19
The complex Klein–Gordon field

This chapter is a supplement to the material for a real scalar field. Much of the previous chapter applies here too.

19.1 The action

The free-field action is given by

$$S = \int (\mathrm{d}x) \Big\{ \hbar^2 c^2 (\partial^\mu \phi_A)^\dagger (\partial_\mu \phi_A) + m^2 c^4 \phi_A^* \phi_A \\ + V(\phi_A^\dagger \phi_A) - J_A^\dagger \phi_A - J_A \phi_A^\dagger \Big\}. \qquad (19.1)$$

The field now has effectively twice as many components as the real scalar field, coming from the real and imaginary parts.

19.2 Field equations and continuity

Since the complex field and its complex conjugate are independent variables, there are equations of motion for both of these. Varying S first with respect to ϕ_A^*, we obtain

$$\delta S = \int (\mathrm{d}x) \delta\phi_A^* \left(-\hbar^2 c^2 \Box\, \phi_A + m^2 c^4 \phi_A + V'(\phi_A^2, \phi_A) - J_A \right) \\ + \frac{1}{c} \int \mathrm{d}\sigma^\mu \delta\phi_A^* (\partial_\mu \phi_A) = 0, \qquad (19.2)$$

which gives rise to the field equation

$$\left(-\Box + \frac{m^2 c^2}{\hbar^2} \right) \phi_A(x) = (\hbar^2 c^2)^{-1} J_A(x), \qquad (19.3)$$

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and the continuity condition over a spacelike hyper-surface,

$$\Delta(\partial_\sigma \phi(x)) = \Delta \Pi_\sigma = 0, \tag{19.4}$$

identifies the conjugate momentum Π_σ. Conversely, the variation with respect to ϕ_A gives the conjugate field equations, which give rise to the field equation

$$\left(-\Box + \frac{m^2c^2}{\hbar^2}\right)\phi_A^*(x) = (\hbar^2c^2)^{-1}J_A^*(x), \tag{19.5}$$

and the corresponding continuity condition over a spacelike hyper-surface,

$$\Delta(\partial_\sigma \phi^*(x)) = \Delta \Pi_\sigma^* = 0. \tag{19.6}$$

19.3 Free-field solutions

The free-field solutions for the complex scalar field have the same form as those of the real scalar field,

$$\phi(x) = \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\phi(k)\mathrm{e}^{\mathrm{i}kx}\ \delta\left(\hbar^2c^2k^2 + m^2c^4\right); \tag{19.7}$$

however, the Fourier coefficients are no longer restricted by eqn. (18.15).

19.4 Formal solution by Green functions

The formal solution to the field equation may be expressed in terms of a Green function $G_{AB}(x,x')$ by

$$\phi_A(x) = \int (\mathrm{d}x')G_{AB}(x,x')J_B(x'), \tag{19.8}$$

where the Green function satisfies the equation

$$\left(-\overset{x}{\Box} + \frac{m^2c^2}{\hbar^2}\right)G_{AB}(x,x') = \delta(x,x')\delta_{AB}. \tag{19.9}$$

Similarly,

$$\phi_A^\dagger(x) = \int (\mathrm{d}x')J_B^\dagger(x')G_{BA}(x',x). \tag{19.10}$$

Note that, although the fields are designated as conjugates ϕ and $\phi^\dagger$, this relationship is not necessarily preserved by the choice of boundary conditions. If the time-ordered, or Feynman Green function is used (which represents virtual processes), then the resulting fields do not remain conjugate to one another over time. The retarded Green function does preserve the conjugate relationship between the fields, since it is real.

19.5 Conserved norm and probability

Let s be independent of x, and consider the phase transformation of the kinetic part of the action:

$$
\begin{aligned}
S &= \int (\mathrm{d}x)\hbar^2 c^2 \left\{(\partial^\mu \mathrm{e}^{-\mathrm{i}s}\phi^*)(\partial_\mu \mathrm{e}^{\mathrm{i}s}\phi)\right\} \\
\delta S &= \int (\mathrm{d}x) \left[(\partial^\mu \phi^*(-\mathrm{i}\delta s)\mathrm{e}^{-\mathrm{i}s})(\partial_\mu \phi \mathrm{e}^{\mathrm{i}s}) + c.c.\right] \\
&= \int (\mathrm{d}x)\delta s(\partial_\mu J^\mu),
\end{aligned}
\tag{19.11}
$$

where

$$
J^\mu = -\mathrm{i}\hbar^2 c^2(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*). \tag{19.12}
$$

The conserved 'charge' of this symmetry can now be used as the definition of the inner product between fields:

$$
(\phi_1, \phi_2) = \mathrm{i}\hbar c \int \mathrm{d}\sigma^\mu(\phi_1^*\partial_\mu\phi_2 - (\partial_\mu\phi_1)^*\phi_2), \tag{19.13}
$$

or, in non-covariant form,

$$
(\phi_1, \phi_2) = \mathrm{i}\hbar c \int \mathrm{d}\sigma\,(\phi_1^*\partial_0\phi_2 - (\partial_0\phi_1)^*\phi_2). \tag{19.14}
$$

This is now our notion of probability.

19.6 The energy–momentum tensor

The application of Noether's theorem for spacetime translations leads to a symmetrical energy–momentum tensor. Although the sign of the energy is ambiguous for the Klein–Gordon field, we can define a Hamiltonian with the interpretation of an energy density which is positive definite, from the zero–zero component of the energy–momentum tensor. Using the action and the formula (11.44), we have

$$
\begin{aligned}
\theta_{00} &= \frac{\partial\mathcal{L}}{\partial(\partial^0\phi_A)}(\partial_0\phi_A) + \frac{\partial\mathcal{L}}{\partial(\partial^0\phi_A^*)}(\partial_0\phi_A^*) - \mathcal{L}g_{00} \\
&= \hbar^2c^2\left[(\partial_0\phi_A^*)(\partial_0\phi_A) + (\partial_i\phi_A^*)(\partial_i\phi_A)\right] + m^2c^4 + V(\phi).
\end{aligned}
\tag{19.15}
$$

Thus, the last line defines the Hamiltonian density $\mathcal{H}$, and the Hamiltonian is given by

$$
H = \int \mathrm{d}\sigma\mathcal{H}. \tag{19.16}
$$

The off-diagonal spacetime components define a momentum:

$$
\begin{aligned}
\theta_{0i} = \theta_{i0} &= \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_A)}(\partial_i \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_A^*)}(\partial_i \phi_A^*) \\
&= \hbar^2 c^2 \left\{ (\partial_0 \phi_A^*)(\partial_i \phi_A) + (\partial_0 \phi_A)(\partial_i \phi_A^*) \right\}. \qquad (19.17)
\end{aligned}
$$

Taking the integral over all space enables us to integrate by parts and show that this quantity is the expectation value (inner product) of the momentum:

$$
\begin{aligned}
\int \mathrm{d}\sigma \, \theta_{0i} &= \hbar^2 c \int \mathrm{d}\sigma \left(\phi^* \partial_i \partial_0 \phi - (\partial_0 \phi^*) \partial_i \phi \right) \\
&= -(\phi, p_i c \phi), \qquad (19.18)
\end{aligned}
$$

where $p = -\mathrm{i}\hbar\partial_i$. The diagonal space components are given by

$$
\begin{aligned}
\theta_{ii} &= \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A)}(\partial_i \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A^*)}(\partial_i \phi_A^*) - \mathcal{L} \\
&= 2\hbar^2 c^2 (\partial_i \phi^*)(\partial_i \phi) - \mathcal{L}, \qquad (19.19)
\end{aligned}
$$

where i is not summed. Similarly, the off-diagonal 'stress' components are given by

$$
\begin{aligned}
\theta_{ij} &= \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A)}(\partial_j \phi_A) + \frac{\partial \mathcal{L}}{\partial(\partial^i \phi_A^*)}(\partial_j \phi_A^*) \\
&= \hbar^2 c^2 \left\{ (\partial_i \phi_A^*)(\partial_j \phi_A) + (\partial_j \phi_A^*)(\partial_i \phi_A) \right\} \\
&= \hbar^{-1} c (\phi_A, p_i p_j \phi_A). \qquad (19.20)
\end{aligned}
$$

We see that the trace over spatial components in $n+1$ dimensions is

$$
\sum_i \theta_{ii} = \mathcal{H} - 2m^2 c^4 \phi_A^2 - 2V(\phi) + (n-1)\mathcal{L}, \qquad (19.21)
$$

so that the full trace gives

$$
\theta^\mu_{\ \mu} = g^{\mu\nu}\theta_{\nu\mu} = -2m^2 c^4 \phi_A^2 - 2V(\phi) + (n-1)\mathcal{L}. \qquad (19.22)
$$

This vanishes for $m = V = 0$ in $1+1$ dimensions.

19.7 Formulation as a two-component real field

The real and imaginary components of the complex scalar field can be parametrized as a two-component vector or real fields φ_A, where $A = 1, 2$. Define

$$
\Phi(x) = \frac{1}{\sqrt{2}} \left(\varphi_1(x) + \mathrm{i}\varphi_2(x) \right). \qquad (19.23)
$$

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Substituting in, and comparing real and imaginary parts, one finds

$$D_\mu = \partial_\mu + \mathrm{i}eA_\mu, \tag{19.24}$$

or

$$D_\mu\varphi_A = \partial_\mu\varphi_A - e\epsilon_{AB}A_\mu\varphi_B. \tag{19.25}$$

The action becomes

$$S = \int (\mathrm{d}x)\left\{\frac{1}{2}(D^\mu\varphi_A)(D_\mu\varphi_A) + \frac{1}{2}m^2\varphi_A\varphi_A - J_A\varphi_A\right\}. \tag{19.26}$$

Notice that the operation charge conjugation is seen trivially here, due to the presence of ϵ_{AB}, as the swapping of field labels.

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