Beltrami Equation with Coefficient in Sobolev and Besov Spaces

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Abstract. Our goal in this work is to present some function spaces on the complex plane $\mathbb{C}$, $X(\mathbb{C})$, for which the quasiregular solutions of the Beltrami equation, $\overline{\partial} f(z) = \mu(z)\partial f(z)$, have first derivatives locally in $X(\mathbb{C})$, provided that the Beltrami coefficient μ belongs to $X(\mathbb{C})$.

1 Introduction

A function $f\colon \mathbb{C} \to \mathbb{C}$ is called μ-quasiregular if it belongs to the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ (functions with distributional first order derivatives locally in L^2) and satisfies the Beltrami equation

$$(1.1) \qquad \overline{\partial} f(z) = \mu(z)\partial f(z), \quad \text{a.e. } z \in \mathbb{C},$$

where μ, called the Beltrami coefficient of f, is a Lebesgue measurable function on the complex plane $\mathbb{C}$ satisfying $\|\mu\|_\infty < 1$.

If, in addition, f is a homeomorphism, then we say that f is μ-quasiconformal. Quasiconformal and quasiregular mappings are a central tool in modern geometric function theory and have had a strong impact in other areas.

It is well known that quasiregular functions are locally in some Hölder class (Mori's Theorem), and moreover they actually belong to $W^{1,p}_{\mathrm{loc}}$ for some $p > 2$. In this paper we are interested in studying how the regularity of the Beltrami coefficient affects the regularity of the solutions of (1.1). Thus, if the Beltrami coefficient μ belongs to the Hölder class $C^{l,s}$, $0 < s < 1$, using Schauder estimates (see for instance [AIM, Chap. 15]), then μ-quasiregular functions belong to $C^{l+1,s}_{\mathrm{loc}}$. For the borderline cases $s = 0$ and $s = 1$, the $C^{l+1,s}$ regularity fails (*e.g.*, [AIM, p. 390]). If $\mu \in W^{1,p}$, $2 < p < \infty$, then one can read in Ahlfors' book [Ah, p. 56] the result that quasiregular functions are locally in $W^{2,p}$. The cases $\mu \in W^{1,p}$, $p \leq 2$, were studied in [CFMOZ]; for instance, when $p = 2$ one gets that the solutions are locally in $W^{2,q}$ for every $q < 2$. In [CFR, Theorem 4.3] some results on the case of homogeneity $sp < 2$, $W^{s,p}$, are obtained and applied to study the Calderón's inverse problem.

Recently, in connection with the planar conductivity equation, the regularity of solutions of the conjugate Beltrami equation, $\overline{\partial} f = \nu \overline{\partial f}$, has been studied when the dilation coefficient ν is real and in $W^{1,p}$, $p > 2$ (see [BFL, Corollary 1]).

Our goal in this paper is to present some function spaces X for which all quasiregular solutions of (1.1) have first derivatives locally in X, provided that the Beltrami coefficient belongs to X. These function spaces will enjoy the additional property of being an algebra (that is, the product of two functions in X is again in X), and this feature will play an important role in our arguments. We deal with Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{C})$ and Besov spaces $B^s_{p,q}(\mathbb{C})$ with $s > 0$, $1 < p < \infty$, $1 < q < \infty$, and $sp > 2$. Let $A^s_{p,q}(\mathbb{C})$ denote any of the function spaces with the indices as we have determined. In any case, the condition $sp > 2$ ensures that we have bounded continuous functions and multiplication algebras (*e.g.*, [RS, 4.6.4]). In Section 2 we will give the precise definitions of these function spaces involved in the statement of the our first theorem.

Theorem 1.1   *Suppose that $\mu \in A^s_{p,q}(\mathbb{C})$ is compactly supported with $\|\mu\|_\infty = k < 1$. Then any $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ satisfying the Beltrami equation (1.1) has first derivatives locally in $A^s_{p,q}(\mathbb{C})$.*

When the Beltrami coefficient is compactly supported there is a unique $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ solution of (1.1) normalized by the condition $z + O(1/z)$ near infinity. Moreover, it is a homeomorphism of the complex plane. It is called the principal solution of (1.1). By Stoilow's Factorization Theorem (*e.g.*, [AIM, section 5.5]), for any quasiregular function f there exists a holomorphic function h such that $f = h \circ \phi$, where ϕ is the associated principal solution. Therefore, we will only concentrate on principal solutions. As is well known ([AIM, p. 165]), ϕ is given explicitly by the formula

$$\phi(z) = z + \mathrm{C}(h)(z),$$

where the operator

$$(1.2) \qquad \mathrm{C}\,h(z) = \frac{1}{\pi} \int_{\mathbb{C}} h(z - w) \frac{1}{w}\,\mathrm{d}w$$

is the Cauchy transform of h. When $h \in L^p$, $1 < p < \infty$, one has the identity $\bar{\partial}\mathrm{C}(h) = h$. Consequently, our theorem immediately follows from next proposition.

Proposition 1.2   *Suppose that μ is compactly supported with $\|\mu\|_\infty = k < 1$ and that $\phi(z) = z + \mathrm{C}(h)(z)$ is the principal solution of the Beltrami equation (1.1). Let $s > 0$, $1 < p < \infty$, $1 < q < \infty$, and $sp > 2$. If $\mu \in A^s_{p,q}(\mathbb{C})$, then $h \in A^s_{p,q}(\mathbb{C})$.*

Sketch of the proof   The Beurling transform is the principal value convolution operator

$$Bf(z) = -\frac{1}{\pi}\,\mathrm{p.v.} \int_{\mathbb{C}} f(z - w) \frac{1}{w^2}\,\mathrm{d}w.$$

The Fourier multiplier of B is $\frac{\bar{\xi}}{\xi}$, or, in other words,

$$\widehat{Bf}(\xi) = \frac{\bar{\xi}}{\xi}\,\widehat{f}(\xi).$$

Thus B is an isometry on $L^2(\mathbb{C})$, and it is well known that B, as any Calderón–Zygmund convolution operator, is bounded on $A^s_{p,q}(\mathbb{C})$.

Recall the relation between the Cauchy and the Beurling transforms: $\partial C = B$. Thus, $\partial \phi = 1 + B(h)$ and $\overline{\partial} \phi = h$, and consequently the function h is determined by the equation

$$(I - \mu B)(h) = \mu.$$

So, we only need to invert the Beltrami operator $I - \mu B$ on the corresponding function space. This task is completed in Section 3. ∎

For the critical case $sp = 2$, we consider a Riesz potential space $I_1(L^{2,1}(\mathbb{C}))$, the set of functions with first order derivatives in the Lorentz space $L^{2,1}(\mathbb{C})$. Even though close to L^2, the Lorentz space $L^{2,1}(\mathbb{C})$ is strictly contained in L^2. This small improvement on the derivatives allows us to have continuous functions vanishing at infinity (by the way, recall that functions with first order derivatives in L^2 may not be continuous).

Proposition 1.3 *Suppose that $\mu \in I_1(L^{2,1}(\mathbb{C}))$ is compactly supported with $\|\mu\|_\infty = k < 1$, and $\phi(z) = z + C(h)(z)$ is the principal solution of the Beltrami equation* (1.1). *Then $h \in I_1(L^{2,1}(\mathbb{C}))$.*

As we mentioned previously, Proposition 1.3 does not hold when the Beltrami coefficient only has first derivatives in L^2. However, the analogous result remains valid if we replace $I_1(L^{2,1}(\mathbb{C}))$ by $I_s(L^{\frac{2}{s},1}(\mathbb{C}))$, $0 < s < 2$.

The main result of [MOV] identifies a class of non-smooth Beltrami coefficients that determine bilipschitz quasiconformal mappings. In particular, we proved the following result.

Theorem 1.4 ([MOV]) *Let Ω be a bounded domain of $\mathbb{C}$ with boundary of class $\mathcal{C}^{1,\varepsilon}$, $0 < \varepsilon < 1$, and let $\mu \in \mathcal{C}^{0,\varepsilon}(\Omega)$ with $\|\mu\|_\infty < 1$. Let $\phi(z) = z + C(h)(z)$ be the principal solution of the Beltrami equation* (1.1). *Then $h \in \mathcal{C}^{0,\varepsilon'}(\Omega)$ for any $\varepsilon' < \varepsilon$ and moreover ϕ is bilipschitz.*

Now, we replace the Hölder smoothness of the Beltrami coefficient by a Sobolev (or Besov) condition restricted on a domain. (See definitions in the next section).

Theorem 1.5 *Let $0 < s < \varepsilon < 1$ and $1 < p < \infty$ such that $sp > 2$ and let Ω be a bounded domain of $\mathbb{C}$ with boundary of class $\mathcal{C}^{1,\varepsilon}$. Suppose that μ is supported in $\overline{\Omega}$ with $\|\mu\|_\infty = k < 1$ and that $\phi(z) = z + C(h)(z)$ is the principal solution of the Beltrami equation* (1.1).

(i) *If $\mu \in W^{s,p}(\Omega)$, then $h \in W^{s,p}(\Omega)$.*
(ii) *If $\mu \in B^s_{p,p}(\Omega)$, then $h \in B^s_{p,p}(\Omega)$.*

The proof runs in parallel to that of the above propositions, but now a new obstacle appears: the boundedness of the Beurling transform on $W^{s,p}(\Omega)$ (or $B^s_{p,p}(\Omega)$). In general, it is not clear if Calderón–Zygmund convolution operators are bounded on $W^{s,p}(\Omega)$ (or $B^s_{p,p}(\Omega)$). Of course, the answer depends on the operator and on the boundary of the domain. We will study this question in domains Ω of $\mathbb{R}^n$, $n \geq 2$.

In $\mathbb{R}^n$ we consider the kernel $K(x) = \omega(x)/|x|^n$, $x \neq 0$, where ω is a homogeneous function of degree 0, with zero integral on the unit sphere and $\omega \in \mathcal{C}^1(S^{n-1})$. Then the singular integral

$$Tf(x) = \text{p.v.} \int f(y)K(x - y) \, dy$$

is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. (Really the condition $\omega \in \mathcal{C}^1(S^{n-1})$ could be weakened, but it is enough for our purpose). On the other hand, Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ ($= F^s_{p,2}(\mathbb{R}^n)$) are described as spaces of Bessel potentials, that is, $f \in W^{s,p}$ if and only if $f = G_s * g$, where G_s denotes the Bessel kernel of order s and $g \in L^p$ (e.g., [St, Chapt. 5]). Remember that the Bessel kernel of order s, G_s, is the L^1 function with Fourier transform $(1 + |\xi|^2)^{-\frac{s}{2}}$. Then, because T is a convolution operator, one has the identity

$$T(f) = T(G_s * g) = G_s * (Tg),$$

and one gets the boundedness of T on $W^{s,p}$, $1 < p < \infty$. But if one takes $f \in W^{s,p}(\Omega)$, Ω a domain of $\mathbb{R}^n$, then

$$T_\Omega f(x) := \text{p.v.} \int_\Omega f(y)K(x - y) \, dy$$

clearly belongs to $L^p(\Omega)$. However, perhaps $T_\Omega f \notin W^{s,p}(\Omega)$. For instance, let Q denote a rectangle in $\mathbb{C}$ and χ_Q denote its characteristic function. A computation shows that the Beurling transform of χ_Q, $B\chi_Q$, has logarithmic singularities at the vertices of the rectangle, and, therefore, its first derivatives belong to $L^p(Q)$ only if $p < 2$ (e.g., [AIM, p. 147]). For positive results, we restrict our attention to operators with even kernel, that is, $K(-x) = K(x)$. In Section 4 we will deal with Theorem 1.6.

Theorem 1.6 *Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{1,\beta}$, $\beta > 0$, and let T be an even smooth homogeneous Calderón–Zygmund operator.*

(i)   *If $T\chi_\Omega \in B^s_{p,p}(\Omega)$, $0 < s < 1$, $n < sp < \infty$, then $T_\Omega : B^s_{p,p}(\Omega) \to B^s_{p,p}(\Omega)$.*
(ii) *If $T\chi_\Omega \in W^{s,p}(\Omega)$, $0 < s < 1$, $n < sp < \infty$, then $T_\Omega : W^{s,p}(\Omega) \to W^{s,p}(\Omega)$.*
(iii) *If $T\chi_\Omega \in W^{1,p}(\Omega)$, $n < p < \infty$, then $T_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$.*

In any case the norm operator depends on the domain Ω and the Calderón–Zygmund constant of the kernel of T (see (2.5) for the definition).

The result reduces the study of the boundedness of the operator T_Ω to the behaviour of T_Ω on the function χ_Ω. Thus, we have a necessary and sufficient condition of type $T(1)$. In the proof of Theorem 1.6, we follow the method of Y. Meyer in [Me], where he studied the continuity of generalised Calderón–Zygmund operators on Sobolev spaces $W^{s,p}(\mathbb{R}^n)$.

Since T is bounded on L^p, using complex and real interpolation, one could think that items (i) and (ii) of the above theorem are a consequence of the third one. But this it not the case, because the conditions on items (i) and (ii) are weaker than $T\chi_\Omega \in W^{1,p}(\Omega)$.

When Ω is a bounded domain of $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{1,\varepsilon}$, $0 < s < \varepsilon < 1$, and $n < sp < \infty$ then T_Ω is bounded on $W^{s,p}(\Omega)$ and $B^s_{p,p}(\Omega)$ (see details in Section 4). In particular, the assumptions on the domain Ω in the statement of Theorem

1.5 are to ensure that the Beurling transform is bounded on the corresponding function space. Recently, V. Cruz and X. Tolsa [CT, To] have shown that if the outward unit normal N on $\partial\Omega$ belongs to the Besov space $B_{p,p}^{s-1/p}(\partial\Omega)$, then $B\chi_\Omega \in W^{s,p}(\Omega)$.

In Section 2 we shall introduce some basic notation and set up some necessary preliminaries. The proof of Propositions 1.2 and 1.3 are in Section 3. In Section 4 we study even smooth homogeneous Calderón–Zygmund operators on domains. The proof of the Theorem 1.5 is explained in Section 5.

As usual, the letter C will denote a constant, which may be different at each occurrence and which is independent of the relevant variables under consideration.

2 Preliminaries

We start by reviewing some basic facts concerning Triebel–Lizorkin spaces and Besov spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the usual Schwartz class of rapidly decreasing $\mathcal{C}^\infty$-functions and $\widehat{g}$ stand for the Fourier transform of g. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\psi}(\xi) = 1$ if $|\xi| \leq 1$ and $\widehat{\psi}(\xi) = 0$ if $|\xi| \geq 3/2$. We set $\psi_0 = \psi$ and $\widehat{\psi}_j(\xi) = \widehat{\psi}(2^{-j}\xi) - \widehat{\psi}(2^{-j+1}\xi)$, $j \in \mathbb{N}$. Since $\sum_{j=0}^{\infty} \widehat{\psi}_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n$, the $\widehat{\psi}_j$ form a dyadic resolution of unity. Then, for $f \in L_{\text{loc}}^1(\mathbb{R}^n)$, $1 \leq p, q < \infty$, and $s > 0$, one defines the norms

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty} \|2^{js}\psi_j * f\|_p^q \right)^{\frac{1}{q}} \quad \text{and} \quad \|f\|_{F_{p,q}^s} = \left\| \left(\sum_{j=0}^{\infty} |2^{js}\psi_j * f|^q \right)^{\frac{1}{q}} \right\|_p$$

The Besov space $B_{p,q}^s(\mathbb{R}^n)$ consists of the functions such that $\|f\|_{B_{p,q}^s} < \infty$, while the functions in the Triebel–Lizorkin space $F_{p,q}^s(\mathbb{R}^n)$ are those such that $\|f\|_{F_{p,q}^s} < \infty$.

The spaces $F_{p,2}^s$, $1 < p < \infty$, are known as Sobolev spaces of fractional order or Bessel-potential spaces and we prefer denote them by $W^{s,p}$. Since $p \geq 1$ and $q \geq 1$, both $B_{p,q}^s$ and $F_{p,q}^s$ are Banach spaces. A systematic treatment of these spaces may be found in [Tri1], [RS], and [Gr, Chapt. 6]. A remarkable fact when $sp > n$ is that $B_{p,q}^s$ and $F_{p,q}^s$ form an algebra with respect to pointwise multiplication, that is,

$$(2.1) \qquad \|f \cdot g\|_{A_{p,q}^s} \leq C\|f\|_{A_{p,q}^s}\|g\|_{A_{p,q}^s},$$

where $A_{p,q}^s$ denotes the corresponding Besov space or Triebel–Lizorkin space (*e.g.*, [RS, 4.6.4]). Moreover, functions in these spaces satisfy some Hölder condition and so they are continuous functions with

$$\|f\|_\infty \leq C\|f\|_{A_{p,q}^s}.$$

We say that a bounded domain $\Omega \subset \mathbb{R}^n$ has a boundary of class $\mathcal{C}^{1,\varepsilon}$ if $\partial\Omega$ is a C^1 hyper-surface whose unit normal vector satisfies a Lipschitz (Hölder) condition of order ε as a function on the surface. To state an alternative condition, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. Then Ω has a boundary of class $\mathcal{C}^{1,\varepsilon}$ if for each point $a \in \partial\Omega$ one may find a ball $B(a, r)$ and a function $x_n = \varphi(x')$, of class $\mathcal{C}^{1,\varepsilon}$, such that, after a rotation if necessary, $\Omega \cap B(a, r)$ is the part of $B(a, r)$ lying below the graph of φ. Thus we get

$$(2.2) \qquad \Omega \cap B(a, r) = \left\{ x \in B(a, r) : x_n < \varphi(x_1, \ldots, x_{n-1}) \right\}.$$

We say that Ω is a bounded Lipschitz domain if the function φ in (2.2) is of class $\mathcal{C}^{0,1}$.

In general, if one has a function space X defined on $\mathbb{R}^n$ and a domain $\Omega \subset \mathbb{R}^n$, one defines the space $X(\Omega)$ as the restrictions of functions of X from $\mathbb{R}^n$ to Ω. In addition, the restriction space is endowed with the quasi-norm quotient. In the cases that we are considering we have an intrinsic characterization of elements of $X(\Omega)$. We will use these characterizations in the proofs of Theorems 1.5 and 1.6. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $1 < p < \infty$ and $0 < s < 1$. Then

(a) (*e.g.*, [Tar, p. 169]) $f \in B_{p,p}^s(\Omega)$ if and only if $f \in L^p(\Omega)$ and

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \mathrm{d}x\mathrm{d}y < \infty.$$

(b) (*e.g.*, [Str, p. 1051]) $f \in W^{s,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and

$$(2.3) \qquad \int_\Omega \left(\int_\Omega \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}} \mathrm{d}x \right)^{\frac{p}{2}} \mathrm{d}y < \infty.$$

(c) (*e.g.*, [Br2, p. 703]) $f \in W^{1,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and

$$(2.4) \qquad \lim_{\alpha \to 0} \alpha \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+p-\alpha}} \mathrm{d}x\mathrm{d}y < \infty.$$

A smooth (of class $\mathcal{C}^1$) homogeneous Calderón–Zygmund operator is a principal value convolution operator of type

$$T(f)(x) = \text{p.v.} \int f(y) \, K(x - y) \, \mathrm{d}y,$$

where

$$K(x) = \frac{\omega(x)}{|x|^n}, \quad x \neq 0,$$

$\omega(x)$ being a homogeneous function of degree 0, continuously differentiable on $\mathbb{R}^n \setminus \{0\}$ and with zero integral on the unit sphere. Note that one trivially has

$$|K(x - y)| \leq \frac{C}{|x - y|^n}$$

and

$$|K(x - y) - K(x - y')| \leq C \frac{|y - y'|}{|x - y|^{n+1}} \quad \text{whenever } |x - y| \geq 2|y - y'|.$$

The Calderón–Zygmund constant of the kernel of T is defined as

$$(2.5) \qquad \|T\|_{CZ} = \||K(x)\,|x|^n\|_\infty + \|\nabla K(x)\,|x|^{n+1}\|_\infty.$$

The operator T is said to be even if the kernel is even, namely, if $\omega(-x) = \omega(x)$ for all $x \neq 0$. The even character of T gives the cancellation $T(\chi_B)\chi_B = 0$ for each ball B, which should be understood as a local version of the global cancellation property $T(1) = 0$ common to all smooth homogeneous Calderón–Zygmund operators. This extra cancellation property is essential for proving Lemma 4.2 and Theorem 1.6.

It is well known that Calderón–Zygmund convolution operators are bounded on $L^p(\mathbb{R}^n)$ and also on $W^{s,p}(\mathbb{R}^n)$ (because $W^{s,p} = G_s * L^p$). Using the method of real interpolation, one easily gets that these operators are also bounded on $B_{p,q}^s(\mathbb{R}^n)$ (see also [Gr, 6.7.2] for a direct proof). The boundedness of Calderón–Zygmund convolution operators on $F_{p,q}^s(\mathbb{R}^n)$ was proved in [FTW, Theorem 3.7] (see [JHL, Theorem 1.2] for a nice proof). Summarizing, if $s > 0$ and $1 < p, q < \infty$, we have

$$(2.6) \qquad \|Tf\|_{A_{p,q}^s} \leq C\|f\|_{A_{p,q}^s},$$

where C is a constant that depends on s, p, q, n and $\|T\|_{CZ}$.

Lorentz spaces are defined on measure spaces (Y, m), but we only need the case where $Y = \mathbb{C}$ and m is the Lebesgue planar measure. The classical definition of Lorentz spaces uses the rearrangement function. For any measurable function f we define its nonincreasing rearrangement by

$$f^*(t) := \inf\big\{\, s : m\{z \in \mathbb{C} : |f(z)| > s\} \leq t \,\big\}.$$

For $1 \leq p, q < \infty$, the Lorentz space $L^{p,q}(\mathbb{C})$ is the set of functions f such that $\|f\|_{L^{p,q}} < \infty$, with

$$\|f\|_{L^{p,q}(\mathbb{C})} := \begin{cases} \big(\int_0^\infty [t^{1/p} f^*(t)]^q t^{-1} dt\big)^{1/q}, & \text{for } 1 \leq q < \infty, \\ \sup_{t > 0} t^{1/p} f^*(t), & \text{for } q = \infty. \end{cases}$$

A second definition of Lorentz spaces, which is equivalent to the first one, is given by real interpolation between Lebesgue spaces:

$$(L^{p_0}, L^{p_1})_{\theta, q} = L^{p,q},$$

where $1 \leq p_0 < p < p_1 \leq \infty$, $1 \leq q \leq \infty$, $0 < \theta < 1$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Lorentz spaces inherited from Lebesgue spaces the stability property of the multiplication by bounded function, that is, if $f \in L^\infty$ and $g \in L^{p,q}$ then $fg \in L^{p,q}$ and we have

$$(2.7) \qquad \|fg\|_{L^{p,q}} \leq \|f\|_\infty \|g\|_{L^{p,q}}.$$

Let $1 \leq p, q < \infty$ and consider $0 < \alpha < 2$. The Lorentz potential space, $I_\alpha(L^{p,q}(\mathbb{C}))$, is the set of functions f such that $f = I_\alpha * g$, where $g \in L^{p,q}(\mathbb{C})$ and $I_\alpha(x) = c_\alpha |x|^{\alpha-2}$ is the Riesz potential of order α. The norm in this space is given by

$$\|f\|_{I_\alpha(L^{p,q}(\mathbb{C}))} = \|g\|_{L^{p,q}}.$$

Note that when $\alpha = 1$, one has $\|f\|_{I_1(L^{p,q}(\mathbb{C}))} \approx \|\nabla f\|_{L^{p,q}}$.

It is well known [St2] that functions f of $I_1(L^{2,1}(\mathbb{C}))$ are continuous, and there exists a constant C such that

$$\|f\|_\infty \le C\|f\|_{I_1(L^{2,1}(\mathbb{C}))}. \tag{2.8}$$

In general $I_\alpha(L^{\frac{2}{\alpha},1}(\mathbb{C}))$ are embedded in $\mathcal{C}_0$, the space of continuous functions vanishing at the infinity (see [Ba]). Again, a remarkable property of these spaces $I_\alpha(L^{\frac{2}{\alpha},1}(\mathbb{C}))$ is that they are multiplication algebras, that is,

$$\|fg\|_{I_\alpha(L^{\frac{2}{\alpha},1})} \le C\|f\|_{I_\alpha(L^{\frac{2}{\alpha},1})}\|g\|_{I_\alpha(L^{\frac{2}{\alpha},1})}. \tag{2.9}$$

Finally, note that Calderón–Zygmund convolution operators are bounded on $L^{p,q}(\mathbb{R}^n)$ and so also on Lorentz potential space, $I_\alpha(L^{p,q}(\mathbb{C}))$, with constant depending on (2.5).

3　Invertibility of the Beltrami Operator

As we mentioned in the introduction, in order to prove Proposition 1.2 (and then Theorem 1.1) and Proposition 1.3 we only have to consider the invertibility of the Beltrami operator $I - \mu B$ on $A^s_{p,q}(\mathbb{C})$ and on $I_1(L^{2,1})(\mathbb{C})$. Following the idea of Iwaniec [Iw, pp. 42–43] we define

$$P_m = I + \mu B + \cdots + (\mu B)^m,$$

so that we have

$$(I - \mu B)P_{n-1} = P_{n-1}(I - \mu B) = I - (\mu B)^n = I - \mu^n B^n + K,$$

where $K = \mu^n B^n - (\mu B)^n$ can be easily seen to be a finite sum of operators that contain as a factor the commutator $[\mu, B] = \mu B - B\mu$. In Lemma 3.2 (and in Lemma 3.3) we will prove that $[\mu, B]$ is compact on $A^s_{p,q}(\mathbb{C})$ (and on $I_1(L^{2,1})(\mathbb{C})$) , so that K is also compact. In Lemma 3.1 we will check that the operator norm of $\mu^n B^n$ on $A^s_{p,q}(\mathbb{C})$ (and on $I_1(L^{2,1})(\mathbb{C})$) is small if n is large. Therefore, $I - \mu B$ is a Fredholm operator on $A^s_{p,q}(\mathbb{C})$ (and on $I_1(L^{2,1})(\mathbb{C})$). Clearly $I - t\mu B$, $0 \le t \le 1$, is a continuous path from the identity to $I - \mu B$. By the index theory of Fredholm operators on Banach spaces (e.g., [Sch]), the index is a continuous function of the operator. Hence $I - \mu B$ has index 0. On the other hand, $I - \mu B$ is injective on $A^s_{p,q}(\mathbb{C})$ (and on $I_1(L^{2,1})(\mathbb{C})$), because by [Iw, p. 43] it is injective on $L^p(\mathbb{C})$ for all $1 < p < \infty$. This concludes the proof that $I - \mu B$ is invertible.

Lemma 3.1　(i)　*The operator norm of $\mu^n B^n$ on $A^s_{p,q}(\mathbb{C})$ is small if n is large.*
(ii)　*The operator norm of $\mu^n B^n$ on $I_1(L^{2,1}(\mathbb{C}))$ is small if n is large.*

Proof　Let

$$b_n = \frac{((-1)^n n)}{\pi}\frac{(\bar{z}^{n-1})}{z^{n+1}}$$

be the kernel of iterated Beurling transform B^n. Then, the Calderón-Zygmund constant of B^n is

$$\||b_n(z)|z|^2\|_\infty + \||\nabla b_n(z)|z|^3\|_\infty \le Cn^2.$$

(i) This is an easy consequence of well-known results. Since

$$\|g^m\|_{A^s_{p,q}} \le C\|g\|_\infty^{m-1}\|g\|_{A^s_{p,q}}$$

(see [RS, Theorem 5.3.2/4]), using (2.1) and (2.6), we have

$$\|\mu^n B^n(f)\|_{A^s_{p,q}} \le C\|\mu^n\|_{A^s_{p,q}}\|B^n(f)\|_{A^s_{p,q}}$$

$$\le C\|\mu^n\|_{A^s_{p,q}} n^2 \|f\|_{A^s_{p,q}}$$

$$\le Cn^2 \|\mu\|_\infty^{n-1}\|\mu\|_{A^s_{p,q}}\|f\|_{A^s_{p,q}},$$

and the norm becomes small if n is big enough, because $\|\mu\|_\infty = k < 1$.

(ii) Using $\|f\|_{I_1(L^{2,1})} \approx \|\nabla f\|_{L^{2,1}}$, (2.9), (2.7), and the boundedness of Calderón–Zygmund convolution operators, we have

$$\|\mu^n B^n(f)\|_{I_1(L^{2,1})} \le C\|\mu^n\|_{I_1(L^{2,1})}\|B^n(f)\|_{I_1(L^{2,1})}$$

$$\le C\|\mu^n\|_{I_1(L^{2,1})} n^2 \|f\|_{I_1(L^{2,1})}$$

$$\le Cn^3 \|\mu\|_\infty^{n-1}\|\mu\|_{I_1(L^{2,1})}\|f\|_{I_1(L^{2,1})},$$

and the norm becomes small if n is big enough, because $\|\mu\|_\infty = k < 1$. ∎

Lemma 3.2   *The commutator $[\mu, B]$ is compact on $A^s_{p,q}(\mathbb{C})$.*

Proof   First we have

$$\|[\mu, B]f\|_{A^s_{p,q}} = \|\mu Bf - B(\mu f)\|_{A^s_{p,q}}$$

$$\le \|\mu\|_{A^s_{p,q}}\|Bf\|_{A^s_{p,q}} + C\|\mu f\|_{A^s_{p,q}}$$

$$\le C\|\mu\|_{A^s_{p,q}}\|f\|_{A^s_{p,q}},$$

and so the commutator is bounded in $A^s_{p,q}$.

Using that the limit of compact operators is a compact operator, we can assume that $\mu \in \mathcal{C}_c^\infty(\mathbb{C})$, with its support contained in the disk $D(0, R)$. Now we use a trick from [AIM, p. 145]. Consider an arbitrary function $g = \mathrm{C}\,f$ with $f \in A^s_{p,q}$, where $\mathrm{C}\,f$ denotes the Cauchy transform of f (see (1.2)). As $\partial g = B(f)$, $\bar{\partial}g = f$, and $B(\bar{\partial}(\mu g)) = \partial(\mu g)$,

$$\mu B(f) - B(\mu f) = \mu \partial g - B(\mu \bar{\partial}g) = \mu \partial g - B(\bar{\partial}(\mu g)) + B(\bar{\partial}\mu\, g)$$

$$= \mu \partial g - \partial(\mu g) + B(\bar{\partial}\mu\, g) = B(\bar{\partial}\mu\, g) - \partial\mu\, g$$

$$= B(\bar{\partial}\mu\, \mathrm{C}\,f) - \partial\mu\, \mathrm{C}\,f.$$

From this representation one can see that $[\mu, B]$ is compact. Given $\varphi \in \mathcal{C}_c^\infty(D(0, R))$, the operator $\varphi\, \mathrm{C}\,f$ is a compact operator on $A^s_{p,q}(\mathbb{C})$, because by the lifting property (see [RS, 2.1.4]) $\varphi\, \mathrm{C}\,f \in A^{s+1}_{p,q}(\mathbb{C})$, obviously $\varphi\, \mathrm{C}\,f(z) = 0$ if $|z| \ge R$, and the inclusion of $A^{s+1}_{p,q}(D(0, R))$ into $A^s_{p,q}(D(0, R))$ is compact (*e.g.*, [RS, 2.4.4]). ∎

Lemma 3.3 *The commutator $[\mu, B]$ is compact on $I_1(L^{2,1}(\mathbb{C}))$.*

Proof As above we have

$$(3.1) \qquad \|[\mu, B]f\|_{I_1(L^{2,1})} = \|\mu Bf - B(\mu f)\|_{I_1(L^{2,1})}$$

$$\leq \|\mu\|_{I_1(L^{2,1})}\|Bf\|_{I_1(L^{2,1})} + C\|\mu f\|_{I_1(L^{2,1})}$$

$$\leq C\|\mu\|_{I_1(L^{2,1})}\|f\|_{I_1(L^{2,1})},$$

and so the commutator is bounded in $I_1(L^{2,1})$. So, by density, we only need to prove the compactness of the commutator when $\mu \in \mathcal{C}_c^\infty$.

On the other hand,

$$\|[\mu, B]f\|_{I_1(L^{2,1})} = \sum_{j=1}^{2} \|\partial_j(\mu B(f) - B(\mu f))\|_{L^{2,1}}$$

$$= \sum_{j=1}^{2} \|[\partial_j\mu, B]f + [\mu, B](\partial_j f)\|_{L^{2,1}}$$

$$\leq \sum_{j=1}^{2} \|[\partial_j\mu, B]f\|_{L^{2,1}} + \|[\mu, B](\partial_j f)\|_{L^{2,1}}.$$

Since the commutator is a compact operator in L^p when μ is smooth [U] and using real interpolation of compact operators [CoP], we have that $[\mu, B]\colon L^{2,1}(\mathbb{C}) \to L^{2,1}(\mathbb{C})$ is compact.

Therefore we only have to prove that $[a, B]\colon I_1(L^{2,1}(\mathbb{C})) \to L^{2,1}(\mathbb{C})$ is a compact operator when $a \in \mathcal{C}_c^\infty(B(0, R))$ for some $R > 0$. Given $\eta > 0$ we consider a regularization of the Beurling transform

$$B^\eta f(z) = -\frac{1}{\pi} \, \text{p.v.} \int f(z - w)K_\eta(w)\mathrm{d}w,$$

where $K_\eta(z) = \varphi_\eta(z)/z^2$ and $0 \leq \varphi_\eta(z) \leq 1$ is a radial $\mathcal{C}^\infty$ function satisfying $\varphi_\eta(|z|) = 0$ if $|z| < \frac{\eta}{2}$ and $\varphi_\eta(|z|) = 1$ if $|z| > \eta$. It is easy to check that B^η is a convolution Calderón–Zygmund operator with constants depending on η.

In the rest of this proof we will use the estimate (2.8) without any mention. For any $f \in I_1(L^{2,1})$, the function $[a, B - B^\eta](f)$ has compact support. On the other hand,

$$|[a, B - B^\eta](f)(z)| = \Big|\frac{-1}{\pi} \int (a(z) - a(y))\Big(\frac{1}{(z - y)^2} - \frac{\varphi_\eta(z - y)}{(z - y)^2}\Big)f(y)\mathrm{d}y\Big|$$

$$\leq C\|f\|_\infty\|\nabla a\|_\infty \int_{|z-y|<\eta} \frac{1}{|z - y|} \, \mathrm{d}y \leq C\eta\|f\|_{I_1(L^{2,1})}\|\nabla a\|_\infty.$$

Consequently the operator $[a, B^\eta]$ tends to $[a, B]$ when $\eta \to 0$. To prove that $[a, B^\eta]\colon I_1(L^{2,1}) \to L^{2,1}$ is compact we will use the Fréchet–Kolgomorov Theorem for Lorentz spaces (*e.g.*, [Br1, p. 111] for L^p spaces).

By (3.1), the image by $[a, B^\eta]$ of the unit ball of $I_1(L^{2,1}(\mathbb{C}))$ is uniformly bounded in $L^{2,1}(\mathbb{C})$. To get the equicontinuity, take $f \in I_1(L^{2,1})$ and $|z - w| < \frac{\eta}{8}$. Then

$$[a, B^\eta] f(z) - [a, B^\eta] f(w)$$

$$= \frac{-1}{\pi} \big((a(z) - a(w)) \int_\mathbb{C} \frac{\varphi_\eta(z - \xi)}{(z - \xi)^2} f(\xi) \mathrm{d}\xi$$

$$+ \frac{-1}{\pi} \int_\mathbb{C} \Big(\frac{\varphi_\eta(z - \xi)}{(z - \xi)^2} - \frac{\varphi_\eta(w - \xi)}{(w - \xi)^2} \Big) \big(a(w) - a(\xi) \big) f(\xi) \mathrm{d}\xi$$

$$= \theta_1(z, w) + \theta_2(z, w).$$

Since B^η is a convolution Calderón–Zygmund operator,

$$|\theta_1(z, w)| = \frac{1}{\pi} \big| \big(a(z) - a(w) \big) B^\eta f(z) \big| \le C_\eta |z - w| \|\nabla a\|_\infty \|f\|_{I_1(L^{2,1})}$$

and

$$|\theta_2(z, w)| = \frac{1}{\pi} \Big| \int_{\mathbb{C} \setminus B(z, \frac{\eta}{4})} \Big(\frac{\varphi_\eta(z - \xi)}{(z - \xi)^2} - \frac{\varphi_\eta(w - \xi)}{(w - \xi)^2} \Big) \big(a(w) - a(\xi) \big) f(\xi) \mathrm{d}\xi \Big|$$

$$\le C |z - w| \|f\|_\infty \|a\|_\infty \Big\{ \int_{|z - \xi| > \frac{\eta}{8}} \frac{1}{|z - \xi|^3} \mathrm{d}\xi + \int_{2\eta > |z - \xi| > \frac{\eta}{8}} \frac{\|\nabla \varphi_\eta\|_\infty}{|z - \xi|^2} \mathrm{d}\xi \Big\}$$

$$\le \frac{C}{\eta} |z - w| \|f\|_{I_1(L^{2,1})}.$$

Therefore

$$(3.2) \qquad \big| [a, B^\eta] f(z) - [a, B^\eta] f(w) \big| \le C |z - w| \|f\|_{I_1(L^{2,1})},$$

where the constant C depends on a and η.

On the other hand, if $|z| > M > 2R$,

$$\big| [a, B^\eta] f(z) \big| = \Big| \int_\mathbb{C} \big(a(z) - a(w) \big) \frac{\varphi_\eta(z - w)}{(z - w)^2} f(w) \mathrm{d}w \Big|$$

$$\le \|f\|_\infty \|a\|_\infty \int_{|w| < R} \frac{1}{|z - w|^2} \mathrm{d}w$$

$$\le C \|f\|_{I_1(L^{2,1})} \|a\|_\infty \frac{1}{|z|^2},$$

and then

$$(3.3) \qquad \big\| [a, B^\eta](f) \chi_{\mathbb{C} \setminus B(0, M)} \big\|_{L^{2,1}} \le C \|f\|_{I_1 L^{2,1}} \|a\|_\infty \big\| \frac{1}{|z|^2} \chi_{\mathbb{C} \setminus B(0, M)} \big\|_{L^{2,1}},$$

which tends to 0 as $M \to 0$. Combining (3.2) and (3.3), by the Fréchet–Kolgomorov Theorem for Lorentz spaces, one gets that $[a, B^\eta]$ is a compact operator from $I_1(L^{2,1}(\mathbb{C}))$ to $L^{2,1}(\mathbb{C})$ as desired. ∎

4 Calderón–Zygmund Operators on Domains

In this section we will prove Theorem 1.6. Let $X(\Omega)$ denote any of the function spaces in the statement of Theorem 1.6 and let $f \in X(\Omega)$. It is clear from the Calderón–Zygmund theory that $T_\Omega f \in L^p(\Omega)$. So, in order to study the behaviour of T_Ω on $X(\Omega)$, we must deal with $T_\Omega f(x) - T_\Omega f(y)$, because we have a characterization of $X(\Omega)$ using first differences. Throughout this section, as appropriate, one must consider the integrals in the sense of principal value. Following [Me] we consider the next decomposition.

Lemma 4.1 *Let $\psi \in \mathcal{C}_c^\infty$ such that $\psi(u) = 1$ on $|u| \leq 2$ and $\psi(u) = 0$ if $|u| \geq 4$. Define $\eta(u) = 1 - \psi(u)$. Then*

$$T_\Omega f(y) - T_\Omega f(x) := \sum_{i=1}^{4} g_i(x, y) + f(x)\big(T\chi_\Omega(y) - T\chi_\Omega(x) \big),$$

where

$$g_1(x, y) = \int_\Omega \big(K(y - u) - K(x - u) \big) \big(f(u) - f(x) \big)\, \eta\Big(\frac{u - x}{|y - x|} \Big)\, \mathrm{d}u,$$

$$g_2(x, y) = -\int_\Omega K(x - u)\big(f(u) - f(x) \big)\, \psi\Big(\frac{u - x}{|y - x|} \Big)\, \mathrm{d}u,$$

$$g_3(x, y) = \int_\Omega K(y - u)\big(f(u) - f(y) \big)\, \psi\Big(\frac{u - x}{|y - x|} \Big)\, \mathrm{d}u,$$

$$g_4(x, y) = \big(f(y) - f(x) \big) \int_\Omega K(y - u)\, \psi\Big(\frac{u - x}{|y - x|} \Big)\, \mathrm{d}u.$$

Proof Note that if $\tilde\psi(w) + \tilde\eta(w) = 1$, we can write

$$T_\Omega f(x) = f(x) T_\Omega \tilde\psi(x) + \int_\Omega K(x - w)(f(w) - f(x))\, \tilde\psi(w)\mathrm{d}w$$

$$+ \int_\Omega K(x - w) f(w) \tilde\eta(w)\mathrm{d}w,$$

and then

$$T_\Omega f(y) - T_\Omega f(x) = \int_\Omega (K(y - u) - K(x - u))(f(u) - f(x))\, \tilde\eta(u)\mathrm{d}u$$

$$- \int_\Omega K(x - u)(f(u) - f(x))\, \tilde\psi(u)\mathrm{d}u$$

$$+ \int_\Omega K(y - u)(f(u) - f(y))\, \tilde\psi(u)\mathrm{d}u$$

$$+ (f(y) - f(x)) \int_\Omega K(y - u)\, \tilde\psi(u)\mathrm{d}u$$

$$+ f(x)(T\chi_\Omega(y) - T\chi_\Omega(x)).$$

Given $x \neq y$, take $\tilde{\psi}(u) = \psi(\frac{u-x}{|y-x|})$ and $\tilde{\eta}(u) = \eta(\frac{u-x}{|y-x|})$, and that is what we wished to prove. ∎

Let $B = B(x_0, r)$ be the ball in $\mathbb{R}^n$ of center x_0 and radius r and let φ_B denote a smooth function supported in B such that $\|\varphi_B\|_\infty \leq 1$ and $\|\nabla \varphi_B\|_\infty \leq r^{-1}$. To deal with the term g_4 we will use the next lemma, which is an application of the Main Lemma of [MOV].

Lemma 4.2 *Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{1,\beta}$, $\beta > 0$, and let T be an even smooth homogeneous Calderón–Zygmund operator. Then there exists a constant $C = C(\Omega)$ such that $\|T_\Omega \varphi_B\|_\infty \leq C$.*

Proof Since the $\mathcal{C}^{0,\beta}$ norm of φ_B is bounded by $1 + r^{-\beta}$, by the Main Lemma of [MOV] we have

$$|T_\Omega \varphi_B(x)| \leq C(1 + r^{-\beta}), \qquad \text{for all } x \in \mathbb{C} \text{ and}$$

$$|T_\Omega \varphi_B(x) - T_\Omega \varphi_B(y)| \leq C r^{-\beta}|x - y|^\beta, \qquad \text{for all } x, y \in \Omega.$$

Associated with the domain Ω there is a $r_0 > 0$ satisfying (2.2). Then if $3r \geq r_0$, one has $|T_\Omega \varphi_B(x)| \leq C(1 + (\frac{3}{r_0})^\beta)$ for all $x \in \mathbb{C}$. If $3r < r_0$, we write

$$T_\Omega \varphi_B(x) = \int_\Omega K(x - y)\varphi_B(y)\mathrm{d}y = \int_{\Omega \cap 3B} K(x - y)\varphi_B(y)\mathrm{d}y$$

$$= \int_{\Omega \cap 3B} K(x - y)\big(\varphi_B(y) - \varphi_B(x)\big)\,\mathrm{d}y + \varphi_B(x)\int_{\Omega \cap 3B} K(x - y)\mathrm{d}y$$

$$= p(x) + q(x).$$

For $p(x)$ we have

$$|p(x)| \leq C \int_{\Omega \cap 3B} \frac{|\varphi_B(x) - \varphi_B(y)|}{|x - y|^n}\mathrm{d}y \leq C\|\nabla \varphi_B\|_\infty \int_{3B} \frac{\mathrm{d}y}{|x - y|^{n-1}} \leq C.$$

If $x \notin B$, $q(x) = 0$, and for $x \in B$ one can prove

$$\left| \int_{\Omega \cap 3B} K(x - y)\mathrm{d}y \right| \leq C(\Omega),$$

proceeding as in the proof of the Main Lemma of [MOV, pp. 408–410]. ∎

Let us continue with the proof of Theorem 1.6. In the case that $f \in B^s_{p,p}(\Omega)$, $0 < s < 1, n < sp < \infty$, we have to prove that

$$(4.1) \qquad \int_\Omega \int_\Omega \frac{|T_\Omega f(x) - T_\Omega f(y)|^p}{|x - y|^{n+sp}}\mathrm{d}x\mathrm{d}y < \infty.$$

By Lemma 4.1,

$$T_\Omega f(y) - T_\Omega f(x) = \sum_{i=1}^4 g_i(x, y) + f(x)\big(T\chi_\Omega(y) - T\chi_\Omega(x)\big),$$

and we will study each term separately. Since f is bounded (because $n < sp < \infty$) and $T\chi_\Omega \in B_{p,p}^s(\Omega)$,

$$\int_\Omega \int_\Omega \frac{|f(x)(T\chi_\Omega(y) - T\chi_\Omega(x))|^p}{|x - y|^{n+sp}} \mathrm{d}x\mathrm{d}y \leq$$

$$\|f\|_\infty \int_\Omega \int_\Omega \frac{|T\chi_\Omega(y) - T\chi_\Omega(x)|^p}{|x - y|^{n+sp}} \mathrm{d}x\mathrm{d}y < \infty.$$

Fix t such that $s < t < 1$. Using the properties of the kernel K and Hölder's inequality (with $\frac{1}{p} + \frac{1}{q} = 1$), we have

$$|g_1(x, y)| \leq C \int_{\Omega \cap \{|u-x|>2|x-y|\}} |K(x - u) - K(y - u)||f(u) - f(x)|\mathrm{d}u$$

$$\leq C \int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|x - y|}{|x - u|^{n+1}} |f(u) - f(x)|\mathrm{d}u$$

$$= C|x - y| \int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|}{|x - u|^{\frac{n}{p}+t}} \frac{1}{|x - u|^{\frac{n}{q}-t+1}} \mathrm{d}u$$

$$\leq C|x - y| \left(\int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \mathrm{d}u \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_{\{|u-x|>2|x-y|\}} \frac{\mathrm{d}u}{|x - u|^{n-tq+q}} \right)^{\frac{1}{q}}$$

$$\leq C|x - y|^t \left(\int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \mathrm{d}u \right)^{\frac{1}{p}}.$$

Thus,

$$\frac{|g_1(x, y)|^p}{|x - y|^{n+sp}} \leq \frac{C}{|x - y|^{n+sp-tp}} \int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \mathrm{d}u,$$

and then, by the Fubini's theorem,

$$\int_\Omega \int_\Omega \frac{|g_1(x, y)|^p}{|x - y|^{n+sp}} \mathrm{d}x\mathrm{d}y$$

$$\leq C \int_\Omega \int_\Omega \frac{1}{|x - y|^{n+sp-tp}} \int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \mathrm{d}u\mathrm{d}x\mathrm{d}y$$

$$= C \int_\Omega \int_\Omega \int_{\Omega \cap \{|u-x|>2|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+p-sp}} \frac{1}{|x - y|^{n+sp-tp}} \mathrm{d}y\mathrm{d}u\mathrm{d}x$$

$$\leq C \int_\Omega \int_\Omega \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \frac{1}{|x - u|^{sp-tp}} \mathrm{d}u\mathrm{d}x$$

$$= C \int_\Omega \int_\Omega \frac{|f(u) - f(x)|^p}{|x - u|^{n+sp}} \mathrm{d}u\mathrm{d}x < \infty.$$

Since the terms g_2 and g_3 are symmetric, we only consider one of them. Take t such that $0 < t < s$. As before, using the properties of the kernel K and the Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$),

$$
\begin{aligned}
|g_2(x, y)| &\le C \int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|}{|x - u|^n} \, du \\
&= C \int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|}{|x - u|^{\frac{n}{p}+t}} \frac{1}{|x - u|^{\frac{n}{q}-t}} \, du \\
&\le C \left(\int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \, du \right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_{\{|x-u| < 4|x-y|\}} \frac{du}{|x - u|^{n-tq}} \right)^{\frac{1}{q}} \\
&\le C|x - y|^t \left(\int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \, du \right)^{\frac{1}{p}}.
\end{aligned}
$$

Then

$$
\frac{|g_2(x, y)|^p}{|x - y|^{n+sp}} \le \frac{C}{|x - y|^{n+sp-tp}} \int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - u|^{n+tp}} \, du,
$$

and therefore

$$
\begin{aligned}
&\int_\Omega \int_\Omega \frac{|g_2(x, y)|^p}{|x - y|^{n+sp}} \, dx dy \\
&\le C \int_\Omega \int_\Omega \int_{\Omega \cap \{|x-u| < 4|x-y|\}} \frac{|f(u) - f(x)|^p}{|x - y|^{n+sp-tp}|x - u|^{n+tp}} \, dy du dx \\
&\le C \int_\Omega \int_\Omega \frac{|f(u) - f(x)|^p}{|x - u|^{n+sp}} \, du dx < \infty.
\end{aligned}
$$

Finally, by Lemma 4.2 we have

$$
\left| \int_\Omega K(y - u) \, \psi\left(\frac{u - x}{|y - x|} \right) du \right| \le C,
$$

and consequently

$$
\int_\Omega \int_\Omega \frac{|g_4(x, y)|^p}{|x - y|^{n+sp}} \, du dx \le C \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx dy < \infty.
$$

Combining all these inequalities we get (4.1).

Using the characterizations (2.3) and (2.4) one can see that the proofs for $f \in W^{s,p}(\Omega)$ or $f \in W^{1,p}(\Omega)$ are very similar to the one we just explained for $f \in B_{p,p}^s(\Omega)$.

Remark If Ω is a bounded domain of $\mathbb{R}^n$ with boundary of class $\mathcal{C}^{1,\varepsilon}$, $\varepsilon > 0$, and T is an even smooth homogeneous Calderón–Zygmund operator, we have (see [MOV, Main Lemma])

$$|T(\chi_\Omega)(x) - T(\chi_\Omega)(y)| \le C|x - y|^\varepsilon, \qquad \forall x, y \in \Omega.$$

Therefore $T(\chi_\Omega)$ belongs to $W^{s,p}(\Omega)$ and $B^s_{p,p}(\Omega)$ for any $s \in (0, \varepsilon)$.

5 Proof of Theorem 1.5

Consider the Beurling transform restricted on the domain Ω of class $\mathcal{C}^{1,\varepsilon}$,

$$B_\Omega g(z) = -\frac{1}{\pi} \, \text{p.v.} \int_\Omega \frac{g(w)}{(z - w)^2} \mathrm{d}w.$$

By [MOV, Main Lemma], $|B(\chi_\Omega)(z) - B(\chi_\Omega)(w)| \le C|z - w|^\varepsilon$ for all $z, w \in \Omega$. Now, applying Theorem 1.6 we have that B_Ω is bounded on the spaces $B^s_{p,p}(\Omega)$ and $W^{s,p}(\Omega)$. Let us denote by $X(\Omega)$ any of these two spaces. We will show that the Beltrami operator $I - \mu B_\Omega$ is invertible on $X(\Omega)$. Then taking $h = (I - \mu B_\Omega)^{-1}(\mu)$ we get the conclusions.

As in the proof of the Proposition 1.2, we claim that $I - \mu B_\Omega$ is a Fredholm operator on $X(\Omega)$. Define $P_m = I + \mu B_\Omega + \cdots + (\mu B_\Omega)^m$ so that

$$(I - \mu B_\Omega) P_{m-1} = P_{m-1}(I - \mu B_\Omega) = I - \mu^m (B_\Omega)^m + R,$$

where $R = \mu^m(B_\Omega)^m - (\mu B_\Omega)^m$ can easily be seen to be a finite sum of operators that contain the commutator $[\mu, B_\Omega]$ as a factor. We will prove that $[\mu, B_\Omega]\colon X(\Omega) \to X(\Omega)$ is a compact operator. On the other hand, for $z \in \Omega$,

$$\left(I - \mu^m(B_\Omega)^m\right) f(z) = \left(I - \mu^m(B^m)_\Omega\right) f(z) + \mu^m(z)\left((B^m)_\Omega f(z) - (B_\Omega)^m f(z)\right)$$

$$= \left(I - \mu^m(B^m)_\Omega\right) f(z) + \mu^m(z) K_m f(z),$$

where B^m is the m-iterated Beurling transform and $K_m f := (B^m)_\Omega f - (B_\Omega)^m f$. As in the proof of Lemma 3.1, if $F \in X(\Omega)$, we get

(5.1) $$\|\mu^m F\|_{X(\Omega)} \le C m \|\mu\|_\infty^{m-1} \|\mu\|_{X(\Omega)} \|F\|_{X(\Omega)}.$$

Recall that the kernel of B^m is $((-1)^m m \bar{z}^{m-1})/(\pi z^{m+1})$, and then, by Theorem 1.6, if $f \in X(\Omega)$, we have

(5.2) $$\|(B^m)_\Omega f\|_{X(\Omega)} \le C \, m^2 \|f\|_{X(\Omega)}.$$

Consequently, combining (5.1) and (5.2),

$$\|\mu^m (B^m)_\Omega f\|_{X(\Omega)} \le C m^3 \|\mu\|_\infty^{m-1} \|\mu\|_{X(\Omega)} \|f\|_{X(\Omega)},$$

which implies that $I - \mu^m(B^m)_\Omega$ is invertible if m is large. Assume for a moment that the operators K_m are compacts on $X(\Omega)$. Thus, $I - \mu B_\Omega$ is a Fredholm operator and in addition has index zero. Since $X(\Omega) \subset L^p(\Omega)$ we also have that $I - \mu B_\Omega$ is injective (see [Iw]) and therefore invertible on $X(\Omega)$.

The compactness of the operators $[\mu, B_\Omega]$ and K_m on $X(\Omega)$ follows parallel arguments. Since $X(\Omega)$ is an algebra and the Beurling transform B_Ω is bounded on $X(\Omega)$, we have

$$\|[\mu, B_\Omega]f\|_X = \|\mu B_\Omega f - B_\Omega(\mu f)\|_X \le C\|\mu\|_X \|f\|_X.$$

Moreover, because the domain Ω is Lipschitz, there exists a sequence of functions $\mu_j \in C^\infty(\overline{\Omega})$ such that μ_j converges to μ in $X(\Omega)$. So, we are reduced to proving the compactness when $\mu \in C^\infty(\overline{\Omega})$. In this case, the kernel of the commutator

$$[\mu, B_\Omega]f(z) = -\frac{1}{\pi} \int_\Omega \frac{\mu(z) - \mu(w)}{(z-w)^2} f(w)\mathrm{d}w := \int_\Omega k(z, w)f(w)\mathrm{d}w$$

clearly satisfies

$$|k(z, w)| \le \frac{C}{|z-w|} \qquad \text{for all } z, w \in \Omega,$$

$$|k(z', w) - k(z, w)| \le C\frac{|z-z'|}{|z-w|^2} \qquad \text{if } |z-w| > 2|z-z'|.$$

Then a simple computation gives (see [MOV, p. 419]), for $z_1, z_2 \in \Omega$,

$$(5.3) \qquad \big|[\mu, B_\Omega]f(z_1) - [\mu, B_\Omega]f(z_2)\big| \le C|z_1 - z_2|\left(1 + \log\frac{d}{|z_1 - z_2|}\right)\|f\|_\infty,$$

where d denotes the diameter of Ω. From (5.3) one immediately gets that $[\mu, B_\Omega]f$ belongs to $B_{p,p}^\beta(\Omega)$ and to $W^{\beta,p}(\Omega)$ for any $\beta < 1$. The compact embedding $W^{\beta,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$, $s < \beta$, (and $B_{p,p}^\beta(\Omega) \hookrightarrow B_{p,p}^s(\Omega)$) gives the compactness for the commutator (*e.g.*, [Tri2, Proposition 7]).

We have $K_m f = (B^m)_\Omega f - (B_\Omega)^m f$. To prove that K_m is compact on $X(\Omega)$ we will proceed by induction. For $m \ge 2$,

$$\begin{aligned}
(B_\Omega)^m f = B_\Omega\big((B_\Omega)^{m-1}f\big) &= B\big([(B_\Omega)^{m-1}f]\chi_\Omega\big) \\
&= B\big(\big[B^{m-1}(f\chi_\Omega) - K_{m-1}f\big]\chi_\Omega\big) \\
&= B\big(B^{m-1}(f\chi_\Omega) - \big(B^{m-1}(f\chi_\Omega)\big)\chi_{\Omega^c} - (K_{m-1}f)\chi_\Omega\big) \\
&= B^m(f\chi_\Omega) - B\big(\chi_{\Omega^c}B^{m-1}(f\chi_\Omega)\big) - B_\Omega(K_{m-1}f).
\end{aligned}$$

It is then enough to prove that, for $m \ge 1$, the operator

$$Q_m f := B\big(\big(B^m(f\chi_\Omega)\big)\chi_{\Omega^c}\big)$$

is compact in $X(\Omega)$. For $z \in \Omega$, we write

$$
\begin{aligned}
Q_m f(z) &= B((B^m(f\chi_\Omega))\chi_{\Omega^c})(z) \\
&= -\frac{1}{\pi} \int_{\Omega^c} \frac{B^m(f\chi_\Omega)(w)}{(z-w)^2} \mathrm{d}w \\
&= -\frac{1}{\pi} \int_{\Omega^c} \frac{1}{(z-w)^2} \frac{(-1)^m m}{\pi} \int_\Omega \frac{\overline{(w-\xi)}^{m-1}}{(w-\xi)^{m+1}} f(\xi)\,\mathrm{d}\xi\,\mathrm{d}w \\
&= \int_\Omega K_m(z,\xi)f(\xi)\mathrm{d}\xi,
\end{aligned}
$$

where

$$
K_m(z,\xi) := \frac{(-1)^{m+1}}{\pi^2} \int_{\Omega^c} \frac{1}{(z-w)^2} \frac{m\,\overline{(w-\xi)}^{m-1}}{(w-\xi)^{m+1}} \mathrm{d}w.
$$

In [MOV, pp. 418–419], it is proved that if Ω is a bounded domain of class $\mathcal{C}^{1,\varepsilon}$ and $f \in L^\infty(\Omega)$, then

$$
|Q_m f(z)| \le C d^\varepsilon \|f\|_\infty, \quad z \in \Omega,
$$

$$
|Q_m f(z_1) - Q_m f(z_2)| \le C|z_1 - z_2|^\varepsilon \left(1 + \log\frac{d}{|z_1 - z_2|}\right)\|f\|_\infty, \quad z_1, z_2 \in \Omega,
$$

where d denotes the diameter of Ω and C depends on m and Ω.

Consequently, if $f \in X(\Omega)$, then $Q_m f$ belongs to $B^\beta_{p,p}(\Omega)$ and to $W^{\beta,p}(\Omega)$ for any $\beta < \varepsilon$. Choose β such that $s < \beta < \varepsilon$. Again, the compact embeddings $W^{\beta,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and $B^\beta_{p,p}(\Omega) \hookrightarrow B^s_{p,p}(\Omega)$ give the compactness of Q_m.

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