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THE REPRESENTATION RING OF THE TWISTED QUANTUM DOUBLE OF A FINITE GROUP

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ABSTRACT. We provide an isomorphism between the Grothendieck ring of modules of the twisted quantum double of a finite group, and a product of centres of twisted group algebras of centralizer subgroups. It follows that this Grothendieck ring is semisimple. Another consequence is a formula for the characters of this ring in terms of representations of twisted group algebras of centralizer subgroups.

Introduction. Let G be a finite group and k an algebraically closed field. Given a 3-cocycle $\omega: G \times G \times G \longrightarrow k^\times$, there is a quasi-Hopf algebra $D^\omega(G)$ with underlying vector space $(kG)^* \otimes_k kG$, where $(kG)^*$ is the Hopf algebra dual to the group algebra kG [1, 4, 7]. As an algebra, $D^\omega(G)$ is a crossed product of the Hopf algebras $(kG)^*$ and kG. There is also a construction of this algebra from the ordinary quantum double $D(G)$, originally defined by Drinfel'd [9], analogous to that of a twisted group algebra from a group algebra. We have therefore chosen to call $D^\omega(G)$ the twisted quantum double of G.

In the case $k = \mathbf{C}$, it is indicated by Dijkgraaf, Pasquier, and Roche in [7] that there is a connection between the representations of $D^\omega(G)$ and conformal field theory; in this context, tensor products of these representations are of interest. Applications of such representations to generalized Thompson series and Moonshine are given in [1] by Bantay. In [14], Mason discusses a conjectured equivalence of categories between $D^\omega(G)$-modules and modules of a certain vertex operator algebra, and in [8], Dong and Mason prove a special case of this conjecture.

The algebra $D^\omega(G)$ turns out to be an example of a quasitriangular quasi-Hopf algebra. For our purposes, this implies that there is a representation ring $R\big(D^\omega(G)\big)$ which is both associative and commutative. The sum and product of this ring are given by direct sum and tensor product of representations. In this exposition, we will analyze the structure of this ring, or more precisely of the Grothendieck ring $\mathcal{R}\big(D^\omega(G)\big)$, the quotient of $R\big(D^\omega(G)\big)$ by the ideal of short exact sequences.

In Section 1, we give an explicit definition of $D^\omega(G)$, and discuss some of its properties. In particular, we show that $D^\omega(G)$ is semisimple if and only if the characteristic p of

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k does not divide the order of G. In the non-semisimple case, the Grothendieck ring is a proper quotient of the representation ring, and is easier to study. In Section 1 we also state a characterization of indecomposable $D^\omega(G)$-modules, which is a simultaneous generalization of the case $k = \mathbf{C}$ in [7], and of the ordinary quantum double $D(G)$ over an arbitrary field in [14].

Our main result (Theorem 4.4) describes the structure of the Grothendieck ring $\mathcal{R}\left(D^\omega(G)\right)$, and implies that it is semisimple. More precisely, we show that there is an isomorphism from $\mathcal{R}\left(D^\omega(G)\right)$ to the product of centres of twisted group algebras of centralizer subgroups of G. As a consequence, we give its set of characters, that is algebra homomorphisms from $\mathcal{R}\left(D^\omega(G)\right)$ to $\mathbf{C}$. These characters distinguish modules up to their composition factors, and are analogous to Brauer characters of the group, which correspond to the characters of the Grothendieck ring $\mathcal{R}(kG)$ of the group algebra kG [2]. They are expressed in terms of characters of twisted group algebras of centralizer subgroups of G, that is trace functions of modules for these algebras. Such modules correspond to projective representations of the centralizer subgroups. Our theorem generalizes a result for the ordinary quantum double $D(G)$ given in [16].

We first present our results in the special case $k = \mathbf{C}$ in Section 2. In this case the Grothendieck ring described above is equal to the representation ring, and proofs are straightforward. For the case of an arbitrary algebraically closed field k, we start with the same approach. However, now $D^\omega(G)$ is an algebra over k, whereas its representation ring is an algebra over $\mathbf{C}$. The situation is complicated by the necessity of lifting certain function values from k to $\mathbf{C}$. It also appears necessary to introduce a representation group, that is a covering group such that any projective representation (or representation of a twisted group algebra) may be lifted to an ordinary representation of this group. In Section 3, we present an analog of Brauer characters for twisted group algebras, and results relating these Brauer characters to those of representation groups. In Section 4, we use these results to prove our main theorem.

All our modules will be finite dimensional over k, and all tensor products will be over k unless otherwise indicated. We caution that we will use the term "character" in two different ways. At times, it will refer to a homomorphism from an algebra to $\mathbf{C}$. At other times, it will refer to the trace function of a module. In [3], Benson and Parker deal with this confusion by referring to the former as a "species." Here we hope that it will be clear from context which meaning is intended.

1. **Preliminaries.** We will define the twisted quantum double as in [1, 4, 7], although we do not require the underlying field to be $\mathbf{C}$. Fix a finite group G and an algebraically closed field k of characteristic p. Let $\omega: G \times G \times G \longrightarrow k^\times$ be a 3-cocycle; that is

$$\omega(a,b,c)\omega(a,bc,d)\omega(b,c,d) = \omega(ab,c,d)\omega(a,b,cd)$$

for all $a, b, c, d \in G$. We assume that ω is normalized so that $\omega(a,b,c)$ is equal to 1 whenever one of a, b, or c is equal to the identity element 1 of G. The *twisted quantum double* $D^\omega(G)$ of G with respect to ω is the following quasi-Hopf algebra with underlying

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vector space $(kG)^* \otimes_k kG$. (For the definition of a quasi-Hopf algebra, see [4] or [10].) Let $\{\delta_g \otimes \bar{x}\}_{g,x\in G}$ be the canonical basis, where δ_g is the function dual to $g \in G$, so that $\delta_g(g) = 1$ and $\delta_g(h) = 0$ when $h \in G$, $h \neq g$. We will abbreviate $\delta_g \otimes \bar{x}$ by $\delta_g\bar{x}$.

The product is given by

$$(\delta_g\bar{x})(\delta_h\bar{y}) = \theta_g(x,y)\delta_g\delta_{xhx^{-1}}\overline{xy} = \begin{cases} \theta_g(x,y)\delta_g\overline{xy} & \text{if } g = xhx^{-1} \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}$. The element $1_{D^\omega(G)} = \sum_{g\in G}\delta_g\bar{1}$ is the multiplicative identity. We point out that this is the algebra structure given by the crossed product [15] of the Hopf algebras $(kG)^*$ and kG with respect to the conjugation action of G on $(kG)^*$ and the linear map $\sigma: kG \otimes kG \longrightarrow (kG)^*$ defined by

$$\sigma(x \otimes y) = \sum_{g\in G} \theta_g(x,y)\delta_g$$

for all $x, y \in G$.

When we use the notation δ_g, we will mean $\delta_g\bar{1}$, and by $\bar{x}$ we will mean $\sum_{g\in G}\delta_g\bar{x}$. Thus the elements $\{\delta_g\}_{g\in G}$ generate a subalgebra of $D^\omega(G)$ isomorphic to $(kG)^*$, but the elements $\{\bar{x}\}_{x\in G}$ do not in general generate a subalgebra isomorphic to kG. However, the elements $\bar{x}$ are invertible, with $\bar{x}^{-1} = \sum_{g\in G}\theta_g(x^{-1},x)^{-1}\delta_g\overline{x^{-1}}$.

The coproduct $\Delta: D^\omega(G) \longrightarrow D^\omega(G) \otimes D^\omega(G)$ is given by

$$\Delta(\delta_g\bar{x}) = \sum_{h\in G} \gamma_x(h, h^{-1}g)(\delta_h\bar{x}) \otimes (\delta_{h^{-1}g}\bar{x}),$$

where $\gamma_x(h,\ell) = \frac{\omega(h,\ell,x)\omega(x,x^{-1}hx,x^{-1}\ell x)}{\omega(h,x,x^{-1}\ell x)}$. This coproduct is not coassociative in general, but quasi-coassociative; this means that there is an invertible element

$$\phi = \sum_{g,h,k\in G} \omega(g,h,k)\delta_g\bar{1} \otimes \delta_h\bar{1} \otimes \delta_k\bar{1}$$

in $D^\omega(G)^{\otimes 3}$ such that $(\Delta \otimes \mathrm{id})\Delta(a) = \phi(\mathrm{id} \otimes \Delta)\Delta(a)\phi^{-1}$ for all $a \in D^\omega(G)$.

A counit ϵ and a coinverse s are given by

$$\epsilon(\delta_g\bar{x}) = \delta_{g,1} \quad \text{and} \quad s(\delta_g\bar{x}) = \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}\delta_{x^{-1}g^{-1}x}\overline{x^{-1}},$$

where $\delta_{g,1}$ is the Kronecker delta. These maps make the algebra $D^\omega(G)$ into a quasi-Hopf algebra [1, 4, 7]. Verification of the details involves the following identities, which result from the 3-cocycle identity for ω:

$$\theta_z(a,b)\theta_z(ab,c) = \theta_{a^{-1}za}(b,c)\theta_z(a,bc),$$
$$\theta_y(a,b)\theta_z(a,b)\gamma_a(y,z)\gamma_b(y^a,z^a) = \theta_{yz}(a,b)\gamma_{ab}(y,z),$$
$$\gamma_z(a,b)\gamma_z(ab,c)\omega(a^z,b^z,c^z) = \gamma_z(b,c)\gamma_z(a,bc)\omega(a,b,c),$$

for all $a, b, c, y, z \in G$. If ω is trivial, we obtain the ordinary quantum double $D(G)$ of G, which is a Hopf algebra. The quantum double construction was originally defined by Drinfel'd for any Hopf algebra [9].

We will next define the representation ring. Let U and V be right $D^\omega(G)$-modules. Then $U \otimes V$ is a right $D^\omega(G)$-module via the pullback of the natural action of $D^\omega(G) \otimes D^\omega(G)$ on $U \otimes V$ with respect to the coproduct $\Delta: D^\omega(G) \longrightarrow D^\omega(G) \otimes D^\omega(G)$. This results in a right $D^\omega(G)$-module since Δ is an algebra homomorphism. The field k is a right $D^\omega(G)$-module via the pullback of the action of k on itself by right multiplication with respect to the counit $\epsilon: D^\omega(G) \longrightarrow k$. Up to isomorphism, this *trivial* module k is a multiplicative identity with respect to tensor product of modules; this follows from the counit property of a quasi-Hopf algebra.

Let $R\big(D^\omega(G)\big)$ denote the *representation ring* of $D^\omega(G)$, that is the $\mathbf{C}$-algebra generated by isomorphism classes of finite dimensional right $D^\omega(G)$-modules with direct sum for addition and tensor product for multiplication. Then $R\big(D^\omega(G)\big)$ is a ring with identity given by the isomorphism class of the trivial module. Associativity of $R\big(D^\omega(G)\big)$ follows from the quasi-coassociativity of the coproduct for $D^\omega(G)$. By abuse of language and notation, we will consider $D^\omega(G)$-modules to be elements of the representation ring, when we really mean their isomorphism classes.

We now define $R_0\big(D^\omega(G)\big)$, the *ideal of short exact sequences*, to be the ideal generated by all $U - U' - U''$ where $0 \longrightarrow U' \longrightarrow U \longrightarrow U'' \longrightarrow 0$ is a short exact sequence of $D^\omega(G)$-modules. Notice that $R_0\big(D^\omega(G)\big)$ is in fact already generated by these elements as a vector space, since tensor products of modules are taken over the field k. The *Grothendieck ring* $\mathcal{R}\big(D^\omega(G)\big)$ is defined to be the quotient of $R\big(D^\omega(G)\big)$ by $R_0\big(D^\omega(G)\big)$. This $\mathbf{C}$-algebra has a basis consisting of the images of the irreducible $D^\omega(G)$-modules.

The quasi-Hopf algebra $D^\omega(G)$ is quasitriangular [4, 10], with

$$R = \sum_{g,h \in G} \delta_g \bar{1} \otimes \delta_h \bar{g} \quad \text{and} \quad R^{-1} = \sum_{g,h \in G} \theta_{ghg^{-1}}(g, g^{-1})^{-1} \delta_g \bar{1} \otimes \delta_h \overline{g^{-1}}.$$

That is, $R\Delta(a)R^{-1} = \sigma\big(\Delta(a)\big)$ for all $a \in D^\omega(G)$, where σ is the twist automorphism interchanging two factors. If U and V are $D^\omega(G)$-modules, this equation yields an isomorphism between the modules $U \otimes V$ and $V \otimes U$, given by the twist σ followed by the natural action of R. Thus the representation ring $R\big(D^\omega(G)\big)$ is commutative.

We note that if $\beta: G \times G \longrightarrow k^\times$ is a normalized 2-cochain, and $\delta\beta$ its coboundary

$$\delta\beta(a, b, c) = \beta(b, c)\beta(a, bc)\beta(ab, c)^{-1}\beta(a, b)^{-1},$$

then $D^{\omega\delta\beta}(G)$ is isomorphic to $D^\omega(G)$ as an algebra. The map $\iota: D^\omega(G) \longrightarrow D^{\omega\delta\beta}(G)$ defined by

$$\iota(\delta_g \bar{x}) = \frac{\beta(g, x)}{\beta(x, g^x)} \delta_g \bar{x}$$

is an algebra isomorphism. As a quasi-Hopf algebra, $D^{\omega\delta\beta}(G)$ is obtained from $D^\omega(G)$ by the twist element

$$F = \sum_{g,h \in G} \beta(g, h)^{-1} \delta_g \bar{1} \otimes \delta_h \bar{1}.$$

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That is, $\Delta\big(\iota(a)\big) = F\Big(\iota \otimes \iota\big(\Delta(a)\big)\Big)F^{-1}$ for all $a \in D^\omega(G)$ [4, 7]. The map $\iota: D^\omega(G) \longrightarrow D^{\omega\delta\beta}(G)$ induces an isomorphism of representation rings

$$\iota^*: R\big(D^{\omega\delta\beta}(G)\big) \xrightarrow{\sim} R\big(D^\omega(G)\big).$$

We point out that $H^3(G, k^\times)$ is finite of order dividing $|G|_{p'}$ (that part of $|G|$ not divisible by the characteristic p of k), and that any 3-cocycle ω is cohomologous to a 3-cocycle of finite order dividing $|G|_{p'}$. This can be shown by an argument exactly analogous to that for 2-cocycles in [6, Lemma 11.38]. In addition, a 3-cocycle of finite order cohomologous to ω may be chosen to be normalized as well. We will assume from now on that ω is normalized and of finite order dividing $|G|_{p'}$.

If x commutes with g and h, then

$$\theta_x(g,h) = \gamma_x(g,h) = \frac{\omega(x,g,h)\omega(g,h,x)}{\omega(g,x,h)},$$

where these are the functions arising in the coproduct and product of $D^\omega(G)$ defined above. A calculation shows that this function $\theta_x: C(x) \times C(x) \longrightarrow k^\times$ is a 2-cocycle; that is $\theta_x(ab,c)\theta_x(a,b) = \theta_x(a,bc)\theta_x(b,c)$ for all $a, b, c \in C(x)$.

As an algebra, $D^\omega(G)$ is semisimple if and only if the characteristic p of k does not divide the order of G; we outline an argument here. If p does not divide $|G|$, then both kG and $(kG)^*$ are semisimple, and Maschke's Theorem implies that $D^\omega(G)$ is semisimple (see [13]). On the other hand, $D^\omega(G)/\ker(\epsilon)$ and kt, where $t = \sum_{g\in G} \delta_1 \bar{g}$, are both isomorphic to the trivial module k as $D^\omega(G)$-modules. If $D^\omega(G)$ is semisimple, then k occurs only once in a decomposition of the right regular module $D^\omega(G)$ into a direct sum of irreducible $D^\omega(G)$-modules. In particular, any composition series of $D^\omega(G)$ contains exactly one factor isomorphic with k. This implies that p does not divide $|G|$, as otherwise we have $t \in \ker(\epsilon)$.

In the case $k = \mathbf{C}$, a description of the irreducible $D^\omega(G)$-modules is given in [7]; in case ω is trivial, a description of the indecomposable $D^\omega(G)$-modules over an arbitrary field k is given in [14]. These results are easily extended. The irreducible (respectively, indecomposable) $D^\omega(G)$-modules are indexed by pairs (U, g) where g is a representative of a conjugacy class of G and U is an irreducible (respectively, indecomposable) $k^{\theta_g}C(g)$-module. Here $k^{\theta_g}C(g)$ denotes the twisted group algebra with basis $\{x_h\}_{h\in C(g)}$ and product $x_h x_\ell = \theta_g(h, \ell)x_{h\ell}$. This characterization of $D^\omega(G)$-modules arises from an equivalence of categories, which we describe next, following the approach in [14]. It may be helpful to notice first that any $D^\omega(G)$-module U is a direct sum of subspaces $U = \sum_{x\in G} U \cdot \delta_x$, and the invertible elements $\bar{g}$ of $D^\omega(G)$ permute these subspaces, as $\delta_x \bar{g} = \bar{g}\delta_{g^{-1}xg}$.

For each $x \in G$, there are subspaces $D^\omega(x) = \sum_{g\in G} k\delta_x\bar{g}$ and $S^\omega(x) = \sum_{g\in C(x)} k\delta_x\bar{g}$ of $D^\omega(G)$. Note that $S^\omega(x)$ is a subalgebra, with identity element δ_x, isomorphic to the twisted group algebra $k^{\theta_x}C(x)$. Accordingly, we identify these two algebras in the following lemma. If K is a conjugacy class of elements of G, let $D^\omega(K) = \sum_{x\in K} D^\omega(x)$. Then $D^\omega(G) \simeq \oplus_K D^\omega(K)$ is a direct sum of two-sided ideals, where K ranges over all

conjugacy classes of G. Therefore an indecomposable $D^\omega(G)$-module is a module for precisely one of the $D^\omega(K)$. The irreducible (indecomposable) $D^\omega(G)$-modules are thus characterized by the equivalence of categories given in the following lemma, whose proof is straightforward. We point out that $D^\omega(x)$ is free as a left $S^\omega(x)$-module, so the equivalence of categories in the lemma preserves short exact sequences.

LEMMA 1.1. *Let K be a conjugacy class of G, $x \in K$, and* Mod-$k^{\theta_x}C(x)$ *and* Mod-$D^\omega(K)$ *the categories of right $k^{\theta_x}C(x)$- and $D^\omega(K)$-modules, respectively. Let $U \in$* Mod-$k^{\theta_x}C(x)$ *and $V \in$* Mod-$D^\omega(K)$. *The maps*

$$U \longmapsto U \otimes_{S^\omega(x)} D^\omega(x) \quad \text{and} \quad V \longmapsto V \cdot \delta_x$$

define an equivalence of categories between Mod-$k^{\theta_x}C(x)$ *and* Mod-$D^\omega(K)$, *where $D^\omega(K)$ acts on $U \otimes_{S^\omega(x)} D^\omega(x)$ by right multiplication in the second factor.*

2. **The case $k = \mathbf{C}$.** In this section, we present results about the representation ring $R\big(D^\omega(G)\big)$ in the special case $k = \mathbf{C}$. These results will be generalized to an arbitrary algebraically closed field in Section 4.

Certain characters of $D^\omega(G)$ are introduced in [1]; these are the functions from $G \times G$ to $\mathbf{C}$ defined for each $D^\omega(G)$-module U by sending a pair (h, g) to the trace $\mathrm{Tr}(\delta_h \bar{g}, U)$ of the action of $\delta_h \bar{g}$ on U. With these characters, we will build certain functions from $R\big(D^\omega(G)\big)$ to $\mathbf{C}$ which are in fact algebra homomorphisms, or characters of the representation ring.

We first define, for each $g \in G$, a linear function $f_g : R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\theta_g}C(g)$ by

$$f_g(U) = \sum_{h \in C(g)} \mathrm{Tr}(\delta_h \bar{g}, U) x_h$$

for any $D^\omega(G)$-module U. We will see that f_g is an algebra homomorphism with image the centre $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$ of $\mathbf{C}^{\theta_g}C(g)$. In fact, the product of all f_g, taken over a set of representatives g of conjugacy classes of G, provides an algebra isomorphism from $R\big(D^\omega(G)\big)$ to $\prod_g Z\big(\mathbf{C}^{\theta_g}C(g)\big)$, as is proved in Theorem 2.2 below.

LEMMA 2.1. *The function $f_g : R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\theta_g}C(g)$ is an algebra homomorphism.*

PROOF. It may be checked that f_g takes the trivial $D^\omega(G)$-module $\mathbf{C}$ to the identity x_1 of $\mathbf{C}^{\theta_g}C(g)$. Now let U and V be $D^\omega(G)$-modules. Then

$$\begin{aligned} f_g(U) f_g(V) &= \sum_{h, \ell \in C(g)} \theta_g(h, \ell)\, \mathrm{Tr}(\delta_h \bar{g}, U)\, \mathrm{Tr}(\delta_\ell \bar{g}, V) x_{h\ell} \\ &= \sum_{h, \ell \in C(g)} \theta_g(h, h^{-1}\ell)\, \mathrm{Tr}(\delta_h \bar{g}, U)\, \mathrm{Tr}(\delta_{h^{-1}\ell} \bar{g}, V) x_\ell, \end{aligned}$$

where in the second sum, ℓ has been replaced by $h^{-1}\ell$.

On the other hand,

$$f_g(U \otimes V) = \sum_{\ell \in C(g)} \mathrm{Tr}(\delta_\ell \bar{g}, U \otimes V) x_\ell.$$

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The action of $\delta_\ell \bar{g}$ on $U \otimes V$ is given by $\Delta(\delta_\ell \bar{g}) = \sum_{h \in G} \gamma_g(h, h^{-1}\ell)\delta_h \bar{g} \otimes \delta_{h^{-1}\ell}\bar{g}$. Therefore

$$\begin{aligned} f_g(U \otimes V) &= \sum_{\ell \in C(g), h \in G} \gamma_g(h, h^{-1}\ell) \operatorname{Tr}(\delta_h \bar{g}, U) \operatorname{Tr}(\delta_{h^{-1}\ell}\bar{g}, V) x_\ell \\ &= \sum_{h, \ell \in C(g)} \theta_g(h, h^{-1}\ell) \operatorname{Tr}(\delta_h \bar{g}, U) \operatorname{Tr}(\delta_{h^{-1}\ell}\bar{g}, V) x_\ell \\ &\quad + \sum_{\ell \in C(g), h \in G - C(g)} \gamma_g(h, h^{-1}\ell) \operatorname{Tr}(\delta_h \bar{g}, U) \operatorname{Tr}(\delta_{h^{-1}\ell}\bar{g}, V) x_\ell, \end{aligned}$$

as θ_g is equal to γ_g on $C(g) \times C(g)$. Comparing the two calculations, we need only see that the second sum above is equal to zero. But when $h \notin C(g)$, the element $\delta_h \bar{g}$ is nilpotent, so its trace on any module is zero. ∎

We next show that the image of f_g is contained in $Z\left(\mathbf{C}^{\theta_g} C(g)\right)$. We first give a basis of $Z\left(\mathbf{C}^{\theta_g} C(g)\right)$. We say that $h \in C(g)$ is θ_g-*regular* if $\theta_g(h, c) = \theta_g(c, h)$ for all $c \in C_{C(g)}(h)$. It may be checked that there is a basis of $Z\left(\mathbf{C}^{\theta_g} C(g)\right)$ indexed by conjugacy classes of θ_g-regular elements h of $C(g)$. Each basis element is a sum $\sum a_{h^y} x_{h^y}$ over all elements h^y in the $C(g)$-conjugacy class of h, with coefficients satisfying $a_{h^y} = \theta_g(h, y)/\theta_g(y, h^y)$.

Letting $h, y \in C(g)$, note that

$$\delta_{h^y} \bar{g} = \bar{y}^{-1} \delta_h \bar{y} \bar{g} = \bar{y}^{-1} \left(\frac{\theta_h(y, g)}{\theta_h(g, y)} \delta_h \bar{g} \right) \bar{y}.$$

If U is any $D^\omega(G)$-module then, we have

$$\operatorname{Tr}(\delta_{h^y} \bar{g}, U) = \frac{\theta_h(y, g)}{\theta_h(g, y)} \operatorname{Tr}(\delta_h \bar{g}, U) = \frac{\theta_g(h, y)}{\theta_g(y, h^y)} \operatorname{Tr}(\delta_h \bar{g}, U).$$

The second equality follows from the definition of θ, as $h, y \in C(g)$. The 2-cocycle property of θ_g on $C(g)$ yields

$$\frac{\theta_g(h, cy)}{\theta_g(cy, h^{cy})} = \frac{\theta_g(h, c)\theta_g(h, y)}{\theta_g(c, h)\theta_g(y, h^y)}$$

whenever $c \in C_{C(g)}(h)$. These equations imply that $\operatorname{Tr}(\delta_{h^y} \bar{g}, U) = 0$ if h is not θ_g-regular, and that $f_g(U)$ is in $Z\left(\mathbf{C}^{\theta_g} C(g)\right)$.

Next we will need to consider characters of a twisted group algebra $\mathbf{C}^\alpha H$, where α is a 2-cocycle on a finite group H. For facts about characters of finite dimensional algebras, we refer to [6, Section 9B], and note that $\mathbf{C}^\alpha H$ is semisimple [11]. Specifically, a character is a function from $\mathbf{C}^\alpha H$ to $\mathbf{C}$ given by the trace function on a $\mathbf{C}^\alpha H$-module. These functions are not algebra homomorphisms in general. The character of an arbitrary $\mathbf{C}^\alpha H$-module is the sum of the characters of its irreducible direct summands. The irreducible characters, that is characters corresponding to irreducible modules, are linearly independent over $\mathbf{C}$.

We will consider the vector space generated by $\mathbf{C}^\alpha H$-modules with direct sum for addition. By the trace function of an element of this space, we will mean the corresponding linear combination of trace functions of $\mathbf{C}^\alpha H$-modules.

THEOREM 2.2. *The product π of the maps f_g induces an isomorphism of algebras*

$$R\big(D^\omega(G)\big) \xrightarrow{\sim} \prod_g Z\big(\mathbf{C}^{\theta_g} C(g)\big),$$

the product taken over a set of representatives g of conjugacy classes of G. In particular, the representation ring $R\big(D^\omega(G)\big)$ is semisimple.

PROOF. Lemma 2.1 and the discussion following it show that each f_g is an algebra homomorphism with image contained in $Z\big(\mathbf{C}^{\theta_g} C(g)\big)$. It remains to prove that π is a bijection. Let $a \in R\big(D^\omega(G)\big)$ with $\pi(a) = 0$. Fix $x \in G$ and let $g \in C(x)$. Then $f_g(a) = 0$ implies that $\mathrm{Tr}(\delta_x \bar{g}, a) = 0$, where we have extended the trace function linearly to $R\big(D^\omega(G)\big)$. But $\mathrm{Tr}(\delta_x \bar{g}, a) = \mathrm{Tr}(\delta_x \bar{g}, a\delta_x)$, where here we may consider $a\delta_x$ as an element of the vector space generated by $\mathbf{C}^{\theta_x} C(x)$-modules via the equivalence of categories given in Lemma 1.1. In other words, we have $\mathrm{Tr}(x_g, a\delta_x) = 0$ for all $g \in C(x)$. As the irreducible characters of $\mathbf{C}^{\theta_x} C(x)$ are linearly independent, $a\delta_x = 0$ in the vector space generated by $\mathbf{C}^{\theta_x} C(x)$-modules. But this is true for all $x \in G$. As $D^\omega(G)$-modules are determined by their x-components, we have $a = 0$. Thus we have shown that π is injective.

To complete the proof, we must show that π is also surjective, which will follow once we see that the dimensions of $R\big(D^\omega(G)\big)$ and $\prod_g Z\big(\mathbf{C}^{\theta_g} C(g)\big)$ are the same. By Lemma 1.1, the dimension of $R\big(D^\omega(G)\big)$ is equal to the sum, over a set of representatives g of conjugacy classes of G, of the number of irreducible $\mathbf{C}^{\theta_g} C(g)$-modules. This is also the dimension of $\prod_g Z\big(\mathbf{C}^{\theta_g} C(g)\big)$. ∎

Now we may write down a table consisting of the values of the complete set of characters of the semisimple algebra $R\big(D^\omega(G)\big)$ on the irreducible $D^\omega(G)$-modules. Such a character maps a $D^\omega(G)$-module U to

$$\frac{1}{\deg \rho} \sum_{h \in C(g)} \mathrm{Tr}(\delta_h \bar{g}, U) \rho(x_h),$$

where $g \in G$ and ρ is an irreducible character of $\mathbf{C}^{\theta_g} C(g)$; that is, ρ is the trace function of an irreducible $\mathbf{C}^{\theta_g} C(g)$-module. This table is analogous to the character table of a finite group.

3. **Brauer characters.** Here we present an analog of Brauer characters for a twisted group algebra, and some results about their connection to Brauer characters of a representation group.

Let H be a finite group, k an algebraically closed field of characteristic p, and $\alpha: H \times H \to k^\times$ a 2-cocycle. As before, we denote by $k^\alpha H$ the twisted group algebra that has basis $\{x_h\}_{h \in H}$ with multiplication given by $x_h x_\ell = \alpha(h, \ell) x_{h\ell}$. We will assume that α is normalized so that x_1 is the multiplicative identity of $k^\alpha H$, and also that α is of finite order m dividing $|H|_{p'}$. A calculation shows that the elements x_h of $k^\alpha H$ all have finite order dividing $m \cdot |H|$. The representations of $k^\alpha H$ correspond to projective k-representations of H with associated 2-cocycle α [11, 12].

Let E be a *representation group* for H. That is, there is a central extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} H \longrightarrow 1$$

of H such that any projective k-representation of H may be lifted to an ordinary k-representation of E which on a section of π is projectively equivalent to the original one [6, 12]. We will recall the explicit formula for so lifting a projective representation $\rho_U: H \longrightarrow \mathrm{GL}(U)$, with associated 2-cocycle α, to an ordinary representation $\sigma_U: E \longrightarrow \mathrm{GL}(U)$ [6, 11]. We will denote by $\mathcal{L}(U)$ the vector space U with the structure of a kE-module via σ_U. Fix a section $\{y_g\}_{g\in H}$ of π with $y_1 = 1$, and let $a: H \times H \longrightarrow A$ be the corresponding 2-cocycle provided by multiplication in E. That is, $y_g y_h = a(g,h)y_{gh}$ for all $g, h \in H$. Let $\lambda \in \hat{A}$ be the character such that

$$\lambda\big(a(g,h)\big) = \alpha(g,h)\mu(g)\mu(h)\mu(gh)^{-1}$$

for some function $\mu: G \longrightarrow k^\times$, and all $g, h \in H$. Then $\sigma_U(ay_g) = \lambda(a)\mu(g)\rho_U(g)$ for all $a \in A, g \in H$. Note that μ and λ depend only on α and a, and not on the given representation.

Another way to construct σ_U is by using the isomorphism $kE \simeq \prod_{\alpha\in T} k^\alpha H$, where T is a transversal of the group $B^2(H, k^\times)$ of 2-coboundaries in the group $Z^2(H, k^\times)$ of 2-cocycles [12, Theorem 3.3.5]. The explicit map $kE \longrightarrow k^\alpha H$ is given by sending ay_g to $\lambda(a)\mu(g)x_g$, where λ and μ are as above. A $k^\alpha H$-module may be lifted to kE via this surjection. Morphisms of $k^\alpha H$-modules then become morphisms of kE-modules, and short exact sequences are preserved.

Let $r = |E|'_p$. The r-th roots of unity in k form a cyclic group of order r. For the rest of this section, fix an isomorphism between the group of r-th roots of unity in k and the group of r-th roots of unity in $\mathbf{C}$.

Let U be a $k^\alpha H$-module, and g a p-regular element of H; that is, p does not divide the order of g. The element x_g of $k^\alpha H$ has finite order dividing $r = |E|'_p$, so its eigenvalues on U are r-th roots of unity, and may be lifted to $\mathbf{C}$ via the isomorphism chosen above. Define the *Brauer character* of U to be the function from the set of p-regular elements of H to $\mathbf{C}$ given by sending g to the sum of the lifts of the eigenvalues of x_g on U to $\mathbf{C}$. In case α is the trivial 2-cocycle, this Brauer character is the usual one for H. For a given p-regular element $g \in H$, define the function $t_{g,\alpha}$ on $k^\alpha H$-modules by setting $t_{g,\alpha}(U)$ equal to the Brauer character of U evaluated at g. If E is a representation group of H, we will denote by $t_y(V)$ the Brauer character of a kE-module V evaluated at a p-regular element y of E.

LEMMA 3.1. *Let U be a $k^\alpha H$-module, g a p-regular element of H, and $\mathcal{L}(U)$ the lift of U to a kE-module defined above. Then $t_{y_g}\big(\mathcal{L}(U)\big)$ is a nonzero scalar multiple of $t_{g,\alpha}(U)$. Further, the same nonzero scalar multiple is involved in any such lift, so that $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of $t_{g,\alpha}$.*

PROOF. It follows from the construction of the module $\mathcal{L}(U)$ that the action of x_g on U and that of y_g on $\mathcal{L}(U)$ differ by the scalar $\mu(g)$. It may be checked that $\mu(g)$ is an r-th root of unity, where $r = |E|_{p'}^2$. Lifting eigenvalues to $\mathbf{C}$ then, and summing, we obtain the desired result. ∎

We will need to exploit the connection between induced kE-modules and induced $k^\alpha H$-modules. Let L be a subgroup of H, and consider α also to be a 2-cocycle of L by restriction. If U is a $k^\alpha L$-module, then U may be lifted to an ordinary $k\pi^{-1}(L)$-module $\mathcal{L}(U)$ using the same procedure as for H and E (see [12, Lemma 3.3.1]).

LEMMA 3.2. *Let U be a $k^\alpha L$-module and $U \otimes_{k^\alpha L} k^\alpha H$ the corresponding induced $k^\alpha H$-module. Then $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha H)$ is isomorphic to the induced kE-module $\mathcal{L}(U) \otimes_{kJ} kE$, where $J = \pi^{-1}(L)$.*

PROOF. Let $g_1, \ldots, g_t$ be a transversal of L in H with $g_1 = 1$. Then $y_{g_1}, \ldots, y_{g_t}$ is a transversal of J in E. Define a linear map from $\mathcal{L}(U) \otimes_{kJ} kE$ to $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha H)$ by

$$u \otimes y_{g_i} \longmapsto \mu(g_i)u \otimes x_{g_i},$$

where we have identified the underlying vector spaces of U and $\mathcal{L}(U)$. It may be checked that this map is an isomorphism of kE-modules. ∎

Let g be a p-regular element of H, and consider the function $t_{g,\alpha}$ on $k^\alpha\langle g\rangle$-modules, by replacing H by $\langle g\rangle$ in the above development. We may extend $t_{g,\alpha}$ linearly to the $\mathbf{C}$-vector space generated by $k^\alpha\langle g\rangle$-modules (with direct sum for addition). In the next section, we will use $t_{g,\alpha}$ to define a function generalizing f_g from Section 2. The following lemma will be used in the proof that such a function is an algebra homomorphism.

LEMMA 3.3. *Let g be a p-regular element of H, and L a proper subgroup of $\langle g\rangle$. Then the kernel of $t_{g,\alpha}$ contains any $k^\alpha\langle g\rangle$-module induced from $k^\alpha L$.*

PROOF. Let E be a representation group for $\langle g\rangle$. By Lemma 3.1, $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of $t_{g,\alpha}$. Therefore it suffices to show that $t_{y_g}\big(\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha\langle g\rangle)\big) = 0$ for any $k^\alpha L$-module U. By Lemma 3.2, $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha\langle g\rangle) \simeq \mathcal{L}(U) \otimes_{kJ} kE$ as kE-modules, where $J = \pi^{-1}(L)$ is a proper subgroup of E containing A. But t_{y_g} factors through $\langle y_g\rangle \leq E$; that is $t_{y_g} = \tilde{t}_{y_g} \circ \downarrow^E_{\langle y_g\rangle}$, where $\tilde{t}_{y_g}$ is the corresponding trace function on $k\langle y_g\rangle$-modules, and $\downarrow^E_{\langle y_g\rangle}$ denotes restriction of modules [3]. Using $\uparrow^E_J$ to denote induction of modules, we apply $\downarrow^E_{\langle y_g\rangle}$ to $\mathcal{L}(U) \uparrow^E_J = \mathcal{L}(U) \otimes_{kJ} kE$, and the Mackey Subgroup Theorem yields

$$\mathcal{L}(U) \uparrow^E_J \downarrow^E_{\langle y_g\rangle} \simeq \sum_{\sigma \in J\backslash E/\langle y_g\rangle} \mathcal{L}(U)^\sigma \downarrow^{J^\sigma}_{J^\sigma \cap \langle y_g\rangle} \uparrow^{\langle y_g\rangle}_{J^\sigma \cap \langle y_g\rangle}.$$

Now, as E is generated by y_g and A, and J^σ is a proper subgroup of E containing A, we have $y_g \notin J^\sigma$. Therefore $J^\sigma \cap \langle y_g\rangle$ is a proper subgroup of $\langle y_g\rangle$. Thus $\mathcal{L}(U) \uparrow^E_J \downarrow^E_{\langle y_g\rangle}$ is in the image of induction from proper subgroups of $\langle y_g\rangle$, and so is in the kernel of $\tilde{t}_{y_g}$ [3]. ∎

We return to the situation where ω is a 3-cocycle of a finite group G, of finite order dividing $|G|_{p'}$. The next lemma deals with a certain direct summand of the tensor product $U \otimes V$ of two $D^\omega(G)$-modules U and V.

LEMMA 3.4. *Let U and V be $D^\omega(G)$-modules, $\ell \in G$, and $g \in C(\ell)$. Consider $(U \otimes V)\delta_\ell$ to be a $k^{\theta_\ell}\langle g\rangle$-module via Lemma 1.1 and restriction. Then the subspace*

$$\sum_{h \in G - C(g)} U\delta_h \otimes V\delta_{h^{-1}\ell}$$

of $(U \otimes V)\delta_\ell$ is a $k^{\theta_\ell}\langle g\rangle$-submodule, and is a sum of modules induced from proper subgroups of $\langle g\rangle$.

PROOF. First note that the given subspace is indeed a $k^{\theta_\ell}\langle g\rangle$-submodule, as the sum is over elements h not in $C(g)$, and $\delta_h \bar{g} = \bar{g}\delta_{g^{-1}hg}$. We may assume that U and V are indecomposable $D^\omega(G)$-modules. By Lemma 1.1 then, there are elements $x, y \in G$ such that

$$U \simeq U\delta_x \otimes_{S^\omega(x)} D^\omega(x) \quad \text{and} \quad V \simeq V\delta_y \otimes_{S^\omega(y)} D^\omega(x).$$

Consider a $k^{\theta_\ell}\langle g\rangle$-submodule

$$M = \sum_{(x,y)} U\delta_x \otimes V\delta_y$$

of $\sum_{h \in G - C(g)} U\delta_h \otimes V\delta_{h^{-1}\ell}$, where the pair (x, y) ranges over all $\langle g\rangle$-conjugates of $(h_0, h_0^{-1}\ell)$ for a fixed $h_0 \in G - C(g)$. The $\langle g\rangle$-conjugates of $(h_0, h_0^{-1}\ell)$ form an orbit as a $\langle g\rangle$-set which is isomorphic to $L \backslash \langle g\rangle$ for a proper subgroup L of $\langle g\rangle$, as h_0 is not in $C(g)$. It may be checked that M is induced from the $k^{\theta_\ell}L$-module $U\delta_{h_0} \otimes V\delta_{h_0^{-1}\ell}$; the map from $(U\delta_{h_0} \otimes V\delta_{h_0^{-1}\ell}) \otimes_{k^{\theta_\ell}L} k^{\theta_\ell}\langle g\rangle$ to M defined by

$$(u\delta_{h_0} \otimes v\delta_{h_0^{-1}\ell}) \otimes \delta_\ell \overline{g^j} \mapsto u\delta_{h_0}\overline{g^j} \otimes v\delta_{h_0^{-1}\ell}\overline{g^j}\gamma_{g^j}(h_0, h_0^{-1}\ell)$$

is a $k^{\theta_\ell}\langle g\rangle$-module isomorphism. ∎

4. Characters of the Grothendieck ring. In this section, we will give a generalization of Theorem 2.2 to an arbitrary algebraically closed field k of characteristic p. Let r be the square of the least common multiple of all $|E|_{p'}$, where E ranges over a set of representation groups for the subgroups of G. For the rest of this section, fix an isomorphism between the r-th roots of unity in k and in $\mathbf{C}$.

Given $g \in G$, the values of θ_g are $|G|_{p'}$-th roots of unity in k, by the restriction on the order of ω. Therefore, they may be lifted to $\mathbf{C}$ by our chosen isomorphism of r-th roots of unity in k and in $\mathbf{C}$. Denote by $\Theta_g : C(g) \times C(g) \to \mathbf{C}^\times$ the lift of the map θ_g with respect to this isomorphism, and note that Θ_g is a 2-cocycle on $C(g)$ as well, but with values in $\mathbf{C}^\times$. Thus we may form the twisted group algebra $\mathbf{C}^{\Theta_g}C(g)$.

Now let g be a p-regular element of G. Define a linear function $f_g : R\big(D^\omega(G)\big) \to \mathbf{C}^{\Theta_g}C(g)$ by

$$f_g(U) = \sum_{h \in C(g)} t_{g,\theta_h}(U\delta_h)x_h,$$

where $U\delta_h$ is considered to be a $k^{\theta_h}C(h)$-module via Lemma 1.1, and the functions t_{g,θ_h} are defined in Section 3. We will show that f_g is an algebra homomorphism and that its image is contained in $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$.

We point out that the proofs of results in Section 2 cannot be translated directly in this situation. The functions t_{g,θ_h} require looking at eigenvalues of invertible operators. By contrast, the proof of Lemma 2.1 depended on considering traces of nilpotent operators.

THEOREM 4.1. *The function* $f_g\colon R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\Theta_g}C(g)$ *is an algebra homomorphism.*

PROOF. It may be checked that f_g takes the trivial $D^\omega(G)$-module k to the identity x_1 of $\mathbf{C}^{\Theta_g}C(g)$. Now let U and V be $D^\omega(G)$-modules. Then

$$\begin{aligned} f_g(U)f_g(V) &= \sum_{h,\ell\in C(g)} \Theta_g(h,\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_\ell}(V\delta_\ell)x_{h\ell} \\ &= \sum_{h,\ell\in C(g)} \Theta_g(h,h^{-1}\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_{h^{-1}\ell}}(V\delta_{h^{-1}\ell})x_\ell, \end{aligned}$$

where in the second sum, we have replaced ℓ by $h^{-1}\ell$. On the other hand,

$$\begin{aligned} f_g(U\otimes V) &= \sum_{\ell\in C(g)} t_{g,\theta_\ell}\big((U\otimes V)\delta_\ell\big)x_\ell \\ &= \sum_{\ell\in C(g)} t_{g,\theta_\ell}\Big(\sum_{h\in G} U\delta_h\otimes V\delta_{h^{-1}\ell}\Big)x_\ell. \end{aligned}$$

Now t_{g,θ_ℓ} is the sum of lifts of eigenvalues of $\delta_\ell\bar{g}$ on a module to $\mathbf{C}$, and the action of $\delta_\ell\bar{g}$ on a tensor product is given by

$$\Delta(\delta_\ell\bar{g}) = \sum_{h\in G}\gamma_g(h,h^{-1}\ell)\delta_h\bar{g}\otimes\delta_{h^{-1}\ell}\bar{g}.$$

Note that when $h\in C(g)$, $U\delta_h\otimes V\delta_{h^{-1}\ell}$ is a $k^{\theta_\ell}\langle g\rangle$-submodule of $(U\otimes V)\delta_\ell$, and $\gamma_g(h,h^{-1}\ell)$ lifts to $\Theta_g(h,h^{-1}\ell)$ in $\mathbf{C}$, since $\gamma_g=\theta_g$ on $C(g)\times C(g)$. Therefore

$$\begin{aligned} f_g(U\otimes V) = &\sum_{\ell,h\in C(g)} \Theta_g(h,h^{-1}\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_{h^{-1}\ell}}(V\delta_{h^{-1}\ell})x_\ell \\ &+ \sum_{\ell\in C(g)} t_{g,\theta_\ell}\Big(\sum_{h\in G-C(g)} U\delta_h\otimes V\delta_{h^{-1}\ell}\Big)x_\ell. \end{aligned}$$

By Lemmas 3.3 and 3.4, the second summand above is zero, and so $f_g(U\otimes V)$ is equal to $f_g(U)f_g(V)$. ∎

For the next lemma, we recall the description of $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$ given in Section 2. It has a basis indexed by Θ_g-regular elements h, each basis element being a sum $\sum a_{h^y}x_{h^y}$ over all elements h^y in the $C(g)$-conjugacy class of h, with coefficients satisfying $a_{h^y} = \Theta_g(h,y)/\Theta_g(y,h^y)$. We note that by the construction of Θ_g, h is Θ_g-regular if and only if h is θ_g-regular.

LEMMA 4.2. *The image of* f_g *is contained in* $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$.

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PROOF. Let U be a $D^\omega(G)$-module and $h, y \in C(g)$. As $\delta_{h^y}\bar{g} = \bar{y}^{-1}(\frac{\theta_h(y,g)}{\theta_h(g,y)}\delta_h\bar{g})\bar{y}$, the action of $\frac{\theta_h(y,g)}{\theta_h(g,y)}\delta_h\bar{g}$ on $U\delta_h$ corresponds to the action of $\delta_{h^y}\bar{g}$ on $U\delta_{h^y}$ under the vector space isomorphism sending $U\delta_h$ to $U\delta_{h^y}$ defined by right action by $\bar{y}$. Thus the eigenvalues of $\delta_{h^y}\bar{g}$ on $U\delta_{h^y}$ are $\frac{\theta_h(y,g)}{\theta_h(g,y)}$ times the eigenvalues of $\delta_h\bar{g}$ on $U\delta_h$. So we have

$$t_{g,\theta_{h^y}}(U\delta_{h^y}) = \frac{\Theta_h(y,g)}{\Theta_h(g,y)} t_{g,\theta_h}(U\delta_h).$$

If h is Θ_g-regular, this is what we needed to show, as $\frac{\Theta_h(y,g)}{\Theta_h(g,y)} = \frac{\Theta_g(h,y)}{\Theta_g(y,h^y)}$ by the definitions. If h is not Θ_g-regular, these equations force $t_{g,\theta_h}(U\delta_h) = 0$ (see Section 2). ∎

Finally, we will consider the Grothendieck ring $\mathcal{R}\big(D^\omega(G)\big)$, which is the quotient of $R\big(D^\omega(G)\big)$ by the ideal $R_0(D^\omega(G))$ of short exact sequences defined in Section 1.

LEMMA 4.3. *The ideal $R_0\big(D^\omega(G)\big)$ is the kernel of the homomorphism*

$$\pi: R\big(D^\omega(G)\big) \longrightarrow \prod_g Z\big(\mathbf{C}^{\Theta_g} C(g)\big)$$

given by the product of the maps f_g taken over a set of representatives g of conjugacy classes of p-regular elements of G.

PROOF. First let $0 \longrightarrow U' \longrightarrow U \longrightarrow U'' \longrightarrow 0$ be a short exact sequence of $D^\omega(G)$-modules. Now, $U' = \sum_{x \in G} U'\delta_x$ as a direct sum of vector spaces, and similarly for the others. The $D^\omega(G)$-module maps induce linear maps $U'\delta_x \longrightarrow U\delta_x \longrightarrow U''\delta_x$, which correspond to maps of $k^{\theta_x}C(x)$-modules by Lemma 1.1. That is, for each $x \in G$, we have a short exact sequence of $k^{\theta_x}C(x)$-modules

$$0 \longrightarrow U'\delta_x \longrightarrow U\delta_x \longrightarrow U''\delta_x \longrightarrow 0.$$

Fix $x \in G$ and let g be a p-regular element of $C(x)$. Let E be a representation group of $C(x)$, with $E/A \simeq C(x)$ for a central subgroup A of E. We will use notation as in Section 3. We claim that $t_{g,\theta_x}(U\delta_x - U'\delta_x - U''\delta_x) = 0$. Now, the above short exact sequence of $k^{\theta_x}C(x)$-modules may be lifted, by the discussion at the beginning of Section 3, to a short exact sequence $\mathcal{L}(U\delta_x - U'\delta_x - U''\delta_x)$ of kE-modules. By Lemma 3.1, $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of t_{g,θ_x}. Any short exact sequence is in the kernel of t_{y_g} [3], so we do have $t_{g,\theta_x}(U\delta_x - U'\delta_x - U''\delta_x) = 0$. Therefore, $f_g(U - U' - U'') = 0$ for all p-regular elements g of G, and so $R_0\big(D^\omega(G)\big)$ is contained in the kernel of π.

Now let $b \in R\big(D^\omega(G)\big)$ with $\pi(b) = 0$. Fix $x \in G$ and let g be a p-regular element of $C(x)$. Then $f_g(b) = 0$ implies that $t_{g,\theta_x}(b\delta_x) = 0$ for all $x \in C(g)$, where $b\delta_x$ is considered to be an element of the vector space generated by $k^{\theta_x}C(x)$-modules. Lift $b\delta_x$ to E; that is, lift all modules involved in $b\delta_x$ to E and form the corresponding linear combination $\mathcal{L}(b\delta_x)$. By Lemma 3.1, $\mathcal{L}(b\delta_x)$ is in the kernel of t_{y_g} for all p-regular elements g of $C(x)$. Therefore $\mathcal{L}(b\delta_x)$ is in the kernel of t_{ay_g} for all elements ay_g of E with g a p-regular element and $a \in A$, as a simply acts as the scalar $\lambda(a)\mu(1)$ on any such module lifted

from $C(x)$. But all p-regular elements of E are of this form. This implies that $\mathcal{L}(b\delta_x)$ is in the ideal of short exact sequences of kE-modules [2]. The short exact sequences which appear in $\mathcal{L}(b\delta_x)$ involve only kE-modules which correspond to $k^{\theta_x}C(x)$-modules. Thus $b\delta_x$ is in the vector space generated by short exact sequences of $k^{\theta_x}C(x)$-modules, and this is true for all $x \in G$. By the characterization of $D^\omega(G)$-modules discussed in Section 1, each is determined by its x-components, where x ranges over a set of representatives of conjugacy classes of G. And a short exact sequence of $k^{\theta_x}C(x)$-modules corresponds to a short exact sequence of $D^\omega(G)$-modules, as noted in the text preceding Lemma 1.1. Thus $b \in R_0\big(D^\omega(G)\big)$. ∎

THEOREM 4.4. *The product π of the maps f_g induces an algebra isomorphism*

$$\mathcal{R}\big(D^\omega(G)\big) \xrightarrow{\sim} \prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big),$$

the product taken over a set of representatives g of conjugacy classes of p-regular elements of G. In particular, the Grothendieck ring $\mathcal{R}\big(D^\omega(G)\big)$ is semisimple.

PROOF. By Lemma 4.3, the homomorphism π induces an injection from $\mathcal{R}\big(D^\omega(G)\big)$ to $\prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$. It remains to prove that π is a surjection, which will follow once we see that the dimensions of these two finite dimensional algebras are the same.

The dimension of the Grothendieck ring $\mathcal{R}\big(D^\omega(G)\big)$ is equal to the number of irreducible $D^\omega(G)$-modules. By the characterization given in Lemma 1.1, this is

$$\sum_x \text{(number of irreducible } k^{\theta_x}C(x)\text{-modules)},$$

where the sum is taken over a set of representatives x of conjugacy classes in G. But the number of irreducible $k^{\theta_x}C(x)$-modules is the number of conjugacy classes in $C(x)$ of p-regular θ_x-regular elements [5, 12].

On the other hand, the dimension of $\prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$ is equal to

$$\sum_g \text{(number of } \Theta_g\text{-regular conjugacy classes in } C(g)),$$

the sum taken over a set of representatives g of conjugacy classes of p-regular elements in G.

But these quantities are just two different ways of counting the orbits in the G-set

$$\{(x,g) \mid x \in G, g \text{ is a } p\text{-regular element of } G, xg = gx, \text{ and } x \text{ is } \Theta_g\text{-regular}\},$$

where G acts by conjugation on each factor. This follows from the observation that x is a Θ_g-regular element of $C(g)$ if and only if g is a θ_x-regular element of $C(x)$. To see this, note that when g and x commute,

$$\frac{\theta_g(x,c)}{\theta_g(c,x)} = \frac{\theta_x(c,g)}{\theta_x(g,c)}$$

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whenever $c \in C(x) \cap C(g)$. ∎

The theorem allows us to write down a table consisting of the values of the complete set of characters of the semisimple ring $\mathcal{R}\left(D^{\omega}(G)\right)$ on the images of the irreducible $D^{\omega}(G)$-modules. Such a character maps the image of a $D^{\omega}(G)$-module U to

$$\frac{1}{\deg \rho} \sum_{h \in C(g)} t_{g,\theta_h}(U\delta_h)\rho(x_h),$$

where $g \in G$ and ρ is an irreducible character of $\mathbf{C}^{\Theta_g}C(g)$. These characters therefore distinguish $D^{\omega}(G)$-modules up to their composition factors, and this table of characters is analogous to the Brauer character table of a finite group.

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