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A dichotomy for topological full groups

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Abstract. Given a minimal action α of a countable group on the Cantor set, we show that the alternating full group $\mathsf{A}(\alpha)$ is non-amenable if and only if the topological full group $\mathsf{F}(\alpha)$ is C^*-simple. This implies, for instance, that the Elek–Monod example of non-amenable topological full group coming from a Cantor minimal $\mathbb{Z}^2$-system is C^*-simple.

1 Introduction

Given an action α of a group on the Cantor set X, the *topological full group* of α, denoted by $\mathsf{F}(\alpha)$, is the group of homeomorphisms on X which are locally given by α.

In [JM13], Juschenko and Monod showed that topological full groups of Cantor minimal $\mathbb{Z}$-systems are amenable. Together with results of Matui [Mat06], this gave rise to the first examples of infinite, simple, finitely generated, amenable groups. On the other hand, in [EM13], Elek and Monod constructed an example of a free minimal $\mathbb{Z}^2$-subshift whose topological full group contains a free group.

A group Γ is said to have the *unique trace property* if its reduced C^*-algebra $C^*_r(\Gamma)$ has a unique tracial state and to be *C^*-simple* if $C^*_r(\Gamma)$ is simple. In [BKKO17], Breuillard et al. showed that Γ has the unique trace property if and only if it does not contain any non-trivial amenable normal subgroup, and in [Ken20], Kennedy showed that Γ is C^*-simple if and only if it does not contain any nontrivial amenable uniformly recurrent subgroup (URS).

By using this new characterization of C^*-simplicity, Le Boudec and Matte Bon showed in [LBMB18] that the topological full group of a free minimal action of a countable non-amenable group on the Cantor set is C^*-simple, and asked whether the same conclusion holds if one does not assume freeness. In [BS19], Brix and the author showed that it suffices to assume that the action is topologically free. In [KTD19], Kerr and Tucker-Drob obtained examples of C^*-simple topological full groups coming from actions of amenable groups.

Given an action α of a group on the Cantor set, Nekrashevych introduced in [Nek19] the *alternating full group* of the action, which we denote by $\mathsf{A}(\alpha)$. This is a normal subgroup of $\mathsf{F}(\alpha)$ generated by certain copies of finite alternating groups. It was shown in [Nek19] that if α is minimal, then $\mathsf{A}(\alpha)$ is simple and is contained in every nontrivial normal subgroup of $\mathsf{F}(\alpha)$.

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In [MB18], Matte Bon obtained a classification of URSs of topological full groups. By using this result, we show the following theorem.

Theorem (Theorem 3.5) *Let α be a minimal action of a countable group on the Cantor set. The following conditions are equivalent:*

(i) $\mathsf{A}(\alpha)$ *is non-amenable.*
(ii) *Any group H such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$ is C^*-simple.*
(iii) *There exists a C^*-simple group H such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$.*

As a consequence, we obtain the following corollary.

Corollary (Corollary 3.6) *Let α be a minimal action of a countable group on the Cantor set. Then $\mathsf{F}(\alpha)$ has the unique trace property if and only if it is C^*-simple. If $\mathsf{A}(\alpha) = \mathsf{F}(\alpha)'$, then $\mathsf{F}(\alpha)$ is non-amenable if and only if it is C^*-simple.*

It is still an open problem whether $\mathsf{A}(\alpha)$ always coincides with $\mathsf{F}(\alpha)'$, but in many cases, this is known to be true. For example, it follows from results of Matui [Mat15] that this is the case for free Cantor minimal $\mathbb{Z}^n$-systems. This implies that the example of non-amenable topological full group coming from an action of $\mathbb{Z}^2$ in [EM13] is C^*-simple.

2 Preliminaries

2.1 Topological dynamics

Given a locally compact Hausdorff space X, we denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, and by $\mathcal{P}(X)$ the space of regular probability measures on X.

If Γ is a group acting by homeomorphisms on X, we say that X is a *locally compact Γ-space*. If X admits no nontrivial Γ-invariant closed subspaces, then we say that X (or the action) is *minimal*. Given $U \subset X$, let $\mathrm{St}_\Gamma(U)$ consist of the elements of Γ which fix pointwise U, and $\mathrm{St}_\Gamma(U)^0$ consist of the elements of Γ which fix pointwise a neighborhood of U. To ease the notation, given $x \in X$, we let $\Gamma_x := \mathrm{St}_\Gamma(\{x\})$ and $\Gamma_x^0 := \mathrm{St}_\Gamma(\{x\})^0$.

Denote by $\mathrm{Sub}(\Gamma)$ the set of subgroups of Γ, endowed with the *Chabauty topology*; this is the restriction to $\mathrm{Sub}(\Gamma)$ of the product topology on $\{0,1\}^\Gamma$, where every subgroup $\Lambda \in \mathrm{Sub}(G)$ is identified with its characteristic function $\mathbf{1}_\Lambda \in \{0,1\}^\Gamma$. Notice that the space of amenable subgroups $\mathrm{Sub}_{am}(\Gamma)$ is closed in $\mathrm{Sub}(\Gamma)$. We consider $\mathrm{Sub}(\Gamma)$ as a compact Γ-space under the action by conjugation. A subgroup $\Lambda \leq \Gamma$ is said to be *confined* if $\{e\}$ is not in the closure of the Γ-orbit of Λ.

An *invariant random subgroup* (IRS) is a Γ-invariant regular probability measure on $\mathrm{Sub}(\Gamma)$. We say an IRS is *amenable* if its support is contained in $\mathrm{Sub}_{am}(\Gamma)$. By [BKKO17, Corollary 4.3] and [BDL16, Corollary 1.5], Γ has the unique trace property if and only if its unique amenable normal subgroup is $\{e\}$, if and only if its unique amenable IRS is $\delta_{\{e\}}$.

A *URS* is a Γ-invariant closed minimal subspace $\mathcal{U} \subset \mathrm{Sub}(\Gamma)$. We say $\mathcal{U}$ is *amenable* if every element of $\mathcal{U}$ is amenable. By [Ken20, Theorem 4.1], Γ is C^*-simple if and only if its only amenable URS is $\{\{e\}\}$. Alternatively, Γ is C^*-simple if and only if it does not contain any confined amenable subgroup.

Suppose Γ is countable and X is a minimal compact Γ-space. Let $\mathrm{Stab}_\Gamma\colon X \to \mathrm{Sub}(\Gamma)$ be the map given by $\mathrm{Stab}_\Gamma(x) := \Gamma_x$ and $\mathrm{Stab}^0_\Gamma\colon X \to \mathrm{Sub}(\Gamma)$ be the map given by $\mathrm{Stab}^0_\Gamma(x) := \Gamma^0_x$, for $x \in X$. Notice that Stab_Γ and Stab^0_Γ are Borel measurable and Γ-equivariant. Moreover, the set $Y := \{x \in X : \Gamma_x = \Gamma^0_x\}$ is dense in X and $\mathrm{Stab}_\Gamma(Y)$ is a URS, the so-called *stabilizer URS* of the action $\Gamma \curvearrowright X$ (for a proof of these last claims, see [LBMB18, Section 2]).

2.2 Topological full groups

Fix an action α of a group Γ on the Cantor set X. We say that a homeomorphism $h\colon U \to V$ between clopen subsets $U, V \subset X$ is *locally given by* α if there exist $g_1, \dots, g_n \in \Gamma$ and clopen sets $A_1, \dots, A_n \subset U$ such that $U = \bigsqcup_{i=1}^n A_i$ and $h|_{A_i} = g_i|_{A_i}$, for $1 \le i \le n$. The *topological full group* of α, denoted by $\mathsf{F}(\alpha)$, is the group of homeomorphisms $h\colon X \to X$ which are locally given by α.

Given $d \in \mathbb{N}$, a *d-multisection* is a collection of d disjoint clopen sets $(A_i)_{i=1}^d \subset X$ and d^2 homeomorphisms $(h_{i,j}\colon A_i \to A_j)_{i,j=1}^d$ which are locally given by α and such that, for $1 \le i, j, k \le d$, it holds that $h_{j,k} h_{i,j} = h_{i,k}$ and $h_{i,i} = \mathrm{Id}_{A_i}$.

Given $d \in \mathbb{N}$, let S_d and A_d be the symmetric and alternating groups, respectively. Given a d-multisection $\mathcal{F} = ((A_i)_{i=1}^d, (h_{i,j})_{i,j=1}^d)$ and $\sigma \in S_d$, let $\mathcal{F}(\sigma) \in \mathsf{F}(\alpha)$ be given by $\mathcal{F}(\sigma)|_{A_i} := h_{\sigma(i),i}$, for $1 \le i \le n$ and $\mathcal{F}(\sigma)(x) = x$ for $x \notin \bigsqcup_{i=1}^n A_i$. The *alternating full group* $\mathsf{A}(\alpha)$ is the subgroup of $\mathsf{F}(\alpha)$ generated by

$$\{\mathcal{F}(\sigma) : d \in \mathbb{N}, \mathcal{F} \text{ is a } d\text{-multisection}, \sigma \in A_d\}.$$

Notice that $\mathsf{A}(\alpha)$ is normal in $\mathsf{F}(\alpha)$ and that $\mathsf{A}(\alpha)$ is contained in the derived subgroup $\mathsf{F}(\alpha)'$. If α is a minimal action of a countable group on the Cantor set, then $\mathsf{A}(\alpha)$ is simple [Nek19, Theorem 4.1].

Remark 2.1 Alternatively, $\mathsf{F}(\alpha)$ and $\mathsf{A}(\alpha)$ can be described as groups of bisections of the groupoid of germs of α. This is the point of view adopted in [MB18, Nek19]. Conversely, given an effective groupoid G with unit space $G^{(0)}$ homeomorphic to the Cantor set, denote by α the natural action of the topological full group of G on $G^{(0)}$. Then the topological and alternating full groups of G coincide with $\mathsf{F}(\alpha)$ and $\mathsf{A}(\alpha)$, respectively [NO19, Corollary 4.7].

3 C^*-simplicity of full groups

Given a locally compact Γ-space X and $U \subset X$ open not necessarily invariant, let

$$\mathcal{P}_\Gamma(U) := \{\mu \in \mathcal{P}(U) : \forall g \in \Gamma\ \forall A \in \mathcal{B}(U),\ \ \mu(A \cap g^{-1}U) = \mu(gA \cap U)\}.$$

Alternatively, one can characterize $\mathcal{P}_\Gamma(U)$ as the measures $\mu \in \mathcal{P}(U)$ such that $\mu(gA) = \mu(A)$ for every $g \in \Gamma$ and $A \in \mathcal{B}(U)$ such that $gA \subset U$.

Proposition 3.1 *Let X be a compact Γ-space and $U \subset X$ open such that $X = \Gamma.U$. Then the map $j\colon \mathcal{P}_\Gamma(X) \to \mathcal{P}_\Gamma(U)$ given by $j(\nu) := \frac{\nu|_{\mathcal{B}(U)}}{\nu(U)}$ is a well-defined bijection.*

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Proof Take $g_1, \dots, g_n \in \Gamma$ such that $X = \bigcup_{i=1}^n g_i U$. For $1 \leq i \leq n$, let

$$A_i := U \setminus \bigcup_{j=1}^{i-1} g_i^{-1} g_j U.$$

Then $X = \bigsqcup_{i=1}^n g_i A_i$. Given $\nu \in \mathcal{P}_\Gamma(X)$, obviously $\nu(U) \geq 1/n$, so that j is a well-defined map. Moreover, given $A \in \mathcal{B}(X)$, we have

$$\nu(A) = \sum_{i=1}^n \nu(g_i A_i \cap A) = \sum_{i=1}^n \nu(A_i \cap g_i^{-1} A). \tag{1}$$

Since each A_i is contained in U, this implies that ν is determined by its restriction to $\mathcal{B}(U)$.

If $\nu_1, \nu_2 \in \mathcal{P}_\Gamma(X)$ are such that $j(\nu_1) = j(\nu_2)$, then $\nu_1|_U = \frac{\nu_1(U)}{\nu_2(U)} \nu_2|_U$. Furthermore, by (1), we have

$$1 = \nu_1(X) = \sum_{i=1}^n \nu_1(A_i) = \sum_{i=1}^n \frac{\nu_1(U)}{\nu_2(U)} \nu_2(A_i) = \frac{\nu_1(U)}{\nu_2(U)} \nu_2(X) = \frac{\nu_1(U)}{\nu_2(U)},$$

and hence $\nu_1(U) = \nu_2(U)$. Consequently, $\nu_1 = \nu_2$ and j is injective.

Let us now show that j is surjective. Given $\mu \in \mathcal{P}_\Gamma(U)$ and $A \in \mathcal{B}(X)$, let

$$\nu(A) := \sum_{i=1}^n \mu(A_i \cap g_i^{-1} A).$$

Given $B \in \mathcal{B}(U)$, we have

$$\nu(B) = \sum_{i=1}^n \mu(A_i \cap g_i^{-1} B) = \sum_{i=1}^n \mu(g_i A_i \cap B) = \mu(B),$$

so that $\nu|_{\mathcal{B}(U)} = \mu$.

We claim that ν is Γ-invariant. Fix $A \in \mathcal{B}(X)$ and $g \in \Gamma$, and we will show that $\nu(A) = \nu(gA)$.

For $1 \leq i \leq n$, let $h_i := g^{-1} g_i$, $B_i := A_i \cap g_i^{-1} A$, and $C_i := A_i \cap g_i^{-1} g A = A_i \cap h_i^{-1} A$. By definition of ν, we have $\nu(A) = \sum_{i=1}^n \mu(B_i)$ and $\nu(gA) = \sum_{i=1}^n \mu(C_i)$.

Moreover, one can readily check that $A = \bigsqcup_{i=1}^n g_i B_i = \bigsqcup_{i=1}^n h_i C_i$.

For $1 \leq i, j \leq n$, let $B_{i,j} := B_i \cap g_i^{-1} h_j C_j$ and $C_{i,j} := h_j^{-1} g_i B_i \cap C_j$.

Notice that, for $1 \leq i \leq n$,

$$\bigsqcup_{j=1}^n B_{i,j} = B_i \cap g_i^{-1} A = B_i$$

and, for $1 \leq j \leq n$,

$$\bigsqcup_{i=1}^n C_{i,j} = h_j^{-1} A \cap C_j = C_j.$$

Furthermore, $g_i B_{i,j} = h_j C_{i,j}$, and hence $\mu(B_{i,j}) = \mu(C_{i,j})$, since $B_{i,j}$ and $C_{i,j}$ are contained in U for every i, j. Therefore,

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$$\nu(A)=\sum_{i=1}^{n}\mu(B_i)=\sum_{i,j=1}^{n}\mu(B_{i,j})=\sum_{i,j=1}^{n}\mu(C_{i,j})=\sum_{j=1}^{n}\mu(C_j)=\nu(gA).$$

Finally, we have that $j(\frac{\nu}{\nu(X)})=\frac{\nu|_{\mathcal{B}(U)}/\nu(X)}{\nu(U)/\nu(X)}=\nu|_{\mathcal{B}(U)}=\mu$. ∎

Remark 3.2 Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions on compact spaces. The actions are said to be *Kakutani equivalent* [Li18, Definition 2.14] if there exist clopen sets $A \subset X$ and $B \subset Y$ such that $X=\Gamma.A$, $Y=\Lambda.B$, and the partial transformation groupoids obtained by restriction to A and B are isomorphic. Proposition 3.1 implies that Kakutani equivalence induces a bijection between $\mathcal{P}_\Gamma(X)$ and $\mathcal{P}_\Lambda(X)$.

The proof of the following result is analogous to [NO19, Lemma 4.9(2)].

Lemma 3.3 *Let α be a minimal action of a group Γ on the Cantor set X. Given $U \subset X$ clopen, $x \in U$, and $g \in \Gamma$ such that $g(x) \in U$, there exists a neighborhood V of x and $h \in \mathrm{St}_{\mathsf{A}(\alpha)}(U^c)$ such that $g|_V = h|_V$.*

Proof Case 1: $g(x) \neq x$. Take $k \in \Gamma$ such that $k(g(x)) \in U \setminus \{x, g(x)\}$. Let V be a clopen neighborhood of x such that V, $g(V)$, and $kg(V)$ are disjoint subsets of U. Then the homeomorphisms $h_{2,1} := g|_V$ and $h_{3,1} := kg|_V$ give rise to a 3-multisection $\mathcal{F}$ such that $\mathcal{F}((123))|_V = g|_V$ and $\mathcal{F}((123)) \in \mathrm{St}_{\mathsf{A}(\alpha)}(U^c)$.

Case 2: $g(x)=x$. Take $k \in \Gamma$ such that $k(x) \in U \setminus \{x\}$. By Case 1, there are $h_1, h_2 \in \mathrm{St}_{\mathsf{A}(\alpha)}(U^c)$ and V_1, V_2 neighborhoods of x and $k(x)$, respectively, such that $k|_{V_1} = h_1|_{V_1}$ and $gk^{-1}|_{V_2} = h_2|_{V_2}$. Then $V := V_1 \cap k^{-1}(V_2)$ is a neighborhood of x such that $h_2h_1|_V = g|_V$. ∎

The next lemma uses the same idea of [MB18, Corollary 6.5].

Lemma 3.4 *Let α be a minimal action of a countable group Γ on the Cantor set X and H a group such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$. Then H is not C^*-simple if and only if $\mathsf{A}(\alpha)_x^0$ is amenable for all $x \in X$.*

Proof Suppose H is not C^*-simple. Then H contains a confined amenable subgroup. By [MB18, Theorem 6.1], there exists $Q \subset X$ finite such that $\mathrm{St}^0_{\mathsf{A}(\alpha)}(Q)$ is amenable. Given $x \in X$, take a net $(g_i) \subset \mathsf{F}(\alpha)$ such that $g_i q \to x$ for any $q \in Q$ (existence of such a net (g_i) follows from minimality and proximality of $\mathsf{F}(\alpha) \curvearrowright X$; see [MB18, Lemma 5.12]). Take K a limit point of $g_i\,\mathrm{St}^0_{\mathsf{A}(\alpha)}(Q)g_i^{-1}$. One can readily check that $\mathsf{A}(\alpha)_x^0 \leq K$, and hence $\mathsf{A}(\alpha)_x^0$ is amenable.

Conversely, if $\mathsf{A}(\alpha)_x^0$ is amenable for all $x \in X$, then since $\mathsf{A}(\alpha)_x^0$ is nontrivial for every x, it follows that the stabilizer URS $\mathcal{U}$ of $\mathsf{A}(\alpha) \curvearrowright X$ is a nontrivial amenable URS of $\mathsf{A}(\alpha)$. By [MB18, Theorem 6.1], any element of $\mathcal{U}$ is a confined subgroup of $\mathsf{F}(\alpha)$ (hence of H as well). Therefore, H is not C^*-simple. ∎

Theorem 3.5 *Let α be a minimal action of a countable group Γ on the Cantor set X. The following conditions are equivalent:*

(i) $\mathsf{A}(\alpha)$ *is non-amenable.*

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(ii) *Any group H such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$ is C^*-simple.*
(iii) *There exists a C^*-simple group H such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$.*

Proof The implications (ii) $\implies$ (iii) $\implies$ (i) are immediate.

(i) $\implies$ (ii): Suppose that there exists H non-C^*-simple such that $\mathsf{A}(\alpha) \leq H \leq \mathsf{F}(\alpha)$. By Lemma 3.4, $\mathsf{A}(\alpha)^0_x$ is amenable for every $x \in X$.

Fix a clopen nonempty set U properly contained in X. Since, for any $x \in U^c$, we have $\Lambda := \mathrm{St}_{\mathsf{A}(\alpha)}(U^c) \leq \mathsf{A}(\alpha)^0_x$, it follows that Λ is amenable.

Let $\mu \in \mathcal{P}_\Lambda(U)$, and we claim that $\mu \in \mathcal{P}_\Gamma(U)$. By regularity, it suffices to show that, for any $K \subset U$ compact and $g \in \Gamma$ such that $g(K) \subset U$, it holds that $\mu(gK) = gK$. By Lemma 3.3, there are $h_1, \dots, h_n \in \Lambda$ and a partition $K = \bigsqcup_{i=1}^n K_i$ into compact sets such that $g|_{K_i} = h_i|_{K_i}$ for $1 \leq i \leq n$. Therefore,

$$\mu(gK) = \sum_{i=1}^n \mu(gK_i) = \sum_{i=1}^n \mu(h_i K_i) = \sum_{i=1}^n \mu(K_i) = \mu(K)$$

and $\mu \in \mathcal{P}_\Gamma(U)$.

We conclude from Proposition 3.1 that there is $\nu \in \mathcal{P}_\Gamma(X) = \mathcal{P}_{\mathsf{F}(\alpha)}(X)$. Furthermore, by minimality of the action, ν has full support. Let $\rho := (\mathrm{Stab}^0_{\mathsf{A}(\alpha)})_* \nu$.

Given $g \in \Lambda \setminus \{e\}$, we have $\rho(\{K \in \mathrm{Sub}(\mathsf{A}(\alpha)) : g \in K\}) \geq \nu(U^c) > 0$. Hence, ρ is a nontrivial amenable IRS on $\mathsf{A}(\alpha)$. Since $\mathsf{A}(\alpha)$ is simple, this implies that $\mathsf{A}(\alpha)$ is amenable. ∎

The following is an immediate consequence of Theorem 3.5.

Corollary 3.6 *Let α be a minimal action of a countable group on the Cantor set. Then $\mathsf{F}(\alpha)$ has the unique trace property if and only if it is C^*-simple. If $\mathsf{A}(\alpha) = \mathsf{F}(\alpha)'$, then $\mathsf{F}(\alpha)$ is non-amenable if and only if it is C^*-simple.*

Remark 3.7 If α is an action of a group on the noncompact Cantor set X, then the topological full group $\mathsf{F}(\alpha)$ is the group of homeomorphisms on X which are locally given by α and have compact support. Moreover, $\mathsf{A}(\alpha)$ is defined by requiring that the domains of the partial homeomorphisms of the multisections to be compact-open (as in [MB18, Definition 5.1]). By arguing as in [MB18, Corollary 6.5], the same conclusion of Theorem 3.5 and Corollary 3.6 holds in the noncompact case.

Example 3.8 It follows from [Mat12, Lemma 6.3] and [Mat15, Theorem 4.7] that, given a free minimal action α of $\mathbb{Z}^n$ on the Cantor set, it holds that $\mathsf{F}(\alpha)' = \mathsf{A}(\alpha)$. Hence, the example of non-amenable topological full group coming from a Cantor minimal $\mathbb{Z}^2$-system in [EM13] is C^*-simple.

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