mathcal{M}_1)\mathcal{A}(\mathcal{M}_2 \oplus \mathcal{M}_1)^*]^{-w^*} = \mathcal{B},$$

and

$$[(\mathcal{M}_2 \oplus \mathcal{M}_1)^*\mathcal{B}(\mathcal{M}_2 \oplus \mathcal{M}_1)]^{-w^*} = \mathcal{A}.$$

□

If $\chi = \mathrm{Map}(\mathcal{M}_2 \oplus \mathcal{M}_1)$, we can see that

$$\chi(P \oplus Q) = \chi_2(P) \oplus \chi_1(Q)\ \forall P \in pr(\mathcal{A}_2'),\ Q \in pr(\mathcal{A}_1'),$$
$$\chi^*(P \oplus Q) = \chi_2^*(P) \oplus \chi_1^*(Q)\ \forall P \in pr(\mathcal{B}_2'),\ Q \in pr(\mathcal{B}_1'),$$

Also by Lemma 2.6 in [**3**], $\chi(\mathrm{Lat}(\mathcal{A})) = \mathrm{Lat}(\mathcal{B})$. Since, $\chi_i^*|_{pr(\mathcal{B}_i')}$ extends to a *-isomorphism $\zeta_i : \mathcal{B}_i' \to \mathcal{A}_i'$, $i = 1, 2$, see the discussion above Lemma 2.2 , $\chi^*|_{pr(\mathcal{B}_2')\oplus pr(\mathcal{B}_1')}$ extends to a *-isomorphism

$$\zeta = \zeta_2 \oplus \zeta_1 : \mathcal{B}_2' \oplus \mathcal{B}_1' \to \mathcal{A}_2' \oplus \mathcal{A}_1',$$

satisfying $\zeta(\mathrm{Lat}(\mathcal{B})) = \mathrm{Lat}(\mathcal{A})$. By Theorem 3.3 in [**3**], and its proof if

$$\mathcal{N} = \{T : T\zeta(X) = XT \ \ \forall\ X \in (\mathcal{B}_2' \oplus \mathcal{B}_1')\},$$

then

$$[\mathcal{N}\mathcal{A}\mathcal{N}^*]^{-w^*} = \mathcal{B}, \quad [\mathcal{N}^*\mathcal{B}\mathcal{N}]^{-w^*} = \mathcal{A}.$$

Observe that $\mathcal{N} = \mathcal{N}_2 \oplus \mathcal{N}_1$.

THEOREM 2.3.

$$\mathcal{U} = [\mathcal{N}_2\mathcal{V}\mathcal{N}_1^*]^{-w^*}, \quad \mathcal{V} = [\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1]^{-w^*}.$$

Proof. We have

$$\begin{pmatrix} \mathcal{N}_2 & 0 \\ 0 & \mathcal{N}_1 \end{pmatrix}\begin{pmatrix} 0 & \mathcal{V} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathcal{N}_2^* & 0 \\ 0 & \mathcal{N}_1^* \end{pmatrix} \subset \mathcal{B}.$$

Thus,

$$\mathcal{N}_2\mathcal{V}\mathcal{N}_1^* \subset \mathcal{U}.$$

Similarly,

$$\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1 \subset \mathcal{V} \Rightarrow \mathcal{N}_2\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1\mathcal{N}_1^* \subset \mathcal{N}_2\mathcal{V}\mathcal{N}_1^*.$$

Since the algebras $[\mathcal{N}_i\mathcal{N}_i^*]^{-w^*}$ contain the identity operators, the last relation implies that

$$\mathcal{U} \subset [\mathcal{N}_2\mathcal{V}\mathcal{N}_1^*]^{-w^*} \Rightarrow \mathcal{U} = [\mathcal{N}_2\mathcal{V}\mathcal{N}_1^*]^{-w^*}.$$

Similarly, we can prove

$$\mathcal{V} = [\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1]^{-w^*}.$$

□

3. Hyperreflexivity and weak TRO equivalence. In this section, we fix Hilbert spaces H_1, H_2, K_1, K_2 and essential weak* closed spaces

$$\mathcal{U} \subset B(K_1, K_2), \quad \mathcal{V} \subset B(H_1, H_2)$$

that are weak TRO equivalent. We fix an infinite cardinal I greater than or equal to the maximum of $I_i, i = 1, 2$ where I_i is the cardinal of an orthonormal basis of H_i. We are going to prove that $k(M_I^w(\mathcal{V})) \le k(M_I^w(\mathcal{U}))$. If $k(M_I^w(\mathcal{U})) = \infty$, the inequality is obvious. So we assume throughout this section that $k(M_I^w(\mathcal{U})) < \infty$. From the results of Section 2, there exist von Neumann algebras $\mathcal{B}_i \subset B(K_i), \mathcal{A}_i \subset B(H_i), i = 1, 2$ and *-isomorphisms $\zeta_i : \mathcal{B}_i' \to \mathcal{A}_i', i = 1, 2$ such that if

$$\mathcal{N}_i = \{T \in B(H_i, K_i) : T\zeta_i(P) = PT \ \ \forall\, P \in pr(\mathcal{B}_i')\}, \quad i = 1, 2,$$

then,

$$\mathcal{U} = [\mathcal{N}_2\mathcal{V}\mathcal{N}_1^*]^{-w^*}, \mathcal{V} = [\mathcal{N}_2^*\mathcal{U}\mathcal{N}_1]^{-w^*}, \mathcal{A}_i = [\mathcal{N}_i^*\mathcal{N}_i]^{-w^*}, \mathcal{B}_i = [\mathcal{N}_i\mathcal{N}_i^*]^{-w^*}, i = 1, 2.$$

We also recall the algebras $\mathcal{A}, \mathcal{B}$ defined in Section 2. Since $\mathcal{U}$ is a hyperreflexive space, $\mathcal{B}$ is a reflexive algebra and thus by 2.7.i in [**3**], $\mathcal{A}$ is also a reflexive algebra. Therefore, $\mathcal{V}$ is a reflexive space.

LEMMA 3.1.

$$\mathcal{V} = \{T \in B(H_1, H_2) : P_i \in pr(\mathcal{B}_i'), i = 1, 2, P_2\mathcal{U}P_1 = 0 \Rightarrow \zeta_2(P_2)T\zeta_1(P_1) = 0\}.$$

Proof. We denote by Ω the space

$$\{T \in B(H_1, H_2) : P_i \in pr(\mathcal{B}_i'), i = 1, 2, P_2\mathcal{U}P_1 = 0 \Rightarrow \zeta_2(P_2)T\zeta_1(P_1) = 0\}.$$

Fix $P_i \in pr(\mathcal{B}_i'), i = 1, 2$ such that $P_2\mathcal{U}P_1 = 0$. We recall $\zeta, \mathcal{A}, \mathcal{B}$ from Section 2. We have that $P_2^\perp \oplus P_1 \in \mathrm{Lat}(\mathcal{B})$. Since $\zeta(\mathrm{Lat}(\mathcal{B})) = \mathrm{Lat}(\mathcal{A})$, we take $\zeta_2(P_2)^\perp \oplus \zeta_1(P_1) \in \mathrm{Lat}(\mathcal{A})$. Therefore, $\zeta_2(P_2)\mathcal{V}\zeta_1(P_1) = 0$. It follows that $\mathcal{V} \subset \Omega$.

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Conversely, if $T \in \Omega$ and $P_2^{\perp}\mathcal{U}P_1 = 0$ for $P_i \in pr(\mathcal{B}_i')$, $i = 1, 2$, then

$$\begin{pmatrix} \zeta_2(P_2)^{\perp} & 0 \\ 0 & \zeta_1(P_1)^{\perp} \end{pmatrix}\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \zeta_2(P_2) & 0 \\ 0 & \zeta_1(P_1) \end{pmatrix} = 0, \quad \forall \ P_2 \oplus P_1 \in \mathrm{Lat}(\mathcal{B}).$$

Therefore,

$$\zeta(Q)^{\perp}\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}\zeta(Q) = 0 \ \forall \ Q \ \in \ \mathrm{Lat}(\mathcal{B}).$$

But $\zeta(\mathrm{Lat}(\mathcal{B})) = \mathrm{Lat}(\mathcal{A})$. Thus

$$Q^{\perp}\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}Q = 0 \ \forall \ Q \ \in \ \mathrm{Lat}(\mathcal{A}).$$

Therefore,

$$\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \in \mathcal{A} \Rightarrow T \in \mathcal{V}.$$

We have thus proved $\Omega \subset \mathcal{V} \Rightarrow \Omega = \mathcal{V}$. □

We define the space

$$\mathcal{W} = [\mathcal{V}\mathcal{N}_1^*]^{-w^*} \subset B(K_1, H_2).$$

Lemma 3.2.

$$\mathcal{W} = \{T \in B(K_1, H_2) : P_i \in pr(\mathcal{B}_i'), i = 1, 2, P_2\mathcal{U}P_1 = 0 \Rightarrow \zeta_2(P_2)TP_1 = 0\}.$$

Proof. Define

$$\Omega = \{T \in B(K_1, H_2) : P_i \in pr(\mathcal{B}_i'), i = 1, 2, P_2\mathcal{U}P_1 = 0 \Rightarrow \zeta_2(P_2)TP_1 = 0\}.$$

Fix $P_i \in pr(\mathcal{B}_i')$, $i = 1, 2$ such that $P_2\mathcal{U}P_1 = 0$ and fix $V \in \mathcal{V}$, $S \in \mathcal{N}_1$. We have,

$$\zeta_2(P_2)VS^*P_1 = \zeta_2(P_2)V\zeta_1(P_1)S^*.$$

By Lemma 3.1, $\zeta_2(P_2)V\zeta_1(P_1) = 0$. Thus, $\zeta_2(P_2)VS^*P_1 = 0$. We have thus proved $\mathcal{W} \subset \Omega$.

For the converse, fix $A \in \Omega$ and $S \in \mathcal{N}_1$. If $P_i \in pr(\mathcal{B}_i')$, $i = 1, 2$ such that $P_2\mathcal{U}P_1 = 0$ we have

$$\zeta_2(P_2)AS\zeta_1(P_1) = \zeta_2(P_2)AP_1S = 0S = 0.$$

Thus,

$$\Omega\mathcal{N}_1 \subset \mathcal{V} \Rightarrow \Omega\mathcal{N}_1\mathcal{N}_1^* \subset \mathcal{W}.$$

But $[\mathcal{N}_1\mathcal{N}_1^*]^{-w^*}$ contains the identity operator. Therefore, $\Omega \subset \mathcal{W}$. The proof is complete. □

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LEMMA 3.3.

$$k(\mathcal{W}) \leq k(M_I^w(\mathcal{U})).$$

Proof. Suppose that I_2 is the cardinal of an orthonormal basis of H_2. We have $I_2 \leq I$. By Lemma 1.5, there exists a column $N = (N_i)_{i \in I_2}$ such that and $N_i \in \mathcal{N}_2$ for all i, and $N^*N = I_{H_2}$. Adding zeros, if necessary, we may assume that $N = (N_i)_{i \in I}$. We claim that

$$\mathcal{W} = N^* C_I^w(\mathcal{U}).$$

Indeed,

$$\mathcal{N}_2^* \mathcal{U} = [\mathcal{N}_2^* \mathcal{N}_2 \mathcal{V} \mathcal{N}_1^*]^{-w^*} \subset [\mathcal{V} \mathcal{N}_1^*]^{-w^*} = \mathcal{W}.$$

Thus,

$$N^* C_I^w(\mathcal{U}) \subset \mathcal{W}.$$

Since $N\mathcal{W} \subset C_I^w(\mathcal{U})$, we have

$$N^*N\mathcal{W} \subset N^* C_I^w(\mathcal{U}) \Rightarrow \mathcal{W} \subset N^* C_I^w(\mathcal{U}).$$

So the claim holds. In the sequel, we use the fact

$$k(C_I^w(\mathcal{U})) \leq k(M_I^w(\mathcal{U})).$$

Fix $A \in B(K_1, H_2)$. We have

$$\begin{aligned} d(A, \mathcal{W}) = \inf_{W \in \mathcal{W}} \|A - W\| &= \inf_{U \in C_I^w(\mathcal{U})} \|A - N^*U\| = \\ &\inf_{U \in C_I^w(\mathcal{U})} \|N^*NA - N^*U\| \leq \inf_{U \in C_I^w(\mathcal{U})} \|NA - U\| = \\ &d(NA, C_I^w(\mathcal{U})) \leq k(M_I^w(\mathcal{U})) r_{C_I^w(\mathcal{U})}(NA). \end{aligned}$$

Since $\mathcal{U}$ is a $\mathcal{B}_2 - \mathcal{B}_1$ bimodule, $C_I^w(\mathcal{U})$ is a $M_I^w(\mathcal{B}_2) - \mathcal{B}_1$ bimodule. Therefore, by Lemma 1.2 for any $\epsilon > 0$, there exist $P_i \in pr(\mathcal{B}_i')$ such that $P_2 \mathcal{U} P_1 = 0$ and

$$\begin{gathered} r_{C_I^w(\mathcal{U})}(NA) - \epsilon < \|P_2^I NAP_1\| = \|N\zeta_2(P_2)AP_1\| \leq \\ \|\zeta_2(P_2)AP_1\|. \end{gathered}$$

By Lemma 3.2, $\zeta_2(P_2)\mathcal{W}P_1 = 0$, thus

$$\|\zeta_2(P_2)AP_1\| \leq r_{\mathcal{W}}(A).$$

Since ϵ is arbitrary,

$$r_{C_I^w(\mathcal{U})}(NA) \leq r_{\mathcal{W}}(A).$$

We have thus proved that

$$d(A, \mathcal{W}) \leq k(M_I^w(\mathcal{U})) r_{\mathcal{W}}(A).$$

The proof is complete. □

LEMMA 3.4.

$$k(M_I^w(\mathcal{W})) \leq k(M_I^w(\mathcal{U})).$$

Proof. We can see that the spaces $M_I^w(\mathcal{U})$, $M_I^w(\mathcal{V})$ are weak TRO equivalent:

$$M_I^w(\mathcal{U}) = [M_I^w(\mathcal{N}_2)M_I^w(\mathcal{V})M_I^w(\mathcal{N}_1)^*]^{-w^*},$$

$$M_I^w(\mathcal{V}) = [M_I^w(\mathcal{N}_2)^*M_I^w(\mathcal{U})M_I^w(\mathcal{N}_1)]^{-w^*}.$$

Following the above arguments from the beginning to Lemma 3.3, the space

$$M_I^w(\mathcal{W}) = [M_I^w(\mathcal{V})M_I^w(\mathcal{N}_1)^*]^{-w^*}$$

has hyperreflexivity constant less than or equal to the hyperreflexivity constant of $M_I^w(M_I^w(\mathcal{U}))$ which, since I is infinite, is equal to the hyperreflexivity constant of $M_I^w(\mathcal{U})$. □

LEMMA 3.5.

$$k(\mathcal{V}) \leq k(M_I^w(\mathcal{U})).$$

Proof. Let I_1 be the cardinal of an orthonormal basis of H_1. We can find an infinite column $M = (M_i)_{i\in I_1}$, $M_i \in \mathcal{N}_1$ such that $M^*M = I_{H_1}$, (Lemma 1.5). Adding zeros if necessary, we may assume that $M = (M_i)_{i\in I}$. We have

$$\mathcal{V}\mathcal{N}_1^*\mathcal{N}_1 \subset \mathcal{V} \Rightarrow \mathcal{N}_1^*\mathcal{N}_1\mathcal{V}^* \subset \mathcal{V}^* \Rightarrow \mathcal{N}_1^*\mathcal{W}^* \subset \mathcal{V}^*.$$

Therefore,

$$M^*C_I^w(\mathcal{W}^*) \subset \mathcal{V}^*.$$

On the other hand, $\mathcal{V}^* = M^*M\mathcal{V}^*$. Since $M\mathcal{V}^* \subset C_I^w(\mathcal{W}^*)$, we have

$$\mathcal{V}^* \subset M^*C_I^w(\mathcal{W}^*) \Rightarrow \mathcal{V}^* = M^*C_I^w(\mathcal{W}^*).$$

Choose $T \in B(H_2, H_1)$. Using Lemma 3.4, we have

$$d(T, \mathcal{V}^*) = \inf_{V\in\mathcal{V}} \|T - V^*\| = \inf_{S\in C_I^w(\mathcal{W}^*)} \|T - M^*S\| = \inf_{S\in C_I^w(\mathcal{W}^*)} \|M^*MT - M^*S\| \leq$$
$$\inf_{S\in C_I^w(\mathcal{W}^*)} \|MT - S\| \leq k(M_I^w(\mathcal{U}))r_{C_I^w(\mathcal{W}^*)}(MT).$$

Fix $\epsilon > 0$. Since $\mathcal{W}$ is an $\mathcal{A}_2 - \mathcal{B}_1$ bimodule, there exist $P \in pr(\mathcal{A}_2')$, $Q \in pr(\mathcal{B}_1')$ such that $Q\mathcal{W}^*P = 0$ and

$$r_{C_I^w(\mathcal{W}^*)}(MT) - \epsilon < \|Q^IMTP\| = \|M\zeta_1(Q)TP\| \leq \|\zeta_1(Q)TP\|.$$

We have

$$P\mathcal{W}Q = 0 \Rightarrow P\mathcal{V}\mathcal{N}_1^*Q = 0 \Rightarrow P\mathcal{V}\zeta_1(Q) = 0 \Rightarrow \zeta_1(Q)\mathcal{V}^*P = 0.$$

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Therefore,

$$r_{C_I^w(\mathcal{W}^*)}(MT) - \epsilon < r_{\mathcal{V}^*}(T).$$

Since ϵ is arbitrary, we have

$$r_{C_I^w(\mathcal{W}^*)}(MT) \leq r_{\mathcal{V}^*}(T) \Rightarrow d(T, \mathcal{V}^*) \leq k(M_I^w(\mathcal{U}))r_{\mathcal{V}^*}(T).$$

Therefore $\mathcal{V}^*$, and hence $\mathcal{V}$ has hyperreflexivity constant less than $k(M_I^w(\mathcal{U}))$. $\square$

THEOREM 3.6. *Let $\mathcal{U}, \mathcal{V}, H_1, H_2, K_1, K_2, I$ be as in the beginning of this section. Then*

$$k(M_I^w(\mathcal{V})) \leq k(M_I^w(\mathcal{U})).$$

In the special case that H_1, H_2, K_1, K_2 are separable, we have

$$k(M_\infty^w(\mathcal{V})) = k(M_\infty^w(\mathcal{U})).$$

Proof. The spaces $M_I^w(\mathcal{U})$, $M_I^w(\mathcal{V})$ are weak TRO equivalent:

$$M_I^w(\mathcal{U}) = [M_I^w(\mathcal{N}_2)M_I^w(\mathcal{V})M_I^w(\mathcal{N}_1)^*]^{-w^*},$$
$$M_I^w(\mathcal{V}) = [M_I^w(\mathcal{N}_2)^* M_I^w(\mathcal{U})M_I^w(\mathcal{N}_1)]^{-w^*}.$$

Following the arguments from the beginning to Lemma 3.5, $M_I^w(\mathcal{V})$ has hyperreflexivity constant less than or equal to $k(M_I^w(M_I^w(\mathcal{U})) = k(M_I^w(\mathcal{U}))$.

If $\mathcal{U}, \mathcal{V}$ are separably acting spaces, then by the first part of the proof,

$$k(M_\infty^w(\mathcal{V})) \leq k(M_\infty^w(\mathcal{U})).$$

By symmetry,

$$k(M_\infty^w(\mathcal{U})) \leq k(M_\infty^w(\mathcal{V})).$$

$\square$

4. Isomorphisms and complete hyperreflexivity. In this section, for each reflexive space $\mathcal{X}$ we write $k_c(\mathcal{X})$ for its complete hyperreflexivity constant, $k(M_\infty^w(\mathcal{X})$.

THEOREM 4.1. *Let $\mathcal{B}, \mathcal{A}$ be stably isomorphic CSL algebras acting on the separable Hilbert spaces K, H respectively. If $\mathcal{B}$ is completely hyperreflexive, then $\mathcal{A}$ is also completely hyperreflexive and $k_c(\mathcal{B}) = k_c(\mathcal{A})$.*

Proof. By Theorem 3.2 in [**4**] and the main result of [**6**], the algebras $\mathcal{B}$ and $\mathcal{A}$ are weak TRO equivalent. The conclusion comes from Theorem 3.6. $\square$

COROLLARY 4.2. *Let $\mathcal{B}, \mathcal{A}$ be CSL algebras acting on the separable Hilbert spaces K, H respectively. We assume that $\mathcal{B}$ is completely hyperreflexive. If either*

(i) *$\mathcal{A}$ is not completely hyperreflexive, or*

(ii) *$\mathcal{A}$ is completely hyperreflexive, but $k_c(\mathcal{A}) \neq k_c(\mathcal{B})$,*

then $\mathcal{B}$ and $\mathcal{A}$ cannot be stably isomorphic.

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REMARK 4.3. In view of Theorem 4.1, we remark that two stably isomorphic completely hyperreflexive spaces need not have the same complete hyperreflexivity constant. For example, take $H = l^2(\mathbb{N})$, I_H the corresponding identity operator, $\mathcal{X} = \mathbb{C}I_H$ and $\mathcal{Y} = B(H)$. Since, the space $\mathcal{X}\bar{\otimes}B(H)$ is isomorphic as a dual operator space with $\mathcal{Y}\bar{\otimes}B(H)$, $\mathcal{X}$ and $\mathcal{Y}$ are stably isomorphic. But $k_c(\mathcal{Y}) = 1$ and $k_c(\mathcal{X}) > 1$. (See Lemma 6.11 in [**2**]).

THEOREM 4.4. *Let $\mathcal{L}_i \subset B(H_i), i = 1, 2$ be separably acting reflexive lattices. If there exists a *-isomorphism $\theta : \mathcal{L}_1'' \to \mathcal{L}_2''$ such that $\theta(\mathcal{L}_1) = \mathcal{L}_2$, then the algebras* $\mathrm{Alg}(\mathcal{L}_1), \mathrm{Alg}(\mathcal{L}_2)$ *are weak TRO equivalent, (Theorem 3.3 in [**3**]). Therefore, by Theorem 3.6,* $k_c(\mathrm{Alg}(\mathcal{L}_1)) = k_c(\mathrm{Alg}(\mathcal{L}_2))$.

COROLLARY 4.5. *Let $\mathcal{L}_i \subset B(H_i), i = 1, 2$ be separably acting totally atomic CSL's. If these lattices are isomorphic as CSL's, then the algebras* $\mathrm{Alg}(\mathcal{L}_1), \mathrm{Alg}(\mathcal{L}_2)$ *are weak TRO equivalent, (Theorem 5.3 in [**3**]). Therefore, by Theorem 3.6,* $k_c(\mathrm{Alg}(\mathcal{L}_1)) = k_c(\mathrm{Alg}(\mathcal{L}_2))$.

THEOREM 4.6. *Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras acting on the Hilbert spaces K and H respectively. If $\pi : \mathcal{A}' \to \mathcal{B}'$ is a $*$-isomorphism and I is an infinite cardinal greater than or equal to the cardinal of an orthonormal basis of H, then*

$$k(M_I^w(\mathcal{B})) \leq k(M_I^w(\mathcal{A})).$$

In the special case where $\mathcal{A}, \mathcal{B}$ are separably acting, we have

$$k(M_\infty^w(\mathcal{B})) = k(M_\infty^w(\mathcal{A})).$$

Proof. We define the following TRO:

$$\mathcal{M} = \{M \in B(K, H) : MA = \pi(A)M \ \ \forall\, A \in \mathcal{A}'\}.$$

By Theorem 3.2 in [**3**], we have

$$[\mathcal{M}^*\mathcal{A}\mathcal{M}]^{-w^*} = \mathcal{B}, \quad [\mathcal{M}\mathcal{B}\mathcal{M}^*]^{-w^*} = \mathcal{A}.$$

Thus, $\mathcal{A}, \mathcal{B}$ are weak TRO equivalent. The conclusion comes from Theorem 3.6. □

THEOREM 4.7. *Let $\mathcal{A}$ be a separably acting von Neumann algebra for which the commutant is stable, i.e., $\mathcal{A}'$ and $M_\infty^w(\mathcal{A}')$ are isomorphic. Then, $\mathcal{A}$ is completely hyperreflexive and $k_c(\mathcal{A}) \leq 9$.*

Proof. The algebra $M_\infty^w(\mathcal{A}^\infty)$ is unitarily equivalent to $M_\infty^w(\mathcal{A})^\infty$. The last algebra is hyperreflexive with constant less than nine, [**10**]. Thus, $k(M_\infty^w(\mathcal{A}^\infty)) \leq 9$. Since by hypothesis the commutants $(\mathcal{A}^\infty)'$ and $\mathcal{A}'$ are isomorphic, Theorem 4.6 implies that

$$k(M_\infty^w(\mathcal{A})) = k(M_\infty^w(\mathcal{A}^\infty)) \leq 9.$$

□

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