Patterns among square roots of the 2 × 2 identity matrix

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1. *The problem and its solution*

The 2×2 identity matrix, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, has an infinite number of square roots. The purpose of this paper is to show some interesting patterns that appear among these square roots. In the process, we will take a brief tour of some topics in number theory, including Pythagorean triples, Eisenstein triples, Fibonacci numbers, Pell numbers and Diophantine triples.

For this paper, we will only consider matrices whose entries are integers or rational numbers.

Finding the general form of the square roots is straightforward. We have $\begin{pmatrix} p & q \\ r & s \end{pmatrix}^2 = I$, so $\begin{pmatrix} p^2 + qr & q(p + s) \\ r(p + s) & s^2 + qr \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Since $q(p + s) = 0$ it follows that $q = 0$ or $p + s = 0$ or both.

Similarly, since $r(p + s) = 0$, it follows that $r = 0$, or $p + s = 0$ or both. Note that

$$p^2 + qr = 1. \tag{1}$$

Case 1: $q = 0$ and $r = 0$, so $p + s$ could be 0 but does not have to be. Since $p^2 + qr = s^2 + qr = 1$ we have $p^2 = s^2 = 1$, and we get four matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 2: $q = 0$, $p + s = 0$ and $r \neq 0$. In that case we have $s = -p$ and $p^2 = s^2 = 1$ so we get $\begin{pmatrix} 1 & 0 \\ r & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ r & 1 \end{pmatrix}$, where r is any rational number other than 0.

Case 3: $r = 0$, $p + s = 0$ and $q \neq 0$. In that case we have $s = -p$ and $p^2 = s^2 = 1$, so we get $\begin{pmatrix} 1 & q \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & q \\ 0 & 1 \end{pmatrix}$ where q is any rational number other than 0.

Case 4: $r \neq 0$, $q \neq 0$ and $p = s = 0$. By (1) we have $qr = 1$ and so we get $\begin{pmatrix} 0 & q \\ \frac{1}{q} & 0 \end{pmatrix}$ where q is any rational number other than 0. This includes the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

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Case 5: $p + s = 0$ but none of the matrix elements is 0. By (1) we have $r = \dfrac{1 - p^2}{q}$. We get $\begin{pmatrix} p & q \\ r & -p \end{pmatrix}$ where $r = \dfrac{1 - p^2}{q}$. That is, we get

$$\begin{pmatrix} p & q \\ \frac{1-p^2}{q} & -p \end{pmatrix}. \tag{2}$$

This is the interesting case.

Remember that p, q and r are rational numbers. Define t to be the least common denominator of p, q and r. Then define x, y and z by $p = \dfrac{x}{t}$, $q = \dfrac{y}{t}$ and $r = \dfrac{z}{t}$. Then from (1) we get the quadratic equation

$$x^2 + yz = t^2, \tag{3}$$

and the matrix

$$\begin{pmatrix} \frac{x}{t} & \frac{y}{t} \\ \frac{z}{t} & -\frac{x}{t} \end{pmatrix} \tag{4}$$

is a square root of I_2.

2. *Variations*

If $\begin{pmatrix} p & q \\ r & -p \end{pmatrix}$ is a square root of I_2, then we can construct other square roots by negating the diagonal elements, negating the off-diagonal elements, negating all the elements, and by taking the transpose in each case. We thus construct the following matrices which are also square roots of I_2:

$$\begin{pmatrix} -p & q \\ r & p \end{pmatrix}, \begin{pmatrix} p & -q \\ -r & -p \end{pmatrix}, \begin{pmatrix} -p & -q \\ -r & p \end{pmatrix}, \begin{pmatrix} p & r \\ q & -p \end{pmatrix}, \begin{pmatrix} -p & r \\ q & p \end{pmatrix}, \begin{pmatrix} p & -r \\ -q & -p \end{pmatrix}, \begin{pmatrix} -p & -r \\ -q & p \end{pmatrix}. \tag{5}$$

Furthermore, we can rewrite $p^2 + qr = 1$ as $1^2 + (-q)r = p^2$, which is of the form (3), and use (4) to generate 8 more square roots of I_2:

$$\begin{pmatrix} \frac{1}{p} & \frac{-q}{p} \\ \frac{r}{p} & \frac{-1}{p} \end{pmatrix}, \begin{pmatrix} \frac{-1}{p} & \frac{-q}{p} \\ \frac{r}{p} & \frac{1}{p} \end{pmatrix}, \begin{pmatrix} \frac{1}{p} & \frac{q}{p} \\ \frac{-r}{p} & \frac{-1}{p} \end{pmatrix}, \begin{pmatrix} \frac{-1}{p} & \frac{q}{p} \\ \frac{-r}{p} & \frac{1}{p} \end{pmatrix}, \begin{pmatrix} \frac{1}{p} & \frac{r}{p} \\ \frac{-q}{p} & \frac{-1}{p} \end{pmatrix}, \begin{pmatrix} \frac{-1}{p} & \frac{r}{p} \\ \frac{-q}{p} & \frac{1}{p} \end{pmatrix}, \begin{pmatrix} \frac{1}{p} & \frac{-r}{p} \\ \frac{q}{p} & \frac{-1}{p} \end{pmatrix}, \begin{pmatrix} \frac{-1}{p} & \frac{-r}{p} \\ \frac{q}{p} & \frac{1}{p} \end{pmatrix}. \tag{6}$$

Thus for any matrix in this Article which is a square root of I_2 of the form (2), we can generate 15 others using (5) and (6).

3. *Pythagorean triples*

Recall that a Pythagorean triple is a triple of positive integers (a, b, c) satisfying $a^2 + b^2 + c^2$. Using (2), it is easy to try various values of p and q, and then to look for patterns. One such pattern is that if we let $q = 2$ then the left-hand column of the resulting matrix is suggestive of a

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Pythagorean triple. Two examples are $\begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ -12 & -5 \end{pmatrix}$. We will negate the off-diagonal elements so that the left-hand column elements are both positive. We have:

$$\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}, \begin{pmatrix} 7 & -2 \\ 24 & -7 \end{pmatrix}, \begin{pmatrix} 9 & -2 \\ 40 & -9 \end{pmatrix}, \begin{pmatrix} 11 & -2 \\ 60 & -11 \end{pmatrix}, \ldots$$

Thus if the upper right-hand element is -2, then the left-hand column contains the legs, i.e. the a and b elements, of a Pythagorean triple (a, b, c). The triples here are

$$(3,\ 4,\ 5), (5,\ 12,\ 13), (7,\ 24,\ 25), (9,\ 40,\ 41), (11,\ 60,\ 61), \ldots .$$

Triples of this form are known as the Pythagorean family of Pythagorean triples [1]. They are of the form (a, b, c) where $a = 2n + 1$, $b = 2n(n + 1)$ and $c = b + 1$. Then it is easy to show that $a^2 + (-2)b = 1$. This is of the form (3), and so from (4) we get that $\begin{pmatrix} a & -2 \\ b & -a \end{pmatrix}$ is a square root of I_2.

In general, from (2), the matrix we get is $\begin{pmatrix} p & -2 \\ \frac{1}{2}(p^2 - 1) & -p \end{pmatrix}$. The left-hand column elements give us the Pythagorean triple $\left(p, \frac{1}{2}(p^2 - 1), \frac{1}{2}(p^2 + 1)\right)$ with integer entries if p is odd.

If the upper right-hand element of the matrix is not ± 2 but some other non-zero integer, we can still use the left-hand column entries to generate quickly a Pythagorean triple. From (1) and (2), our matrix is $\begin{pmatrix} p & q \\ r & -p \end{pmatrix}$, where $p^2 + qr = 1$. This latter can be written $-qr = p^2 - 1$.

If we multiply the upper left-hand entry by 2, and the lower left-hand entry by $-q$ then the entries on the left-hand column become $2p$ and $-qr = p^2 - 1$. If p is an odd number, we get the Pythagorean triple $(2p,\ p^2 - 1,\ p^2 + 1)$ which is equivalent to the Pythagorean family mentioned above. If p is even, we get triples of the form $(4n,\ 4n^2 - 1,\ 4n^2 + 1)$. Triples of this form belong to what is known as the Platonic family of Pythagorean triples [1].

Example

In (2), let $q = -3$. We get $\begin{pmatrix} p & -3 \\ \frac{1}{3}(p^2 - 1) & -p \end{pmatrix}$. We will choose p values such that $\frac{1}{3}(p^2 - 1)$ is an integer. Thus $p^2 = 1 \bmod 3$, and so either $p = 1 \bmod 3$ or $p = 2 \bmod 3$. We get the matrices

$$\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 4 & -3 \\ 5 & -4 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}, \begin{pmatrix} 7 & -3 \\ 16 & -7 \end{pmatrix}, \begin{pmatrix} 8 & -3 \\ 21 & -8 \end{pmatrix}, \ldots .$$

Multiplying the upper left-hand entries by 2 and the lower left-hand entries by 3 gives the Pythagorean triples

$$(4, 3, 5), (8, 15, 17), (10, 24, 26), (14, 48, 50), (16, 63, 65), \ldots .$$

An interesting question to ask is: using $q = -2$ can we construct a matrix which is a square root of I_2 and which is associated with *any* Pythagorean triple we name? The answer is yes. Suppose we want to build a matrix which is associated with the Pythagorean triple (a, b, c). That means the left-hand column of the matrix must consist of elements a and b, or multiples of a and b. Then the matrix is of the form

$$\begin{pmatrix} \frac{a}{k} & -2 \\ \frac{b}{k} & -\frac{a}{k} \end{pmatrix} = \begin{pmatrix} \frac{a}{k} & -\frac{2k}{k} \\ \frac{b}{k} & -\frac{a}{k} \end{pmatrix}, \tag{7}$$

where k is a positive constant to be determined. We compare (7) with (4), and (3) becomes $a^2 - 2kb = k^2$. We thus get the quadratic equation $k^2 + 2bk - a^2 = 0$ where k is the unknown. Using the quadratic formula, we have $k = \frac{1}{2}\left(-2b \pm \sqrt{4b^2 + 4a^2}\right) = \frac{1}{2}\left(-2b \pm 2\sqrt{b^2 + a^2}\right) = -b \pm \sqrt{c^2}$. Since k is positive, we get

$$k = c - b. \tag{8}$$

Example

Let us take the Pythagorean triple (20, 21, 29). By (8), we have $k = 29 - 21 = 8$ and so by (7) we get the matrix

$$\begin{pmatrix} \frac{20}{8} & -2 \\ \frac{21}{8} & -\frac{20}{8} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -2 \\ \frac{21}{8} & -\frac{5}{2} \end{pmatrix}.$$

There is another way to connect Pythagorean triples with square roots of I_2. This method was presented by Douglas W. Mitchell in [2]. For the triple (a, b, c) we have $a^2 + b^2 = c^2$ and this is of the form (3). Then (4) becomes the matrix $\begin{pmatrix} \frac{a}{c} & \frac{b}{c} \\ \frac{b}{c} & -\frac{a}{c} \end{pmatrix}$, and this is a square root of I_2, as desired. We can also exchange a and b to get the triple (b, a, c) and use that to construct the matrix $\begin{pmatrix} \frac{b}{c} & \frac{a}{c} \\ \frac{a}{c} & -\frac{b}{c} \end{pmatrix}$. Using (6), we can use these matrices to construct $\begin{pmatrix} \frac{c}{a} & -\frac{b}{a} \\ \frac{b}{c} & -\frac{c}{a} \end{pmatrix}$ and $\begin{pmatrix} \frac{c}{b} & -\frac{a}{b} \\ \frac{a}{b} & -\frac{c}{b} \end{pmatrix}$.

Example

For the Pythagorean triples (3, 4, 5) and (4, 3, 5) we obtain the matrices $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$, $\begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$, $\begin{pmatrix} \frac{5}{3} & -\frac{4}{3} \\ \frac{4}{3} & -\frac{5}{3} \end{pmatrix}$ and $\begin{pmatrix} \frac{5}{4} & -\frac{3}{4} \\ \frac{3}{4} & -\frac{5}{4} \end{pmatrix}$.

4. *Consecutive integers*

The identity $n^2 + \left[-(n-1)\right](n+1) = 1$ is of the form (3). Then from (4) we get that the matrix $\begin{pmatrix} n & -(n-1) \\ n+1 & -n \end{pmatrix}$ is a square root of I_2. And so we get matrices of the form $\begin{pmatrix} p & q \\ r & -p \end{pmatrix}$ where $-q$, p and r are consecutive integers.

The following matrices are all square roots of I_2:

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} 4 & -3 \\ 5 & -4 \end{pmatrix}, \begin{pmatrix} 5 & -4 \\ 6 & -5 \end{pmatrix}, \ldots .$$

5. *120-degree triples and Eisenstein triples*

If a triangle contains an interior angle of 120 degrees, and the lengths of its sides are positive integers (a, b, c) where c lies opposite the 120-degree angle, then $a^2 + b^2 + ab = c^2$ and (a, b, c) is called a 120-degree triple, [3]. Some 120-degree triples are $(3, 5, 7)$, $(7, 8, 13)$ and $(5, 16, 19)$.

Similarly, if a triangle contains an interior angle of 60 degrees, and the lengths of its sides are positive integers (a, b, c) where c lies opposite the 60-degree angle, then $a^2 + b^2 - ab = c^2$ and (a, b, c) is called an Eisenstein triple, also called a 60-degree triple, [3]. Furthermore, it is easy to show that if (a, b, c) is a 120-degree triple, and $d = a + b$ then (a, d, c) and (b, d, c) are Eisenstein triples.

Given the 120-degree triple (a, b, c) and/or the Eisenstein triples (a, d, c) and (b, d, c), where $d = a + b$, it is easy to show that $a^2 + bd = c^2$, $b^2 + ad = c^2$ and $c^2 + ab = d^2$. Then from (3) and (4), the following matrices are square roots of I^2:

$$\begin{pmatrix} \frac{a}{c} & \frac{b}{c} \\ \frac{d}{c} & -\frac{a}{c} \end{pmatrix}, \begin{pmatrix} \frac{b}{c} & \frac{a}{c} \\ \frac{d}{c} & -\frac{b}{c} \end{pmatrix}, \begin{pmatrix} \frac{c}{d} & \frac{a}{d} \\ \frac{b}{d} & -\frac{c}{d} \end{pmatrix}.$$

For instance, for the 120-degree triple $(3, 5, 7)$ we have Eisenstein triples $(3, 8, 7)$ and $(5, 8, 7)$. These allow us to construct the following square roots of I_2:

$$\begin{pmatrix} \frac{3}{7} & \frac{5}{7} \\ \frac{8}{7} & -\frac{3}{7} \end{pmatrix}, \begin{pmatrix} \frac{5}{7} & \frac{3}{7} \\ \frac{8}{7} & -\frac{5}{7} \end{pmatrix}, \begin{pmatrix} \frac{7}{8} & \frac{3}{8} \\ \frac{5}{8} & -\frac{7}{8} \end{pmatrix}.$$

6. *Double-angle triples*

If, in triangle ABC, one interior angle $\angle C$ is twice the measure of another interior angle $\angle A$ and the lengths of the sides are positive integers (a, b, c) then $a^2 + ab = c^2$, [4, 5]. Some of these *double-angle triples* are

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(4, 5, 6), (9, 7, 12), (9, 16, 15), (16, 9, 20). The equation $a^2 + ab = c^2$ is of the form (3), and so from (4) we can set up a matrix which is a square root of I_2 : $\begin{pmatrix} \frac{a}{c} & \frac{a}{c} \\ \frac{b}{c} & -\frac{a}{c} \end{pmatrix}$. So, for the double-angle triples listed, we have

$$\begin{pmatrix} \frac{4}{6} & \frac{4}{6} \\ \frac{5}{6} & -\frac{4}{6} \end{pmatrix}, \begin{pmatrix} \frac{9}{12} & \frac{9}{12} \\ \frac{7}{12} & -\frac{9}{12} \end{pmatrix}, \begin{pmatrix} \frac{9}{15} & \frac{9}{15} \\ \frac{16}{15} & -\frac{9}{15} \end{pmatrix}, \begin{pmatrix} \frac{16}{20} & \frac{16}{20} \\ \frac{9}{20} & -\frac{16}{20} \end{pmatrix}.$$

7. *Lord triples*

Nick Lord [6] showed that given a triangle ABC with obtuse angle $\angle C$ and with integer sides (a, b, c) if a perpendicular is dropped from B to $\overline{AC}$ extended to P, the foot of the perpendicular, then $\angle CAB = 2\angle CBP$ if, and only if, $a^2 + bc = c^2$ [7]. In honour of Lord, we can call triples (a, b, c) satisfying this condition *Lord triples*. Some examples are (2, 3, 4), (3, 8, 9), (6, 5, 9) and (4, 15, 16),

The equation $a^2 + cb = c^2$ is of the form (3), and so we can use (4) to construct a matrix which is a square root of I_2 : $\begin{pmatrix} \frac{a}{c} & 1 \\ \frac{b}{c} & -\frac{a}{c} \end{pmatrix}$.

For the triples listed, we get

$$\begin{pmatrix} \frac{2}{4} & 1 \\ \frac{3}{4} & -\frac{2}{4} \end{pmatrix}, \begin{pmatrix} \frac{3}{9} & 1 \\ \frac{8}{9} & -\frac{3}{9} \end{pmatrix}, \begin{pmatrix} \frac{6}{9} & 1 \\ \frac{5}{9} & -\frac{6}{9} \end{pmatrix}, \begin{pmatrix} \frac{4}{16} & 1 \\ \frac{15}{16} & -\frac{4}{16} \end{pmatrix}.$$

8. *Sequences defined recursively*

We will now consider some matrices involving recursively-defined sequences. Suppose we have a sequence x_n such that x_0 and x_1 are constants, and for all $n \geqslant 1$ we have

$$x_{n+1} = \alpha x_n + \beta x_{n-1} \tag{9}$$

where α and β are non-zero constants. Then it follows that

$$x_n^2 - x_{n+1}x_{n-1} = (-\beta)^{n-1}\left(x_1^2 - x_2 x_0\right). \tag{10}$$

This can be proved by mathematical induction. First, it is obviously true for $n = 1$. Next, using (9) and some algebra, we obtain

$$x_n^2 - x_{n+1}x_{n-1} = -\frac{1}{\beta}\left(x_{n+1}^2 - x_n x_{n+2}\right),$$

and (10) follows immediately.

Equation (10) is a generalisation of Cassini's Identity for Fibonacci numbers $F_m^2 - F_{m+1}F_{m-1} = (-1)^{m-1}$ [8].

If we replace n with $2n + 1$, (10) becomes

$$x_{2n+1}^2 - x_{2n+2}x_{2n} = (-\beta)^{2n}\left(x_1^2 - x_2 x_0\right).$$

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For the case in which $x_0 = 0$ and $x_1 = 1$ this becomes

$$x_{2n+1}^2 - x_{2n+2}x_{2n} = \beta^{2n}. \tag{11}$$

It follows from (3) and (4) that

$$\begin{pmatrix} \frac{x_{2n+1}}{\beta^n} & -\frac{x_{2n}}{\beta^n} \\ \frac{x_{2n+2}}{\beta^n} & -\frac{x_{2n+1}}{\beta^n} \end{pmatrix} \tag{12}$$

is a square root of I_2.

For the case in which $x_0 = 0$, $x_1 = 1$ and $\beta = \pm 1$, (11) becomes $x_{2n+1}^2 - x_{2n+2}x_{2n} = 1$, and so

$$\begin{pmatrix} x_{2n+1} & -x_{2n} \\ x_{2n+2} & -x_{2n-1} \end{pmatrix} \tag{13}$$

is a square root of I_2.

9. *Fibonacci numbers*

By trial and error, it is easy to find the matrices $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $\begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 \\ 8 & -3 \end{pmatrix}$. We recognise the absolute values of the entries as Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, … .

The Fibonacci sequence is defined recursively by $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. By (9), we have $\beta = 1$. We can therefore use (13), and so $\begin{pmatrix} F_{2n+1} & -F_{2n} \\ F_{2n+2} & -F_{2n+1} \end{pmatrix}$ is always a square root of I_2. $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and $\begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}$ are of this form.

The other matrix we had was $\begin{pmatrix} 3 & -1 \\ 8 & -3 \end{pmatrix}$ which is of the form $\begin{pmatrix} F_{2n+2} & -F_{2n} \\ F_{2n+4} & -F_{2n+2} \end{pmatrix}$, where n is a positive integer. To show that this is always a square root of I_2, we need to show that $F_{2n+2}^2 - F_{2n+4}F_{2n} = 1$. Since $F_{2n+2} = F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1} = 3F_{2n} - F_{2n-2}$, and writing $x_n = F_{2n}$, we have $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = 3x_n - x_{n-1}$. Then from (10) it follows that $x_{n+1}^2 - x_{n+2}x_n = 1$, as desired.

10. *Golden triples*

Golden triples have been defined [9] as triples of integers (a, b, c) satisfying

$$a^2 - b^2 + ab = c^2. \tag{14}$$

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If c is fixed at one of some special values, then the a and b values are terms of at least one Fibonacci-like sequence, i.e. a sequence satisfying $G_n = G_{n-1} + G_{n-2}$. For instance, if we let $c = 11$ then some golden triples are (10, 3, 11), (13, 16, 11), (29, 45, 11), (10, 7, 11), (17, 24, 11) and (41, 65, 11). We can use the a and b values to construct two Fibonacci-like sequences:

$$10,\ 3,\ 13,\ 16,\ 29,\ 45,\ \dots \qquad \text{and} \qquad 10,\ 7,\ 17,\ 24,\ 41,\ 65,\ \dots\ .$$

For each of the two sequences, we define the term 10 to be the first term G_1. More generally, G_1 is chosen so that $G_2 \leqslant G_1$ but from that point on the sequence is increasing. Then golden triples are of the form

$$\left(G_{2n-1},\ G_{2n},\ c\right). \tag{15}$$

From (14) and (15), we have $G_{2n-1}^2 - G_{2n}^2 + G_{2n-1}G_{2n} = c^2$. This can be written as $G_{2n-1}^2 - G_{2n}\left(G_{2n} - G_{2n-1}\right) = c^2$ or $G_{2n-1}^2 - G_{2n}G_{2n-2} = c^2$. From this, using (3) and (4) we can construct a matrix which is a square root of I_2: $\begin{pmatrix} \frac{G_{2n-1}}{c} & -\frac{G_{2n-2}}{c} \\ \frac{G_{2n}}{c} & -\frac{G_{2n-1}}{c} \end{pmatrix}$, or, in terms of golden triples, $\begin{pmatrix} \frac{a}{c} & \frac{a-b}{c} \\ \frac{b}{c} & -\frac{a}{c} \end{pmatrix}$.

For the case of $c = 11$ we have

$$\begin{pmatrix} \frac{10}{11} & \frac{7}{11} \\ \frac{3}{11} & -\frac{10}{11} \end{pmatrix}, \begin{pmatrix} \frac{13}{11} & -\frac{3}{11} \\ \frac{16}{11} & -\frac{13}{11} \end{pmatrix}, \begin{pmatrix} \frac{29}{11} & -\frac{16}{11} \\ \frac{45}{11} & -\frac{29}{11} \end{pmatrix}, \dots \text{ and} \begin{pmatrix} \frac{10}{11} & \frac{3}{11} \\ \frac{7}{11} & -\frac{10}{11} \end{pmatrix}, \begin{pmatrix} \frac{17}{11} & -\frac{7}{11} \\ \frac{24}{11} & -\frac{17}{11} \end{pmatrix}, \begin{pmatrix} \frac{41}{11} & -\frac{24}{11} \\ \frac{65}{11} & -\frac{41}{11} \end{pmatrix}, \dots .$$

11. *Pell numbers*

Pell numbers, like Fibonacci numbers, are defined recursively. The definition is

$$P_n = 2P_{n-1} + P_{n-2}, \tag{16}$$

where $P_0 = 0$ and $P_1 = 1$. The first few Pell numbers are 0, 1, 2, 5, 12, 29, 70, . . .

We have the conditions necessary to use (13), and so the matrix $\begin{pmatrix} P_{2n+1} & -P_{2n} \\ P_{2n+2} & -P_{2n+1} \end{pmatrix}$ is a square root of I_2. Some examples are $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, $\begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}$ and $\begin{pmatrix} 29 & -12 \\ 70 & -29 \end{pmatrix}$.

We can ask what happens if we change the coefficient in (16) from 2 to 3: $A_n = 3A_{n-1} + A_{n-2}$. Then we get the sequence 0, 1, 3, 10, 33, 109, 360, ….

We can then construct these matrices, and they are square roots of I_2:

$$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 10 & -3 \\ 33 & -10 \end{pmatrix}, \begin{pmatrix} 109 & -33 \\ 360 & -109 \end{pmatrix}.$$

12. *Jacobsthal numbers*

Jacobsthal numbers, like Fibonacci numbers and Pell numbers, are defined recursively. The definition is $J_n = J_{n-1} + 2J_{n-2}$, where $J_0 = 0$ and $J_1 = 1$. The first few Jacobsthal numbers are 0, 1, 1, 3, 5, 11, 21, 43, 85, …..

Note that for Jacobsthal numbers, the coefficient β in (9) is 2, not 1, so we cannot use (13). We can, however, use (12), and so $\begin{pmatrix} \frac{1}{2^n}J_{2n+1} & -\frac{1}{2^n}J_{2n} \\ \frac{1}{2^n}J_{2n+2} & -\frac{1}{2^n}J_{2n+1} \end{pmatrix}$ is a square root of I_2.

Some examples are:

$$\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} \frac{11}{4} & -\frac{5}{4} \\ \frac{21}{4} & -\frac{11}{4} \end{pmatrix}, \begin{pmatrix} \frac{43}{8} & -\frac{21}{5} \\ \frac{85}{8} & -\frac{43}{8} \end{pmatrix}.$$

13. *Diophantine triples*

A Diophantine triple is a set of three positive integers such that the product of any two of them is 1 less than a perfect square [10]. A Diophantine quadruple is a set of four positive integers having the same property. An example of a Diophantine quadruple is $\{1, 3, 8, 120\}$. Of course, from that we can extract four distinct Diophantine triples. Some other Diophantine triples are

$$\{1, 8, 15\}, \{2, 4, 12\}, \{2, 12, 24\}, \{3, 8, 21\}, \{3, 5, 16\}, \{5, 16, 39\}.$$

It is easy to see that we can use Diophantine triples and quadruples to construct square roots of I_2. Specifically, if a and b are two elements of a Diophantine triple or quadruple, then $\sqrt{ab + 1}$ is an integer and, using $(\sqrt{ab + 1})^2 + (-a)\cdot b = 1$ along with (3) and (4), $\begin{pmatrix} \sqrt{ab+1} & -a \\ b & -\sqrt{ab+1} \end{pmatrix}$ is a square root of I_2. For instance, from $\{1, 3, 8, 120\}$ we can construct $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $\begin{pmatrix} 3 & -1 \\ 8 & -3 \end{pmatrix}$, $\begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}$, $\begin{pmatrix} 11 & -1 \\ 120 & -11 \end{pmatrix}$, $\begin{pmatrix} 19 & -3 \\ 120 & -19 \end{pmatrix}$ and $\begin{pmatrix} 31 & -8 \\ 120 & -31 \end{pmatrix}$. From $\{3, 8, 21\}$ we can construct $\begin{pmatrix} 8 & -3 \\ 21 & -8 \end{pmatrix}$ and $\begin{pmatrix} 13 & -8 \\ 21 & -13 \end{pmatrix}$, as well as $\begin{pmatrix} 5 & -3 \\ 8 & -5 \end{pmatrix}$, which we already had from $\{1, 3, 8, 120\}$.

Note that several of these matrices are of the form $\begin{pmatrix} F_{2n+1} & -F_{2n} \\ F_{2n+2} & -F_{2n+1} \end{pmatrix}$ or $\begin{pmatrix} F_{2n+2} & -F_{2n} \\ F_{2n+4} & -F_{2n+2} \end{pmatrix}$, which we had above in the section on Fibonacci numbers. This indicates a close connection between Fibonacci numbers and at least some Diophantine triples.

Some of the matrix families we have discussed overlap. For instance, the matrix $\begin{pmatrix} 5 & -2 \\ 12 & -5 \end{pmatrix}$ appeared above in the section on Pythagorean triples, it appeared in the section on Pell numbers, and we can also construct it from the 2 and the 12 in the Diophantine triples $\{2,\ 4,\ 12\}$ and $\{2,\ 12,\ 24\}$.

Further research

It would be interesting to consider square roots of I_2 that contain irrational and complex number elements, as well as square roots of higher-rank identity matrices such as I_3.

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