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CHARACTERIZATIONS FOR PRIME SEMILATTICES

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1. Introduction. Throughout this paper when we refer to a semilattice S we shall mean that S is a meet semilattice. We shall denote the infimum of two elements a, b of S by $a \wedge b$, and the supremum, if it exists, by $a \vee b$. A *prime* semilattice is a meet semilattice such that the infimum distributes over all existing finite suprema, in the sense that if $x_1 \vee x_2 \dots \vee x_n$ exists then $(x \wedge x_1) \vee (x \wedge x_2) \dots \vee (x \wedge x_n)$ exists for any x and equals $x \wedge (x_1 \vee x_2 \dots \vee x_n)$. Such semilattices were first studied by Balbes [**1**] and we use his terminology.

A non-empty subset F of S is a *filter* provided that $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$. A proper filter F is prime if, whenever $x_1 \vee x_2 \dots \vee x_n$ exists and is an element of F then $x_i \in F$ for some $i \in \{1, 2, \dots, n\}$. A *semi-ideal* of S is a non-empty subset I of S such that if $b \in I$ and $a \leqq b$ then $a \in I$. We call I an ideal if, further, when $x_1 \vee x_2 \vee \dots \vee x_n$ exists such that $x_i \in I$ for all $i \in \{1, 2, \dots, n\}$, then

$$x_1 \vee x_2 \vee \dots \vee x_n \in I.$$

An ideal P is called prime if $a \wedge b \in P$ implies $a \in P$ or $b \in P$.

Prime semilattices were first characterized by Balbes [**1**]. Recently, several characterizations for prime semilattices were also obtained by Y. S. Pawar and N. K. Thakare [**6**]. Unfortunately, some of the proofs given in their paper [**6**] were wrong. In this paper, we shall correct all these mistakes. Several new characterizations for prime semilattices are obtained. Maximal filters, maximal ideals, prime filters and prime ideals in prime semilattices are studied. We then consider primeness for finite semilattices. We prove that a finite semilattice S is D_2 if and only if S is prime. As B. M. Schein claimed that D_2 is not equivalent to D_n in general, our theorem shows that such counter-example cannot be found in finite semilattices.

2. Ideal extensions.

Definition 2.1. Let I be a semi-ideal of S. An *extension of I by x* is defined to be the set

$$\langle x, I\rangle = \{a \in S | a \wedge x \in I\},$$

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where x is an arbitrary element in S.

This terminology comes from the fact that $I \subset \langle x, I\rangle$ for all $x \in S$. It should be noted that $\langle x, I\rangle$ need not be an ideal of S, but it is always a semi-ideal of S.

Definition 2.2. Let I be an ideal of S. If $\langle x, I\rangle$ is also an ideal of S for some $x \in S$, then $\langle x, I\rangle$ is called the *extended ideal of I by x*. For simplicity, we denote $\langle x, (y]\rangle$ by $\langle x, y\rangle$.

PROPOSITION 2.3. *Any extension of a prime ideal in S is a prime ideal.*

Proof. Let P be a prime ideal of S. If $x \in P$, then $\langle x, P\rangle = S$ and there is nothing to prove. Assume $x \notin P$, and let $y \in \langle x, P\rangle$. Because P is prime, $y \wedge x \in P$ implies $y \in P$. Thus $\langle x, P\rangle = P$ is a prime ideal.

COROLLARY 2.4. *Let I be a non-empty subset of a semilattice S. If*

$$I = \bigcup_{i\in\Gamma} P_i \quad \textit{or} \quad I = \bigcap_{i\in\Gamma} P_i,$$

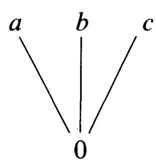
where P_i's are prime ideals, Γ is an index set, then $\langle x, I\rangle$ is an ideal.

Proof. Let $I = \cap_{i\in\Gamma} P_i$. Then

$$\langle x, I\rangle = \langle x, \bigcap_{i\in\Gamma} P_i\rangle = \bigcap_{i\in\Gamma} \langle x, P_i\rangle = \bigcap_{i\in\Gamma'} \langle x, P_i\rangle = \bigcap_{i\in\Gamma'} P_i$$

where $\Gamma' = \{i \in \Gamma : x \notin P_i\}$ since $x \in P_i$ implies $\langle x, P_i\rangle = S$. Trivially, non-empty intersection of ideals is an ideal, thus $\langle x, I\rangle$ is an ideal. Similarly for $I = \cup_{i\in\Gamma} P_i$.

Remark 2.5. The converse of Proposition 2.3 is not generally true, that is in a semilattice, an ideal with a prime extension need not be prime. For example, let S be the semilattice $\{0, a, b, c\}$ with Hasse diagram shown below



Then $I = \{0, a\}$ is an ideal of S but not prime. It is clear that $\langle c, I\rangle = \{0, a, b\}$, $\langle b, I\rangle = \{0, a, c\}$, $\langle a, I\rangle = S$, $\langle 0, I\rangle = S$ are all prime ideals.

PROPOSITION 2.6. *Let I be a subset of S. If for all $x \in S$, $\langle x, I\rangle$ is an ideal of S, then I must be an ideal.*

Proof. Let $y \leqq z$ and $z \in I$. Consider $\langle z, I\rangle$. $\langle z, I\rangle$ is an ideal and $z \in \langle z, I\rangle$, hence $y \in \langle z, I\rangle$ and so $y = y \wedge z \in I$. Now suppose

$x_1 \vee x_2 \vee \ldots \vee x_n$ exists in S and $x_i \in I$ $(i = 1, 2, \ldots, n)$. Consider $\langle x_1 \vee x_2 \ldots \vee x_n, I\rangle$. Clearly,

$$x_i \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) = x_i \in I,$$

therefore

$$x_i \in \langle x_1 \vee x_2 \ldots \vee x_n, I\rangle.$$

Since $\langle x_1 \vee x_2 \ldots \vee x_n, I\rangle$ is an ideal and $x_1 \vee x_2 \ldots \vee x_n$ exists in S, so

$$x_1 \vee x_2 \ldots \vee x_n \in \langle x_1 \vee x_2 \ldots \vee x_n, I\rangle.$$

Hence

$$x_1 \vee x_2 \vee \ldots \vee x_n = (x_1 \vee x_2 \vee \ldots \vee x_n) \wedge (x_1 \vee x_2 \ldots \vee x_n) \in I$$

and I is indeed an ideal of S.

We observe that the ordering of elements in S is also related to the reverse set inclusion of their corresponding ideal extensions. In fact, we have the following proposition.

PROPOSITION 2.7. *Let x, y be elements of S. Then $x \leqq y$ if and only if $\langle x, I\rangle \supseteq \langle y, I\rangle$ for all ideals I of S.*

Proof. ($\Rightarrow$) Let $z \in \langle y, I\rangle$. Then $z \wedge y \in I$. Since $x \leqq y$ and I is an ideal, we have $z \wedge x \in I$. This implies that $z \in \langle x, I\rangle$. Thus $\langle y, I\rangle \subseteq \langle x, I\rangle$.

($\Leftarrow$) Consider $(y]$, the principal ideal generated by y. Obviously, $x \wedge y \in (y]$, so

$$x \in \langle y, y\rangle \subseteq \langle x, y\rangle$$

by hypothesis. Hence $x = x \wedge x \in (y]$, that is, $x \leqq y$.

In fact, Proposition 2.7 holds when the word "ideal" is replaced by "semi-ideal".

3. Characterizations for prime semilattices by ideals. In this section, we shall characterize prime semilattices by ideals.

The following theorem was obtained by Y. S. Pawar and N. K. Thakare in [**6**, Theorem 6, p. 294].

THEOREM 3.1. *For any semilattice S the following are equivalent.*

(1) *S is prime.*

(2) *$\langle a, b\rangle$ is an ideal for all a, b in S.*

(3) *$\langle a, b\rangle$ is an ideal for all $b \leqq a$.*

It should be noted that the proof of this theorem is wrong. Pawar and

Thakare assumed the existence of $(a \wedge x_2) \vee (a \wedge x_2) \vee \ldots \vee (a \wedge x_n)$ at the first instance and then proved that

$$(a \wedge x_1) \vee (a \wedge x_2) \vee \ldots \vee (a \wedge x_n) = a \wedge (x_1 \vee x_2 \ldots \vee x_n).$$

In fact, there is no way of being sure that $(a \wedge x_1) \vee (a \wedge x_2) \ldots \vee (a \wedge x_n)$ exists in a semilattice.

We now give a new characterization for prime semilattices which includes Pawar and Thakare's result as its trivial corollary. Their result is therefore true in spite of the mistake in their proof.

THEOREM 3.2. *For any semilattice S the following are equivalent.*

(i) *S is prime.*

(ii) *$\langle a, I\rangle$ is an ideal for any ideal I and $a \in S$.*

(iii) *$\langle a, I\rangle$ is an ideal for any ideal I such that I is bounded by a, i.e., $i \leqq a$ for any $i \in I$.*

(iv) *$\langle a, b\rangle$ is an ideal for any $b \leqq a$.*

(v) *$\langle a, b\rangle$ is an ideal for any $a, b \in S$.*

Proof. (i) $\Rightarrow$ (ii). If $x \leqq y \in \langle a, I\rangle$, then

$$x \wedge a \leqq y \wedge a \in I.$$

This implies that $x \wedge a \in I$ and hence $x \in \langle a, I\rangle$.

Now assume $x_i \in \langle a, I\rangle$ for all $i \in \{1, 2, \ldots, n\}$ and $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in S. Then

$$(x_1 \vee x_2 \vee \ldots \vee x_n) \wedge a = (x_1 \wedge a) \vee \ldots \vee (x_n \wedge a) \in I$$

as $x_i \wedge a \in I$ for all $i \in \{1, 2, \ldots, n\}$. Hence,

$$(x_1 \vee x_2 \vee \ldots \vee x_n) \in \langle a, I\rangle.$$

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (iv). Since $b \leqq a$, the principal ideal $(b]$ is bounded by a. So by (iii), $\langle a, b\rangle$ is an ideal.

(iv) $\Rightarrow$ (v). Since $\langle a, b\rangle = \langle a, a \wedge b\rangle$, $\langle a, b\rangle$ is an ideal by (iv).

(v) $\Rightarrow$ (i). Assume $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in S and $a \in S$. For any $y \geqq a \wedge x_i$, $i = 1, 2, \ldots, n$, we have $x_i \in \langle a, y\rangle$.

By (v), $\langle a, y\rangle$ is an ideal of S, so

$$x_1 \vee x_2 \vee \ldots \vee x_n \in \langle a, y\rangle.$$

Thus,

$$a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) \in (y],$$

and

$$a \wedge (x_1 \vee x_2 \ldots \vee x_n) \leqq y.$$

Clearly, $a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n)$ is an upper bound of $\{a \wedge x_i\}_{i \in \{1,2,\ldots,n\}}$. This means that $a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n)$ is the least upper bound of $\{a \wedge x_i\}_{i \in \{1,2,\ldots,n\}}$. Hence,

$$a \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) = (a \wedge x_1) \vee (a \wedge x_2) \vee \ldots \vee (a \wedge x_n).$$

S is therefore a prime semilattice.

The following lemma, which has certain interest of its own, is necessary for the characterization of prime semilattice in Section 4.

LEMMA 3.3. *A filter F of a semilattice S is prime if and only if F^c, the set complement of F in S, is a prime ideal.*

The proof of this lemma is easy and therefore it is omitted.

4. Applications of Balbes-Stone theorem. A modified version of Balbes Theorem has already been obtained by C. S. Hoo and K. P. Shum in [**8**]. The corrected version of Balbes' result (Theorem 2.2. of [**8**]) is as follows.

THEOREM 4.1. (Balbes) *In a semilattice S, the following are equivalent.*

(i) *S is a prime semilattice.*

(ii) *If F is a filter in S and I is a non-empty subset of S, disjoint from F and such that $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in I whenever $x_1, x_2, \ldots, x_n \in I$, then there exists a prime filter F' such that $F \subset F'$ and $F' \cap I = \emptyset$.*

(iii) *If F is a filter in S and I is an ideal of S disjoint from F, then there exists a prime filter F' such that $F \subset F'$ and $F' \cap I = \emptyset$.*

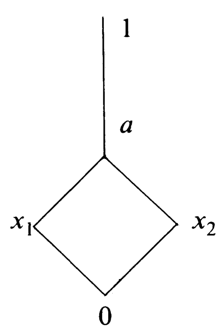
(iv) *If $x \nleq y$, then there exists a prime filter F' such that $x \in F'$ and $y \notin F'$.*

Hoo and Shum [**8**] added the equivalent condition (iii) to Balbes' original statements. In [**6**], Pawar and Thakare tried to produce a proof for (i) $\Rightarrow$ (iii) (Theorem 1 [**6**], p. 292). Unfortunately, the proof supplied by them was wrong. In their proof, they stated the following sentence:

"Further $i_1 \wedge i_2 \wedge \ldots \wedge i_n \geqq (q_1 \wedge q_2 \wedge \ldots \wedge q_n) \wedge (x_1 \vee x_2 \vee \ldots \vee x_n)$" where $i_1, i_2, \ldots, i_n$ are elements of an ideal I and $q_1, q_2, \ldots, q_n$ are elements of a filter Q disjoint from I. [Theorem 1 [**6**], p. 292, line 19].

The above statement is not correct as can be seen from the following counter example.

Example 4.2. Let $S = \{0, x_1, x_2, a, 1\}$ with Hasse diagram shown below.



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It is easily checked that S is a prime semilattice. Clearly $I = \{0, x_1, x_2, a\}$ is an ideal, $Q = \{1\}$ is a filter disjoint from I. Take $i_1 = x_1$, $i_2 = x_2$, $q_1 = 1$, $q_2 = 1$, then $i_1 = q_1 \wedge x_1$, $i_2 = q_2 \wedge x_2$. But

$$i_1 \wedge i_2 = x_1 \wedge x_2 = 0 \nleqq (q_1 \wedge q_2) \wedge (x_1 \vee x_2) = a.$$

Thus the statement given by Pawar and Thakare is incorrect.

We now correct their proof.

We shall call (i) $\Leftrightarrow$ (iii) Stone's Theorem because such theorem was first obtained by M. H. Stone in distributive lattice [**3**, p. 74, Theorem 15].

THEOREM 4.3. (i) $\Leftrightarrow$ (iii).

Proof. (i) $\Rightarrow$ (iii). Let S be a prime semilattice. Then by Zorn's lemma, there exists a filter Q maximal with the property that it contains F and is disjoint from I. Suppose $x_1 \vee x_2 \vee \ldots \vee x_n$ exists in Q with $x_k \notin Q$ for $k = 1, 2, \ldots, n$. Then $[Q \cup \{x_1\})$, $[Q \cup \{x_2\})$, $\ldots [Q \cup \{x_n\})$ have a non-empty intersection with I. Hence, there exists $i_k \in I$, $q_k \in Q$, $k = 1, 2, \ldots, n$ such that

$$i_k \geqq q_k \wedge x_k \geqq q_1 \wedge q_2 \wedge \ldots \wedge q_n \wedge x_k.$$

Because I is an ideal,

$$q_1 \wedge q_2 \wedge \ldots \wedge q_n \wedge x_k \in I.$$

Also, by the primeness of S,

$$(q_1 \wedge q_2 \wedge \ldots \wedge q_n) \wedge (x_1 \vee x_2 \vee \ldots \vee x_n)$$
$$= q_1 \wedge q_2 \wedge \ldots q_n \wedge x_1) \vee \ldots$$
$$\vee (q_1 \wedge q_2 \wedge \ldots \wedge q_n \wedge x_n) \in I,$$

as I is an ideal. Thus $I \cap Q \neq \emptyset$ which contradicts the choice of Q. Hence Q is a prime filter. The proof is completed.

(iii) $\Rightarrow$ (i). This follows as in [**6**].

We now call Theorem 4.3 as Balbes-Stone theorem and apply this theorem to give two new characterizations for prime semilattices.

THEOREM 4.4. *A semilattice S is prime if and only if $\langle x, P\rangle \supseteq \langle y, P\rangle$ for any prime ideal P of $S \Rightarrow x \leqq y$.*

Proof. ($\Rightarrow$) Suppose that $x \nleqq y$. Since S is prime, by Balbes-Stone Theorem (iv), there is a prime filter F such that $x \in F$ but $y \notin F$. By Lemma 3.3 $I = F^c$ is a prime ideal such that $x \notin I$ and $y \in I$. As $y \in I$, so

$$S = \langle y, I\rangle \subseteq \langle x, I\rangle.$$

Then $x \in \langle x, I\rangle$ implies $x \in I = F^c$. This contradicts $x \in F$. Thus $x \leqq y$.

($\Leftarrow$) Suppose that $x \not\leq y$. Then there is a prime ideal I such that $\langle x, I\rangle \not\supseteq \langle y, I\rangle$. Hence, there is an element z such that $z \wedge x \notin I$ and $z \wedge y \in I$. Therefore $x \notin \langle z, I\rangle$ and $y \in \langle z, I\rangle$. By Proposition 2.3, $\langle z, I\rangle$ is a prime ideal. Let $F = \langle z, I\rangle^c$ be the set complement of $\langle z, I\rangle$ in S. Then, by Lemma 3.3, F is a prime filter of S. Thus, $x \in F$ and $y \notin F$. Apply Balbes-Stone Theorem (iv), S is a prime semilattice. The proof is completed.

THEOREM 4.5. *A semilattice S is prime if and only if for any ideal I in S, $I = \cap \{J{:}J$ is a prime ideal containing $I\}$.*

Proof. ($\Leftarrow$) By Corollary 2.4, $\langle x, I\rangle$ is an ideal for any ideal I and $x \in S$. Consequently S is prime, by Theorem 3.2.

($\Rightarrow$) Clearly, $I \subseteq \cap \{J{:}J$ is a prime ideal containing $I\}$. Now assume that there exists an element $x \notin I$ and $x \in J$ for any prime ideal J containing I. Then $[x) \cap I = \emptyset$. By Theorem 4.3, there exists a prime filter F such that $[x) \subseteq F$ and $F \cap I = \emptyset$. Apply Lemma 3.3, we know that $J_0 = F^c$ is a prime ideal containing I. But $x \notin J_0$ contradicts our assumption. Thus

$$I = \cap \{J{:}J \text{ is a prime ideal containing } I\}.$$

The following statement was stated by Grätzer in ([**3**], Corollary 18, p. 75):

"Every ideal of a distributive lattice is the intersection of all prime ideals containing it."

As we notice that the concepts of primeness and distributivity given in ([**3**], p. 36) are exactly the same in a lattice, so by Theorem 4.5, we can modify Grätzer's result as follows:

COROLLARY 4.6. *A lattice L is distributive if and only if every ideal of L is the intersection of all prime ideals containing it.*

5. Filters, ideals and complemented semilattices. Filters and ideals in prime semilattices were studied by Pawar and Thakare in [**6**]. The following theorem was stated by them.

THEOREM 5.1. *Let S be a prime semilattice with* 0 *and* 1 *in which the complement of every maximal ideal is a maximal filter. Then S is complemented.* ([**6**], Theorem 9, p. 296).

Unfortunately, the proof of this theorem provided by Pawar and Thakare is wrong. In this section, we shall amend this result which leads to a thorough study of complemented prime semilattices.

As the condition "complement of every maximal ideal is a maximal filter" is a rather ambiguous statement, we shall study what it can mean. We first obtain the following proposition.

PROPOSITION 5.2. *Let S be a prime semilattice with* 1 *in which N^c (the set complement of N) is a maximal filter for every maximal ideal N of S. Then for any ideal M the following are equivalent.*

(i) *M is a maximal ideal.*
(ii) *$Q = M^c$ is a maximal filter.*
(iii) *$Q = M^c$ is a prime filter.*
(iv) *M is a prime ideal.*
(v) *M is a minimal prime ideal.*
(vi) *Q is a minimal prime filter.*

Note. It is a well-known fact that a maximal ideal of a prime semilattice must be prime. The converse is easily shown to be false in general. However, from Proposition 5.2, we note that if the complement of a maximal ideal is a maximal filter, then there is no difference between maximal and prime ideals in prime semilattices with 1.

Proof. (i) $\Rightarrow$ (ii). This follows by assumption.

(ii) $\Rightarrow$ (iii). Q is a maximal (proper) filter, so there exists an element $x \in S - Q$. Apply Balbes theorem (or more explicitly, apply Theorem 2 in **[6]**), there exists a prime filter F such that $F \supseteq Q$ and $x \notin F$. By the maximality of Q, we must have $Q = F$. Thus Q is a prime filter.

(iii) $\Rightarrow$ (iv). See Lemma 3.3.

(iv) $\Rightarrow$ (i). Suppose M is not a maximal ideal of S. Because S has 1, by Zorn's lemma there is a maximal ideal M_1 such that $M \subsetneq M_1 \subsetneq S$. By assumption, $Q_1 = M_1^c$ is a maximal filter. As $Q = M^c$ is a filter and $Q \supsetneq Q_1$, this contradicts the maximality of Q_1. Hence M is a maximal ideal.

(iv) $\Rightarrow$ (v). Let M' be a prime ideal of S such that $M' \subseteq M \subsetneq S$. Since M' is a maximal ideal (by (i) $\Leftrightarrow$ (iv)) thus $M' = M$. Therefore M is a minimal prime ideal.

(v) $\Rightarrow$ (iv). This is trivial.

(iii) $\Rightarrow$ (vi). Suppose that $Q_1 \subseteq Q \subsetneq S$ and Q_1 is a prime filter. Then, because (iii) $\Leftrightarrow$ (ii), Q_1 is a maximal filter. Therefore $Q_1 = Q$, and so Q is a minimal prime filter.

(vi) $\Rightarrow$ (iii). This is trivial.

The proof is thus completed.

Remark. In **[6]**, Pawar and Thakare also proved (ii) $\Rightarrow$ (iii). But they had to assume that the semilattice has 0. In fact, our proof shows that the assumption of zero is superfluous.

In general, the complement of a maximal ideal M in a prime semilattice with 1 need not be a maximal filter, but it is a filter.

PROPOSITION 5.3. *Let S be a prime semilattice with* 1. *Then for every maximal ideal M, $Q = M^c$ is a filter.*

Proof. Suppose that $x \wedge y \in M^c = Q$. If $x \in M$, then $x \geqq x \wedge y$ implies that $x \wedge y \in M$ since M is an ideal. This contradicts the fact that $x \wedge y \in M^c$. Hence, $x \in Q$ and $y \in Q$. Suppose that $x \in Q, y \in Q$ and $x \wedge y \in M$. Since M is a maximal ideal, we have $(MU\{x\}] = S$. Therefore

$$x \vee m_1 \vee \ldots \vee m_k = 1$$

for some $m_1, m_2, \ldots m_k \in M$ (see [**2**]), and hence

$$y \wedge (x \vee m_1 \vee \ldots \vee m_k) = y.$$

As S is a prime semilattice, so

$$y = (y \wedge x) \vee (y \wedge m_1) \vee \ldots \vee (y \wedge m_k).$$

Because $y \wedge x \in M$, and $y \wedge m_i \in M$ for all $i = 1, \ldots, k$, we have $y \in M$, which is a contradiction. Thus $x \wedge y \in Q$. Hence $Q = M^c$ is a filter.

Note. In proving Theorem 5.1, Pawar and Thakare considered the ideal $\langle a, 0\rangle$ for any $a \in S$ and assumed $a \vee x \neq 1$ for all $x \in \langle a, 0\rangle$. Then they considered the set

$$A = \{ \{a, x\}^u : x \in \langle a, 0\rangle \}$$

and let $J = A^l$, where for any non-empty subset Y of S, Y^u and Y^l denote the set of all upper bounds of Y and the set of all lower bounds of Y respectively. As J is a proper ideal in S and $1 \in S$, $J \subseteq M$ for some maximal ideal M in S. Then they claimed that $a \in S - M$. However, we observe that $0 \in \langle a, 0\rangle$ and in fact a is the smallest element in A, so $J = A^l = (a]$. Therefore $a \in J \subseteq M$. Thus Pawar and Thakare's proof is incorrect. Also, they claimed that $\langle a, 0\rangle \subseteq M$, but such claim is not justifiable. ([**6**], Theorem 9, p. 296, lines 5-12).

In order to amend the mistakes made by Pawar and Thakare [**6**], we find a rather interesting result which, in fact, is a characterization for complemented semilattices. The following lemma, which has certain interest of its own, is crucial for such characterization.

LEMMA 5.4. *Let S be a prime semilattice with* 0 *and* 1, *then the following statements are equivalent*:

(i) *For any maximal ideal M, M^c is a maximal filter.*

(ii) *For any a in S, there exists a sequence of elements $\{t_i\}_{i\in\{1,\ldots,n\}}$ in $\langle a, 0\rangle$ such that*

$$a \vee t_1 \vee t_2 \vee \ldots \vee t_n = 1.$$

Proof. (i) $\Rightarrow$ (ii). First, we claim that

$$((a] \cup \langle a, 0\rangle] = S \quad \text{for all } a \in S.$$

Suppose if possible that $((a] \cup \langle a, 0\rangle] \subsetneq S$. Then, by Zorn's lemma, there exists a maximal ideal M such that

$$((a] \cup \langle a, 0\rangle] \subseteq M \subsetneq S.$$

This is because S has 1. Now let $Q = M^c$, by assumption (i), Q is a maximal filter. Hence $a \notin Q$ implies $[Q \cup \{a\}) = S$. Consequently, we can pick some element $q \in Q$ such that $q \wedge a = 0$, which implies that $q \in Q \cap M = \emptyset$, a contradiction. Therefore we have proved that

$$((a] \cup \langle a, 0\rangle] = S.$$

Thus, there exists $a' \leqq a$, $\{t_i\}_{i=1}^n$ in $\langle a, 0\rangle$ such that

$$a' \vee t_1 \vee \ldots \vee t_n = 1.$$

Clearly $a \vee t_1 \vee \ldots \vee t_n = 1$, so (ii) is established.

(ii) $\Rightarrow$ (i). Let M be a maximal ideal, M^c be its set complement in S. Clearly, M^c is a filter (by Proposition 5.3). Assume that there exists a filter F such that $M^c \subsetneq F \subsetneq S$, then $M \cap F \neq \emptyset$, that is, there exists $a \in M \cap F$. Now, by (ii) there exists $\{t_i\}_{i=1}^n$ in $\langle a, 0\rangle$ such that

$$a \vee t_1 \vee \ldots \vee t_n = 1.$$

If $t_i \in M$ for all $i = 1, \ldots, n$ then

$$1 = a \vee t_1 \vee \ldots \vee t_n \in M,$$

a contradiction. Therefore $t_i \in M^c \subset F$ for some $i \in \{1, \ldots, n\}$, but then $0 = a \wedge t_i \in F$, again, a contradiction. Thus M^c is indeed a maximal filter of S.

Remark 5.5. Let S be a prime semilattice with 0 and 1 in which the statement of Proposition 5.4 (ii) holds. Then statements (i)-(vi) in Proposition 5.2 are all equivalent.

THEOREM 5.6. *Let S be a prime semilattice with* 0 *and* 1, *then the following statements are equivalent.*

(I) (i) *For any maximal ideal M, M^c is a maximal filter.*

(ii) *S is pseudocomplemented.*

(II) *S is complemented.*

Proof. (I) $\Rightarrow$ (II). By Lemma 5.4, for any a in S, there exists a sequence of elements $\{t_i\}_{i\in\{1,\ldots,n\}}$ in $\langle a, 0\rangle$ such that

$$a \vee t_1 \vee t_2 \vee \ldots \vee t_n = 1.$$

Now let a^* be the pseudocomplement of a. Then $t_i \leqq a^*$ for all $i \in \{1, \ldots, n\}$, clearly $a \vee a^* = 1$, as $a^* \wedge a = 0$, therefore S is complemented.

(II) $\Rightarrow$ (I). By virtue of Lemma 5.4, clearly (i) is established. Since S is complemented, for any $a \in S$, there exists $t \in S$ such that $a \wedge t = 0$ and

$a \vee t = 1$. We claim that $a^* = t$, for if $a \wedge x = 0$ then

$$x = (a \vee t) \wedge x = (a \wedge x) \vee (t \wedge x) = t \wedge x,$$

that is, $x \leqq t$. Hence, S is pseudocomplemented.

COROLLARY 5.7. *Let S be a prime semilattice with* 0 *and* 1. *If S is a complemented semilattice, then the six statements as stated in Proposition* 5.2 *are equivalent.*

6. Relative annihilators. In this section, we study semilattices in which

$$\langle a, b\rangle \vee \langle b, a\rangle = S.$$

$\langle a, b\rangle \vee \langle b, a\rangle$ means the ideal generated by $\langle a, b\rangle \cup \langle b, a\rangle$. We shall see that these semilattices can be characterized as those in which the filters containing any given prime filter form a chain. In fact, such characterization for lattices has already been obtained by Mandelker [**5**]. Most of his results can be transferred verbatim to semilattices with only slight modifications.

In [**6**], Pawar and Thakare proved the following theorem.

THEOREM 6.1. *In a prime semilattice if the filters containing the given filter F form a chain then F is prime and*

$$\langle a, b\rangle \vee \langle b, a\rangle = S.$$

Also, they said that it will be interesting to see whether the condition $\langle a, b\rangle \vee \langle b, a\rangle = S$ is also necessary [**6**, Theorem 8, p. 295].

The proof of Theorem 6.1 is essentially taken from the necessity part of Mandelker's theorem [**5**, Theorem 3] as Stone's theorem for distributive lattices also holds for prime semilattice (Theorem 4.3). However, if one goes through Mandelker's proof, it can be seen that the sufficient part of Mandelker's theorem also holds for prime semilattices. This answers the question of Pawar and Thakare without any difficulty. We would like to point out that in proving Theorem 6.1, Pawar and Thakare did not mention the chain condition which is a key step in the proof.

We now extend Mandelker's theorem [**5**, Theorem 3] from lattices to semilattices as follows.

LEMMA 6.2. *In any prime semilattice S, each of the following conditions on a given filter F implies the next.*

(i) *For any element a and b of S, there exists an element x in F such that $a \wedge x$ and $b \wedge x$ are comparable.*

(ii) *The filters containing F form a chain.*

(iii) *The prime filters containing F form a chain.*

(iv) *F is prime.*

(v) *F contains a prime filter.*

Proof. (i) $\Rightarrow$ (ii). Let G and H be filters containing F, and suppose that they are not comparable. Choose $a \in G - H$ and $b \in H - G$. Choose $x \in F \subset G$ such that $a \wedge x$ and $b \wedge x$ are comparable. Without loss of generality, we may assume $a \wedge x \leqq b \wedge x$. Since $x \in G$, we have $a \wedge x \in G$. This implies $b \wedge x \in G$ and hence $b \in G$ for G is a filter. This contradicts the fact that $b \notin G$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (iv). Let the prime filters containing F be denoted by $\{F_\alpha\}$. Since the F_α's form a chain, it is easily seen that $(\cap_\alpha F_\alpha) = F'$ is also a prime filter such that $F \subseteq F'$. If $F \subsetneq F'$, then there exists $x \in F' - F$. Because S is a prime semilattice, by Balbes-Stone Theorem (Theorem 4.3), there exists a prime filter G such that $F \subseteq G$, $x \notin G$. But $F' \subseteq G$, a contradiction. Thus $F = F'$ and so F is a prime filter.

(iv) $\Rightarrow$ (v). This is trivial.

THEOREM 6.3. *Let S be a prime semilattice. The identity*

$$\langle a, b\rangle \vee \langle b, a\rangle = S$$

holds for any a, b in S if and only if all the conditions of the lemma are equivalent.

Proof. ($\Rightarrow$) It suffices to show that (v) implies (i). Let P be a prime filter contained in F and choose $z \in P$. For any a and b in S we have

$$\begin{aligned}\langle a, b\rangle \vee \langle b, a\rangle &= S \\ &= \{s{:}s = z_1 \vee z_2 \vee \ldots \vee z_l, \\ &\qquad z_i \in \langle a, b\rangle \cup \langle b, a\rangle\}.\end{aligned}$$

This is because S is prime [**2**, Theorem 1.1]. Thus

$$z = x_1 \vee x_2 \vee \ldots \vee x_n \vee y_1 \vee y_2 \vee \ldots \vee y_m,$$

where $x_i \in \langle a, b\rangle$, $y_j \in \langle b, a\rangle$. Since P is a prime filter, $z \in P$ implies x_i or $y_j \in P$ for some i, j. Without loss of generality, let $x_1 \in P \subseteq F$. Consider $a \wedge x_1$. Clearly $a \wedge x_1 \leqq b$ as $x_1 \in \langle a, b\rangle$. Thus

$$a \wedge x_1 \leqq b \wedge x_1.$$

Therefore, there exists $x_1 \in F$ such that $a \wedge x_1$ and $b \wedge x_1$ are comparable.

($\Leftarrow$) Suppose that there exist a and b of S which are such that

$$J = \langle a, b\rangle \vee \langle b, a\rangle$$

is a proper ideal. Then by Stone's Theorem for prime semilattices (Theorem 4.3), there exists a prime filter P such that $J \cap P = \emptyset$. Thus P satisfies condition (iv) in Lemma 6.2 and hence satisfies condition (i). Thus, there exists $x \in P$ such that $a \wedge x$ and $b \wedge x$ are comparable.

Without loss of generality, suppose

$$a \wedge x \leqq b \wedge x \leqq b.$$

Then $x \in \langle a, b\rangle \subseteq J$. But x is in P, which contradicts $J \cap P = \emptyset$. Hence

$$J = \langle a, b\rangle \vee \langle b, a\rangle = S.$$

The proof is completed.

Thus the question raised in the paper of Pawar and Thakare [**6**] is now completely solved by Theorem 6.3.

7. m-distributive semilattices. We call a semilattice S *m-distributive* if and only if it satisfies the equation

$$(D_m): y \wedge (x_1 \vee x_2 \ldots \vee x_m)$$
$$= (y \wedge x_1) \vee (y \wedge x_2) \ldots \vee (y \wedge x_m)$$

in the sense that whenever the left hand side exists then so does the right hand side and the two sides are equal. This idea was first put forward by Schein [**7**]. We will denote the class of m-distributive semilattices by D_m for each $m = 2, 3 \ldots$ and the class of semilattices which satisfy (D_m) for each $m = 2, 3, \ldots$ by D_ω. In fact the elements of D_ω are just the prime semilattices. According to [**9**, p. 222], a semilattice S is distributive if whenever $x, a, b \in S$ are such that $x \geqq a \wedge b$, there exists $a', b' \in S$ with $a' \geqq a$, $b' \geqq b$ and $x = a' \wedge b'$. If we denote the class of distributive semilattices by D then the following series of inequalities holds:

$$D_2 \supseteq D_3 \supseteq \ldots \supseteq D_\omega \supseteq D.$$

In 1972, B. M. Schein [**7**] conjectured that D_2 and D_m $(m > 2)$ are not equivalent. Also the referee of [**6**] asked whether D_2 is sufficient for a meet semilattice S to be D_ω. As far as we know, in the literature, Schein's conjecture is not yet solved. In this section, we shall show that $D_2 = D_\omega$ in finite semilattices. Thus a partial answer to the above question is obtained.

THEOREM 7.1. *Let S be a finite semilattice. Then S is D_2 if and only if S is a prime semilattice.*

Proof. ($\Leftarrow$) This is trivial.

($\Rightarrow$) Suppose $t_n = x_1 \vee x_2 \vee \ldots \vee x_n$ exists for some $x_1, \ldots, x_n \in S$. We first claim the existence of $x_1 \vee x_2 \vee \ldots \vee x_k$, for every $1 \leqq k \leqq n$. Now,

$$\{x_1, x_2, \ldots, x_k\}^u \neq \emptyset$$

(where Y^u means the upper bounds of the set Y); this is because

$$t_n \in \{x_1, x_2, \ldots, x_k\}^u.$$

Since S is a finite semilattice,

$$t_k = \wedge\{x_1, x_2, \ldots, x_k\}^u$$

exists. Clearly t_k is the least upper bound of $\{x_1, x_2, \ldots, x_k\}$. Therefore

$$t_k = x_1 \vee x_2 \vee \ldots \vee x_k.$$

Our claim is established. Hence, for all $x \in S$, we have

$$x \wedge (x_1 \vee x_2 \vee \ldots \vee x_n) = x \wedge (t_{n-1} \vee x_n)$$

as $t_{n-1} = x_1 \vee x_2 \ldots \vee x_{n-1}$ exists. Therefore, by D_2, we have

$$\begin{aligned} x \wedge (t_{n-1} \vee x_n) &= (x \wedge t_{n-1}) \vee (x \wedge x_n) \\ &= [x \wedge (t_{n-2} \vee x_{n-1})] \vee (x \wedge x_n) \\ &= (x \wedge t_{n-2}) \vee (x \wedge x_{n-1}) \vee (x \wedge x_n) \\ &= \ldots = (x \wedge x_1) \vee (x \wedge x_2) \vee \ldots \vee (x \wedge x_n). \end{aligned}$$

Thus, S is a prime semilattice.

From Theorem 7.1, it is now clear that a counter-example showing that D_2 is not equal to D_n (as conjectured by Schein in [**7**]) does not hold in finite semilattices. Also, the question asked by the referee in [**6**] is partially answered. However, we are still unable to prove that D_2 is equal to D_n in infinite meet semilattices, although we suspect that this may be so, in contrast to Schein's conjecture.

Finally, we prove a theorem which we feel may provide some useful information in solving Schein's conjecture.

THEOREM 7.2. *Let S be a semilattice which is D_n. Let a, b, $c_1, \ldots, c_k$ $(1 \leqq k \leqq n - 1)$ be elements of S. If $a \lneqq b$ such that*

$$a \vee c_1 \vee \ldots \vee c_k = b \vee c_1 \vee \ldots \vee c_k$$

exists, then there exists $c_i (1 \leqq i \leqq k)$ such that

$$a \wedge c_i \lneqq b \wedge c_i.$$

Proof. Suppose $a \wedge c_i = b \wedge c_i$ for all $1 \leqq i \leqq k$. Then

$$\begin{aligned} b &= b \wedge (b \vee c_1 \vee \ldots \vee c_k) \\ &= b \wedge (a \vee c_1 \vee \ldots \vee c_k) \\ &= (b \wedge a) \vee (b \wedge c_1) \vee \ldots \vee (b \wedge c_k) \\ &= a \vee (a \wedge c_1) \vee \ldots \vee (a \wedge c_k) \\ &= a \end{aligned}$$

in contradiction with $a \lneq b$.

Thus there exists an element c_i such that

$$a \wedge c_i \neq b \wedge c_i.$$

The proof is thus completed.

In closing, we would like to pose the following problem for solution. By virtue of the proof in Theorem 7.1, we see that the statements of Lemma 5.4 (ii) and Theorem 5.6 (II) are equivalent in a finite prime semilattice with 0 and 1. In this case, the complementation of S and the statement of Lemma 5.4 (i) are equivalent. Thus the Theorem 9 in **[6]** is true in the finite case. Our question is: does this hold in the infinite case? In other words, is it true that for any element a of an infinite prime semilattice S with 0 and 1, there exists a sequence of elements $\{t_i]_{i \in \{1,2,\ldots,n\}}$ in $\langle a, 0\rangle$ such that $a \vee t_1 \vee t_2 \vee \ldots \vee t_n = 1$ implies the complementation of S?

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