


ARTICLE

On the Ramsey numbers of daisies II

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Abstract

A $(k+r)$ -uniform hypergraph H on $(k+m)$ vertices is an (r, m, k) -daisy if there exists a partition of the vertices $V(H) = K \cup M$ with $|K| = k$, $|M| = m$ such that the set of edges of H is all the $(k+r)$ -tuples $K \cup P$, where P is an r -tuple of M . We obtain an $(r-2)$ -iterated exponential lower bound to the Ramsey number of an (r, m, k) -daisy for 2-colours. This matches the order of magnitude of the best lower bounds for the Ramsey number of a complete r -graph.

Keywords: Ramsey theory; hypergraphs; stepping-up lemma

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1. Introduction

For a natural number N , we set $[N] = \{1, \dots, N\}$. Given a set X , we denote by $X^{(r)}$ the set of r -tuples of X . For two sets X, Y we say that $X < Y$ if $\max(X) < \min(Y)$. Unless stated otherwise, the elements of a set X will be always displayed in increasing order. That is, if $X = \{x_1, \dots, x_t\}$, then $x_1 < \dots < x_t$.

A $(k+r)$ -uniform hypergraph H on $k+m$ vertices is an (r, m, k) -daisy if there exists a partition of the vertices $V(H) = K \cup M$ with $|K| = k$ and $|M| = m$ such that

$$H = \{K \cup P : P \in M^{(r)}\}$$

We say that the set K is the kernel of H , the elements of $M^{(r)}$ are the petals of H and M is the universe of petals. We will often refer to an edge of H by X and its correspondent petal by P .

Daisies were first introduced by Bollobás, Leader, and Malvenuto in [1]. They were interested in Turán-type questions related to (r, m, k) -daisies, i.e., the maximum number of edges that an $(r+k)$ -graph has with no copy of an (r, m, k) -daisy. In this paper we will study the Ramsey number $D_r(m, k)$ of an (r, m, k) -daisy. The number $D_r(m, k)$ is defined as the minimum integer N such that any 2-colouring of the complete hypergraph $[N]^{(k+r)}$ contains a monochromatic (r, m, k) -daisy.

Those numbers were already studied in [5]. Although the main focus of their paper is on daisies with kernel of non fixed size, they noted that

$$R_{r-k}(\lceil m/(k+1) \rceil - k) \leq D_r(m, k) \leq R_r(m) + k, \quad (1)$$

where $R_r(m)$ is the Ramsey number of the complete graph $K_m^{(r)}$, i.e., the minimum integer N such that any 2-colouring of $[N]^{(r)}$ contains a monochromatic set X of size m .

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A natural question raised in [5] is whether $D_r(m, k)$ behaves similarly $R_r(m)$. Erdős, Hajnal, and Rado (see [3, 4]) and Conlon, Fox, and Sudakov [2] showed that there exists absolute constants c_1, c_2 such that for sufficiently large m ,

$$t_{r-2}(c_1 m^2) \leq R_r(m) \leq t_{r-1}(c_2 m), \tag{2}$$

where $t_i(x)$ is the tower function defined by $t_0(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$. In this paper, we provide for $k \geq 1$ a lower bound of $D_r(m, k)$ in the same order of magnitude as the best current bounds of the Ramsey number $R_r(m)$ for sufficiently large m . We remark here that for $k = 0$, the problem is equivalent to the Ramsey number, since an $(r, m, 0)$ -daisy is just the complete graph $K_m^{(r)}$.

Theorem 1.1. *Let $r \geq 3$ and $k \geq 1$ be integers. There exist integer $m_0 = m_0(r, k)$ and absolute constant c such that*

$$D_r(m, k) \geq t_{r-2}(ck^{-2}m^{2^{4-r}})$$

holds for $m \geq m_0$.

In order to prove Theorem 1.1 we will actually study the Ramsey number of a subfamily of daisies. We say that a hypergraph H is a *simple (r, m, k) -daisy* if H is an (r, m, k) -daisy and its kernel K can be partitioned into $K = K_0 \cup K_1$ such that $K_0 < M < K_1$. We define the Ramsey number of simple (r, m, k) -daisies $D_r^{smp}(m, k)$ as the minimum integer N such that any 2-colouring of the complete hypergraph $[N]^{(k+r)}$ yields a monochromatic copy of a simple (r, m, k) -daisy.

In [5], the authors observed that the Ramsey number of daisies can be bounded from below by the Ramsey number of simple daisies.

Proposition 1.2 ([5], Proposition 5.3). $D_r(m, k) \geq D_r^{smp}(\lceil m/(k+1) \rceil, k)$.

Our main technical result is an $(r - 2)$ -iterated exponential lower bound for the Ramsey number of simple (r, m, k) -daisies. Note that Theorem 1.1 is a corollary from Proposition 1.2 and Theorem 1.3.

Theorem 1.3. *Let $r \geq 3$ and $k \geq 1$ be integers. There exist integer $m_0 = m_0(r, k)$ and absolute positive constant c such that*

$$D_r^{smp}(m, k) \geq t_{r-2}(ck^{2^{4-r}-2}m^{2^{4-r}})$$

holds for $m \geq m_0$.

Our proof is a variant of the stepping-up lemma of Erdős, Hajnal and Rado [3, 4]. There are $k + 1$ distinct simple (r, m, k) -daisies depending on the sizes of K_0 and K_1 . While it is not hard to construct a colouring avoiding a monochromatic copy of one of these simple daisies, the main challenge is to define a colouring that avoids all $k + 1$ simple (r, m, k) -daisies simultaneously. To this end, we will introduce in Section 2 some auxiliary trees using the vertices of our ground set. A big portion of the paper consists on the study of those trees and how to use them to obtain a stepping-up lemma.

The paper is organized as follows. We introduce some auxiliary trees and most of the terminology in Section 2. Section 3 is devoted to give a general overview of the proof. We briefly describe the stepping-up lemma in [3, 4] with our setup and later describe the colouring of the variant. Sections 4 and 5 are the heart of the proof. We prove a key lemma (Lemma 5.1) that allows us to identify an important auxiliary tree containing the petal of an edge and then show how to reduce the stepping-up argument to this tree. We finish the proof of the stepping-up lemma and Theorem 1.3 in Section 6.

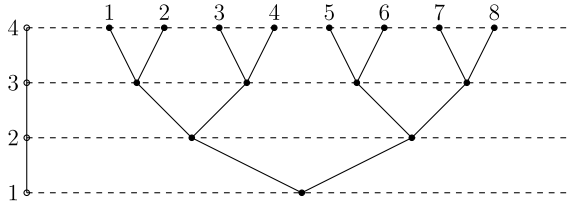


Figure 1. An example of a binary tree $T_{[2^3]}$ with its 4 levels.

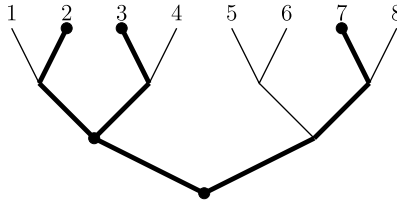


Figure 2. The auxiliary tree T_X for $X = \{2, 3, 7\}$.

2. Auxiliary trees

Given an integer N , we construct a binary tree $T_{[2^N]}$ of height N with $2^{N+1} - 1$ vertices and identify its leaves with the set $[2^N]$. We also identify each level of the tree with the set $[N + 1]$, where the root is at level 1, while the leaves are at level $N + 1$ (see Figure 1). For a vertex $u \in T_{[2^N]}$ we denote its level by $\pi(u)$.

Given two vertices u, v in $T_{[2^N]}$, we say that u is an *ancestor* of v if $\pi(u) < \pi(v)$ and there is a path $u = x_1, x_2, \dots, x_\ell = v$ in $T_{[2^N]}$ such that $\pi(x_i) \neq \pi(x_j)$ for every $1 \leq i, j \leq \ell$. For two vertices $x, y \in [2^N]$ we define the *greatest common ancestor* $a(x, y)$ of x and y as the vertex of $T_{[2^N]}$ of highest level that is an ancestor of both x and y . Also define

$$\delta(x, y) = \pi(a(x, y)).$$

Let $X = \{x_1, \dots, x_t\} \subseteq [2^N]$ with $x_1 < \dots < x_t$ be a subset of the leaves of our binary tree. We define the *auxiliary tree* T_X of X as the subtree of $T_{[2^N]}$ whose vertices are X and all their common ancestors. That is,

$$T_X = X \cup \{a(x_i, x_{i+1}) : 1 \leq i \leq t - 1\}.$$

Note that T_X is a tree of $2t - 1$ vertices (see Figure 2). Moreover, we denote the set of non-leaves by $a(X)$ and its projection by $\delta(X)$, i.e.,

$$a(X) = \{a(x_i, x_{i+1}) : 1 \leq i \leq t - 1\}$$

$$\delta(X) = \{\delta(x_i, x_{i+1}) : 1 \leq i \leq t - 1\}.$$

Since the auxiliary tree T_X is uniquely determined by its ground set X , sometimes we will denote T_X by X .

Given a vertex $u \in a(X)$, we can define the set $X(u)$ of *descendants* of u as the leaves of T_X that have u as an ancestor. That is,

$$X(u) = \{x \in X : u \text{ is an ancestor of } x\}.$$

The set of descendants of u can be partitioned into the left descendants and right descendants as follows: Since T_X is a binary tree, the vertex u has two children u^L and u^R . Let u^L be the left

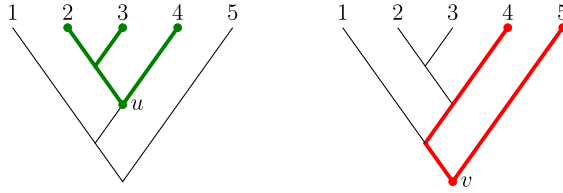


Figure 3. The interval $\{2, 3, 4\}$ is closed, since $X(u) = \{2, 3, 4\}$ for $u = a(2, 4)$. The interval $\{4, 5\}$ is not closed, since $X(v) = \{1, 2, 3, 4, 5\} \neq \{4, 5\}$ for $v = a(4, 5)$.

children of u and u^R be the right children of u . Then we define the left descendants of u by

$$X_L(u) = \begin{cases} u^L & \text{if } u^L \in X, \\ X(u^L) & \text{if } u^L \in a(X), \end{cases}$$

and the right descendants of u by

$$X_R(u) = \begin{cases} u^R & \text{if } u^R \in X, \\ X(u^R) & \text{if } u^R \in a(X), \end{cases}$$

Note that by this definition $X_L(u), X_R(u) \neq \emptyset$ and $\max X_L(u) < \min X_R(u)$.

Although an auxiliary tree is not uniquely determined by its ancestors, we can at least determine the “shape” of the tree T_X by looking at $a(X)$. In a more precise way, the following can be proved by a simple induction.

Fact 2.1. *If X and Y are subsets of $[2^N]$ such that $a(X) = a(Y)$, then $|X| = |Y|$. Moreover, if $X = \{x_1, \dots, x_t\}$ and $Y = \{y_1, \dots, y_t\}$, then $a(x_i, x_{i+1}) = a(y_i, y_{i+1})$ for every $1 \leq i \leq t - 1$.*

Now we devote the rest of the section on classifying our auxiliary trees.

Definition 2.2. *Given $X = \{x_1, \dots, x_t\} \subseteq [2^N]$. We say that an interval $I = \{x_p, \dots, x_q\} \subseteq X$ for some $1 \leq p \leq q \leq t$ is closed in X if the following condition holds:*

$$(\star) I = X(a(x_p, x_q)).$$

In Figure 3, one can see examples of a closed interval and a not closed one. Alternatively, one can replace (\star) by the useful equivalent condition:

$$(\star\star) \text{ For every vertex } y \in X \setminus I, \text{ the vertex } a(x_p, x_q) \text{ is not an ancestor of } y.$$

The following proposition shows that closed intervals cannot have proper intersections.

Proposition 2.3. *Let I_1, I_2 be two intervals in X with $|I_1| \leq |I_2|$. If I_1 and I_2 are closed, then either $I_1 \cap I_2 = \emptyset$ or $I_1 \subseteq I_2$.*

Proof. Suppose that $I_1 \cap I_2$ is a proper intersection. That is, $I_1 \cap I_2 \neq \emptyset, I_1 \setminus I_2 \neq \emptyset$ and $I_2 \setminus I_1 \neq \emptyset$. Write $X = \{x_1, \dots, x_t\}$ and $I_1 = \{x_{p_1}, x_{p_1+1}, \dots, x_{q_1}\}, I_2 = \{x_{p_2}, x_{p_2+1}, \dots, x_{q_2}\}$ for $1 \leq p_1 < p_2 \leq q_1 < q_2 \leq t$. Let $u = a(x_{p_1}, x_{q_1})$ and $v = a(x_{p_2}, x_{q_2})$. We claim that either u is an ancestor of v or v is an ancestor of u . Let $z \in I_1 \cap I_2$. By definition, both u and v are ancestors of z . This means that there exists descending paths connecting z to u and z to v in T_X with vertices in different levels. However, every vertex in T_X has at most one father. Therefore, either the path z to u contains the path z to v or vice-versa. If the path z to u contains the path z to v , then u is an ancestor of v . Hence u is an ancestor of $I_2 \setminus I_1$, which contradicts the fact that I_1 is closed (Condition $(\star\star)$ of Definition 2.2). The other case is analogous. \square

We classify the closed intervals of X by three classes: left combs, right combs, and broken combs (see also Figure 4).

Definition 2.4. *Given a closed interval I in X we say that*

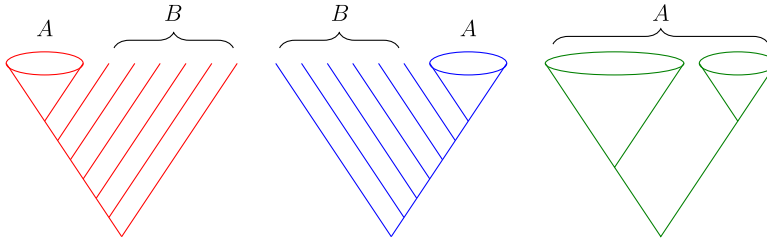


Figure 4. An example of a left, right, and broken comb, respectively.

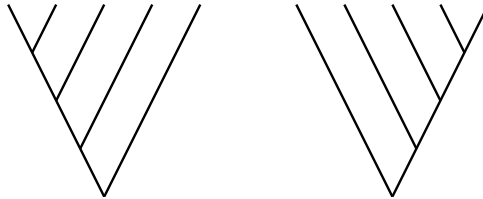


Figure 5. A left and right 1-comb.

- (a) *I is a ℓ -left comb if ℓ is the least positive integer such that there exists a partition $I = A \cup B$ with $|A| = \ell$ and $B \neq \emptyset$ and*
 - (a1) $A < B$.
 - (a2) A is a closed interval in X
 - (a3) If $z = \max(A)$ and $B = \{b_1, \dots, b_s\}$, then $\delta(z, b_1) > \delta(b_1, b_2) > \dots > \delta(b_{s-1}, b_s)$.
- (b) *I is a ℓ -right comb if ℓ is the least positive integer such that there exists a partition $I = A \cup B$ with $|A| = \ell$ and $B \neq \emptyset$ and*
 - (b1) $B < A$.
 - (b2) A is a closed interval in X
 - (b3) If $z = \min(A)$ and $B = \{b_1, \dots, b_s\}$, then $\delta(b_1, b_2) < \dots < \delta(b_{s-1}, b_s) < \delta(b_t, z)$.
- (c) *I is a broken comb if it is neither a left or right comb.*

We will use the convention that an ℓ -left/right comb will be described by its partition $I = A \cup B$ with $|A| = \ell$ that verifies the condition on Definition 2.4. As we can see in the picture above, the set A should be thought as the “handle” of the comb, while the set B should be thought as the “teeth” of the comb. For broken combs we will adopt the same convention by assuming that $B = \emptyset$.

One may remove the use of the projection $\delta(b_i, b_{i+1})$ in conditions (a3) and (b3) of the right/left comb by using the following equivalent alternative conditions:

- (a3*) If $B = \{b_1, \dots, b_s\}$, then the intervals $A \cup \{b_1, \dots, b_i\}$ are closed in X for every $1 \leq i \leq s$
- (b3*) If $B = \{b_1, \dots, b_s\}$, then the intervals $\{b_i, \dots, b_s\} \cup A$ are closed in X for every $1 \leq i \leq s$.

Those conditions have the advantage of describing a comb only using closed intervals. This will be useful later in the proof.

Example 2.5. *A important type of comb in the stepping-up lemma [3, 4] is the 1-left/right comb (see Figure 5). Those are the combs $I = \{y_1, \dots, y_t\}$ satisfying that the sequence $\{\delta(y_i, y_{i+1})\}_{1 \leq i \leq t}$ is monotone. Indeed, the interval I is a 1-left comb if $\delta(y_1, y_2) > \dots > \delta(y_{t-1}, y_t)$, while it is a 1-right comb if $\delta(y_1, y_2) < \dots < \delta(y_{t-1}, y_t)$.*

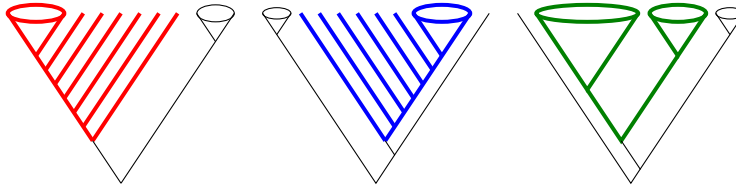


Figure 6. An example of a maximal left, right, and broken comb, respectively.

For the proof of Theorem 1.3 we will be interested in maximal comb structures inside our auxiliary trees.

Definition 2.6. Given $X = \{x_1, \dots, x_t\} \subseteq [2^N]$, a interval $I = \{x_p, \dots, x_q\}$ is a

- (a) Maximal left comb in X if I is a left comb and $I \cup \{x_{q+1}\}$ is not a closed interval in X .
- (b) Maximal right comb in X if I is a right comb and $I \cup \{x_{p-1}\}$ is not a closed interval in X .
- (c) Maximal broken comb in X if I is a broken comb and neither $I \cup \{x_{p-1}\}$ or $I \cup \{x_{q+1}\}$ are closed.

Figure 6 illustrates Definition 2.6. The next proposition shows that given two maximal combs they are either disjoint or one is contained in the “handle” of the other.

Proposition 2.7. Given a closed interval I_1 and a maximal comb $I_2 = A_2 \cup B_2$ with $|I_1| \leq |I_2|$ in a set $X \subseteq [2^N]$, then one of the following holds:

1. $I_1 \cap I_2 = \emptyset$
2. $I_1 \subseteq A_2$.
3. $I_1 = A_2 \cup B_1$ for some initial segment $B_1 \subseteq B_2$

Moreover, condition (3) only holds if I_1 is not a maximal comb.

Proof. By Proposition 2.3 we obtain that either $I_1 \cap I_2 = \emptyset$ or $I_1 \subseteq I_2$. If the first case happens, then I_1 and I_2 satisfy condition (1) and we are done. Hence, we may assume that $I_1 \subseteq I_2$. If I_2 is a broken comb, then by definition $A_2 = I_2$. Thus in this case $I_1 \subseteq A_2$, satisfying condition (2). Now suppose without loss of generality that $I_2 = A_2 \cup B_2$ is a left maximal comb and write $A_2 = \{x_1, \dots, x_\ell\}$, $B_2 = \{y_1, \dots, y_s\}$. If $I_1 \cap B_2 = \emptyset$, then $I_1 \subseteq A_2$ and again condition (2) holds.

At last, it remains to deal with the case that $I_1 \cap B_2 \neq \emptyset$. Since I_1 is an interval of X and $I_1 \subseteq I_2$, then in particular I_1 is an interval of I_2 . Write $I_1 = \{x_p, \dots, x_\ell\} \cup \{y_1, \dots, y_q\}$ for $1 \leq p \leq \ell$ and $1 \leq q \leq s$. By condition (a3*) of Definition 2.4, the set $A_2 \cup \{y_1, \dots, y_{q-1}\}$ is closed. Therefore for any $z \in A_2 \cup \{y_1, \dots, y_{q-1}\}$ the greatest ancestor $a(z, y_q)$ of z and y_q is the same as the greatest ancestor of $a(x_1, y_q)$. In particular, this implies that $a(x_p, y_q)$ is an ancestor for the entire set A_2 . Hence $A_2 \subseteq I_1$ and consequently $I_1 = A_1 \cup B_1$ is a left comb satisfying condition (3), because $A_1 = A_2$ and B_1 is an initial segment of B_2 . Note that I_1 is not maximal in this case, since the set $I_1 \cup \{y_{q+1}\}$ is also a left comb. Thus if I_1 is a maximal comb, then it either satisfies (1) or (2). \square

3. Stepping-up lemma and our colouring

3.1 Erdős–Hajnal–Rado stepping-up lemma

For instructional purposes, we will briefly go over the stepping-up lemma in [3, 4] using our notation. For $k \geq 4$, let $N = R_{k-1}((n - k + 4)/2) - 1$ and $\varphi : [N]^{(k-1)} \rightarrow \{0, 1\}$ be a colouring of the $(k - 1)$ -tuples in $[N]$ with no monochromatic subset of size $(n - k + 4)/2$. Our goal is to find

a colouring $\psi : [2^N]^{(k)} \rightarrow \{0, 1\}$ with no monochromatic subset of size n . This will give us that $R_k(n) > 2^N = 2^{R_{k-1}((n-k+4)/2)-1}$.

Fix an edge $X = \{x_1, \dots, x_k\} \in [2^N]^{(k)}$ and let $\delta_i = \delta(x_i, x_{i+1})$. We describe the colouring ψ by the structure of T_X and the colouring of the vertical projection φ of $[N]$ in the following way

$$\psi(X) = \begin{cases} 0, & \text{if } \delta_{k-3} > \delta_{k-2} < \delta_{k-1} \\ 1, & \text{if } \delta_{k-3} < \delta_{k-2} > \delta_{k-1} \\ \varphi(\{\delta_1, \dots, \delta_{k-1}\}), & \text{otherwise if } |\delta(X)| = k - 1 \\ 0, & \text{otherwise if } |\delta(X)| < k - 1. \end{cases}$$

Suppose by contradiction that ψ contains a monochromatic subset $Y \subset [2^N]$ of size n . We can use the structure of Y to find a large 1-comb.

Proposition 3.1. *There exists an interval I of Y with $|I| \geq (n - k + 6)/2$ such that I is a 1-comb*

Proof. We may assume without loss of generality that Y is monochromatic of colour 0. Write $Y = \{y_1, \dots, y_n\}$ and let $\delta_i^Y = \delta(y_i, y_{i+1})$ for $1 \leq i \leq n - 1$. Since all edges in Y are of colour 0, then for any edge $X = \{x_1, \dots, x_k\} \in Y^{(k)}$ we do not have that

$$\delta(x_{k-3}, x_{k-2}) < \delta(x_{k-2}, x_{k-1}) > \delta(x_{k-1}, x_k). \tag{3}$$

In particular, by taking the edge $\{y_{\ell-k+1}, \dots, y_\ell\}$, inequality (3) implies that $\delta_{\ell-3}^Y < \delta_{\ell-2}^Y > \delta_{\ell-1}^Y$ does not hold for every $k \leq \ell \leq n$. Hence, the sequence $\{\delta_i^Y\}_{i=k-3}^{n-1}$ has no local maximum.

A standard calculus argument says that between two local minimums there is always a local maximum. Therefore, the sequence $\{\delta_i^Y\}_{i=k-3}^{n-1}$ has at most one local minimum, which means that there exists an interval $[p, q]$ of size $(n - k + 4)/2$ such that $\{\delta_i^Y\}_{i \in [p, q]}$ is monotone. Thus by definition the interval $I = \{x_p, x_{p+1}, \dots, x_{q+1}\}$ is a 1-comb of size $(n - k + 6)/2$. \square

Let $I = \{z_1, \dots, z_t\}$ be the 1-comb of Y obtained by Proposition 3.1 and denote $\delta_i^I = \delta(z_i, z_{i+1})$ for $1 \leq i \leq t - 1$. Note that because $\{\delta_i^I\}_{i=1}^{t-1}$ is a monotone sequence, every edge $X \in I^{(k)}$ will be also a 1-comb. Moreover, for every $(k - 1)$ -tuple $Z \in \delta(I)$ there exists an edge $X \in I^{(k)}$ such that $\delta(X) = Z$.

Finally, by the definition of the colouring ψ , if X is a 1-comb, then $\psi(X) = \varphi(\delta(X))$. Thus if $I^{(k)}$ coloured by ψ is monochromatic, then $\delta(I)^{(k-1)}$ coloured by φ is also monochromatic. This implies that $[N]$ has a monochromatic set of size $(n - k + 4)/2$, which contradicts our assumption on φ .

3.2 Overview of the proof

In order to obtain a lower bound for simple daisies, we will define a variant of the stepping-up lemma described in the previous subsection. Suppose for a moment that our goal is to avoid a monochromatic simple (r, m, k) -daisy with in $[2^N]$ with $|K_0| = k_0$ and $|K_1| = k_1$ fixed. Then for every edge $X = \{x_1, \dots, x_{k+r}\}$ of the daisy, we know that the petal of size r of X is $P = \{x_{k_0+1}, \dots, x_{k_0+r}\}$. That is, we know the exact location of the petal prior defining the colouring in our stepping-up lemma. In this case a natural way to define the colouring would be to just assign for every edge X with petal $P \subseteq X$ the colour $\chi(X) = \psi(P)$, where $\psi(P)$ is exactly the stepping-up colouring defined in the previous subsection. Since the petal is the only part of the edge changing when we run through all edges, a similar proof as in the previous subsection works.

Unfortunately, in the original problem we want to avoid all possible monochromatic simple (r, m, k) -daisies, which means that we need to avoid simple daisies for all the values of $|K_0|$ and $|K_1|$. The obstruction now is that the location of the petal within the edge is no longer clear to us.

To fix that we are going to pre-process our potentially monochromatic simple daisy (Lemma 5.1) to satisfy the following property: Every petal P of an edge X is either a closed interval in X or is in the “teeth” of a maximal comb in X . This gives us partial information about the location of the petal. A good strategy then is to define an auxiliary colouring χ_0 for every maximal comb in X and use those colourings to define a colouring for X . This is the content of Section 3.3.

Some technical challenges remain. By Proposition 2.7, the maximal combs in X do not need to be disjoint. Therefore, it might happen that for the colouring χ different maximal combs interfere with each other. To solve that we need to construct a careful colouring taking the issue into consideration. In Section 4 we provide an analysis showing that distinct maximal combs do not interfere with each other in our colouring. Section 5 is devoted to the pre-processing described in the last paragraph. One of the consequences of the section is that for an edge X the colouring $\chi(X)$ is essentially determined by a unique maximal comb inside of it. Finally, we finish the proof in Section 6, by showing, similarly as in Subsection 3.1, that a monochromatic simple daisy in $[2^N]$ corresponds to a monochromatic simple daisy in the vertical colouring of $[N]$.

3.3 A variant of the stepping-up lemma

Let $N = \min_{0 \leq t \leq k-1} \{D_{r-1}^{\text{smp}}(c_k \sqrt{m}, t) - 1\}$ for $r \geq 4$ and c_k some constant depending on k to be defined later and let $\{\varphi_i\}_{r-1 \leq i \leq k+r-1}$ be a family of colourings such that $\varphi_i : [N]^{(i)} \rightarrow \{0, 1\}$ is a 2-colouring of the i -tuples without a monochromatic simple $(r - 1, c_k \sqrt{m}, i - r + 1)$ -daisy. Note that by the choice of N is always possible to find such a family.

Given an $(k + r)$ -tuple $X \in [2^N]^{(k+r)}$ we define

$$\mathcal{I}_X = \{I \subseteq X : I \text{ is a maximal comb in } X\}$$

as the set of maximal combs of X . We will construct now an auxiliary colouring $\chi_0 : \mathcal{I}_X \rightarrow \{0, 1\}$ depending on the structure of T_X and in the family of colourings $\{\varphi_t\}_{0 \leq t \leq k}$. The colouring is divided in several cases depending on the type of the maximal comb I .

Remember that a maximal ℓ -comb is always identified with the partition $I = A \cup B$, where $|A| = \ell$ is the handle and B is the set of teeth of the comb. Also write $I = \{x_1, \dots, x_s\}$ and let $\delta_i^I = \delta(x_i, x_{i+1})$ for $1 \leq i \leq s - 1$. Aiming to simplify the discussion, we will only describe χ_0 for left and broken maximal combs. We define χ_0 for right combs by symmetry. Some Figures are provided to illustrate some of the types (see Figures 7–9).

Type 1: I is broken or left comb, $|I| = r$ and there is no maximal comb $I' = A' \cup B'$ such that $I = A'$.

$$\chi_0(I) = \begin{cases} 0, & \text{if } \delta_{r-3}^I > \delta_{r-2}^I < \delta_{r-1}^I \\ 1, & \text{if } \delta_{r-3}^I < \delta_{r-2}^I > \delta_{r-1}^I \\ \varphi_{|\delta(I)|}(\{\delta_1^I, \dots, \delta_{r-1}^I\}), & \text{otherwise if } |\delta(I)| = r - 1 \\ 0, & \text{otherwise if } |\delta(I)| < r - 1 \end{cases}$$

Type 2: I is left comb, $|I| = r$ and there exists a maximal comb $I' = A' \cup B'$ such that $I = A'$

$$\chi_0(I) = 0$$

Type 3: I is left comb, $\ell = |A| \leq r$ and $r + 1 \leq |I| \leq 2r - 2$.

$$\chi_0(I) = \begin{cases} 0, & \text{if } \delta_{r-3}^I > \delta_{r-2}^I < \delta_{r-1}^I \\ 1, & \text{if } \delta_{r-3}^I < \delta_{r-2}^I > \delta_{r-1}^I \\ \varphi_{|\delta(I)|}(\{\delta_1^I, \dots, \delta_{s-1}^I\}), & \text{otherwise if } |\delta(I)| \geq r - 1 \\ 0, & \text{otherwise if } |\delta(I)| < r - 1 \end{cases}$$

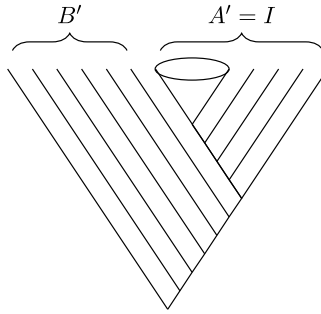


Figure 7. An example of left comb of type 2.

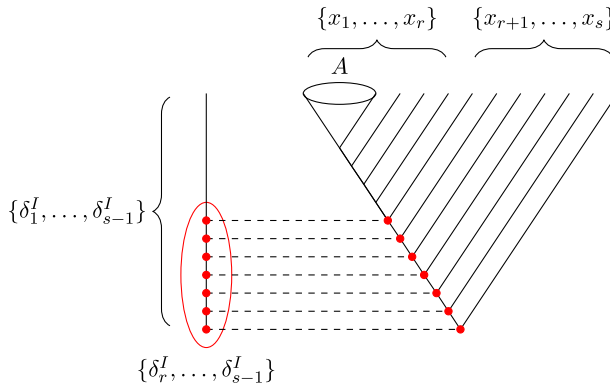


Figure 8. A left comb of Type 4 and its projections.

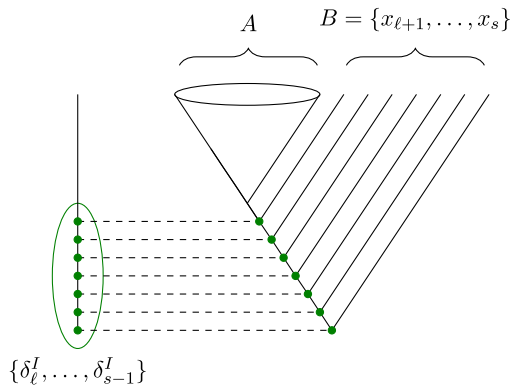


Figure 9. A left comb of Type 5 and its projections.

Type 4: I is left comb, $\ell = |A| \leq r$ and $|I| \geq 2r - 1$.

$$\chi_0(I) = \begin{cases} \varphi_{s-r}(\{\delta_r^I, \dots, \delta_{s-1}^I\}), & \text{if } \delta_{r-3}^I > \delta_{r-2}^I < \delta_{r-1}^I \\ 1 - \varphi_{s-r}(\{\delta_r^I, \dots, \delta_{s-1}^I\}), & \text{if } \delta_{r-3}^I < \delta_{r-2}^I > \delta_{r-1}^I \\ \varphi_{|\delta(I)|}(\{\delta_1^I, \dots, \delta_{s-1}^I\}), & \text{otherwise} \end{cases}$$

Type 5: I is left comb, $\ell = |A| > r$ and $|B| \geq r$.

$$\chi_0(I) = \varphi_{|B|}(\delta_\ell^I, \dots, \delta_{s-1}^I)$$

Type 6: All other broken or left maximal combs.

$$\chi_0(I) = 0$$

Finally, the auxiliary colouring χ_0 define a colouring $\chi : [2^N]^{(k+r)} \rightarrow \{0, 1\}$ as follows:

$$\chi(X) = \sum_{I \in \mathcal{I}_X} \chi_0(I) \pmod{2}$$

4. Colouring data

Given an edge X and a maximal comb $I \subseteq X$, one can determine the colour $\chi_0(I)$ by looking at the type of the maximal comb I . Some of the types do not use information on the ancestors to determine its colouring. For instance, if I is of type 2, then its colour will be always 0. The projection of the ancestors δ_i^I has no influence in defining $\chi_0(I)$. However, if I is of type 1, then the colour crucially depends on the projection of the ancestors.

This observation suggests the following definition. Given an edge $X \in [2^N]^{(k+r)}$ and a maximal comb $I \subseteq X$, let the *colouring data* $F(I)$ of I be defined as the ordered set of ancestors whose projection determine the colouring $\chi_0(I)$. More explicitly, we can define directly the colouring data of I by looking its types. We may assume here that $I = \{x_1, \dots, x_s\}$ is a broken or left maximal comb.

- Type 1,3 and 4: $F(I) = \{a(x_i, x_{i+1})\}_{1 \leq i \leq s-1}$
- Type 2 and 6: $F(I) = \emptyset$
- Type 5: $F(I) = \{a(x_i, x_{i+1})\}_{\ell \leq i \leq s-1}$

Our first observation is that maximal combs with same data have same colour. We say that two combs have the same *orientation* if they are of the same class (e.g., both are left combs).

Proposition 4.1. *Let $X, X' \in [2^N]^{(k+r)}$ be two edges. If I and I' are maximal combs of same type and orientation in X and X' , respectively, such that $F(I) = F(I')$, then $\chi_0(I) = \chi_0(I')$.*

Proof. The proof basically consists of checking the consistency of our definition. If $F(I) = F(I') = \emptyset$, then I and I' are either of type 2 or 6. In both cases $\chi_0(I) = \chi_0(I') = 0$.

If $I = \{x_1, \dots, x_s\}$ and $I' = \{x'_1, \dots, x'_{s'}\}$ are of type 1, 3 or 4, then since $a(I) = F(I) = F(I') = a(I')$ we obtain by Fact 2.1 that $s = s'$ and $a(x_i, x_{i+1}) = a(x'_i, x'_{i+1})$ for every $1 \leq i \leq s$. Therefore $\delta(x_i, x_{i+1}) = \delta(x'_i, x'_{i+1})$ for every $1 \leq i \leq s$ and by the colouring defined in Section 3.3, it follows that $\chi_0(I) = \chi_0(I')$.

The last case that we need to check is when $I = A \cup B = \{x_1, \dots, x_s\}$ and $I' = A' \cup B' = \{x'_1, \dots, x'_{s'}\}$ are of type 5, where $|A| = \ell$ and $|A'| = \ell'$. As usual, we assume that I and I' are left combs. Since $a(\{x_\ell, \dots, x_s\}) = F(I) = F(I') = a(\{x'_{\ell'}, \dots, x'_{s'}\})$, it follows again by Fact 2.1 that $s - \ell = |B| = |B'| = s' - \ell'$ and $a(x_i, x_{i+1}) = a(x'_i, x'_{i+1})$ for $\ell \leq i \leq s - 1$. Thus $\delta(x_i, x_{i+1}) = \delta(x'_i, x'_{i+1})$ for $\ell \leq i \leq s - 1$ and by the colouring of type 5 we obtain that $\chi_0(I) = \chi_0(I')$. \square

Although maximal combs in the same edge do not need to be disjoint, the next result shows that they do not share the same colouring data.

Proposition 4.2. *Let $X \in [2^N]^{(k+r)}$ be an edge. If $I = A \cup B$ and $I' = A' \cup B'$ are maximal combs in X , then $F(I) \cap F(I') = \emptyset$.*

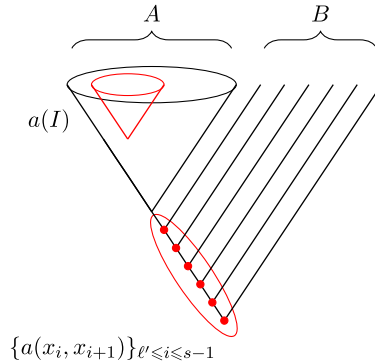


Figure 10. A picture of I and $\{a(x_i, x_{i+1})\}_{\ell' \leq i \leq s-1}$.

Proof. Suppose without loss of generality that $|I| \leq |I'|$. By Proposition 2.7 either $I \cap I' = \emptyset$ or $I \subseteq A'$. If $I \cap I' = \emptyset$, then by the fact that I, I' are closed we obtain that $a(I) \cap a(I') = \emptyset$. Since $F(I) \subseteq a(I)$ by definition, it follows that $F(I) \cap F(I') = \emptyset$.

Now suppose that $I \subseteq A'$. We may assume that $F(I), F(I') \neq \emptyset$ and consequently that I, I' are of type 1, 3, 4 or 5. Since maximal combs of types 1, 3, 4 and 5 have size at least r , our assumption implies that $|I'| \geq |I| \geq r$.

We claim that $|A'| > r$. Suppose that $|A'| = r$. Since $r \leq |I| \leq |A'|$ we obtain that $I = A'$ and $|I| = r$. The maximality of I implies that it is either a left or right maximal comb (otherwise we could extend the comb to $I \cup B$). However, in this case I is of type 2. Thus $F(I) = \emptyset$, which contradicts our assumption on I . Therefore, $|A'| > r$ and consequently I' is of type 5. Write $I' = \{x'_{1}, \dots, x'_{s'}\}$ with $|A'| = \ell'$ and assume that I' is a left comb. Then by definition

$$F(I') = \{a(x'_{i}, x'_{i+1})\}_{\ell' \leq i \leq s'-1}.$$

Since $I \subseteq A'$ we obtain that $F(I) \subseteq a(A')$. By the structure of a left comb (see Figure 10) we have that

$$\delta(x'_{1}, x'_{\ell'}) > \delta(x'_{\ell'}, x'_{\ell'+1}) > \delta(x'_{\ell'+1}, x'_{\ell'+2}) > \dots > \delta(x'_{s'-1}, x'_{s'}).$$

Thus $a(I) \cap \{a(x'_{i}, x'_{i+1})\}_{\ell' \leq i \leq s'-1} = \emptyset$ and consequently $F(I) \cap F(I') = \emptyset$. □

5. Pre-processing

As discussed in Subsection 3.2, we now turn our focus to show that a simple daisy H can be pre-processed in a smaller simple subdaisy H' with the property that for every edge X with petal P we have that either P is a closed interval in X or is part of the “teeth” of a maximal comb in X .

Lemma 5.1. *For any simple (r, m, k) -daisy H with vertex set $V(H) \subseteq [2^N]$, $K_0 < M < K_1$, $|K_0 \cup K_1| = k$ and $|M| = m$, there exists a subset $M' \subseteq M$ of size $|M'| = \frac{1}{2}k^{-1/2}m^{1/2}$ such that the simple $(r, \frac{1}{2}k^{-1/2}m^{1/2}, k)$ -daisy $H' = H[K_0 \cup M' \cup K_1]$ satisfies one of the following (see Figure 11):*

1. M' is a closed interval in $V(H')$.
2. There exists a maximal comb $I = A \cup B$ in $V(H')$ such that $M' \subseteq B$.

Proof. Let $V := V(H)$. Given a closed interval $I \subseteq V$, by condition (ii) of Definition 2.2 there exists a vertex $u \in a(I)$ such that $I = V(u)$. Consider the partition of I given by $I = I^L \cup I^R$, where $I^L = V_L(u)$ are the left descendants of u and $I^R = V_R(u)$ are the right descendants of u . Let u^L be

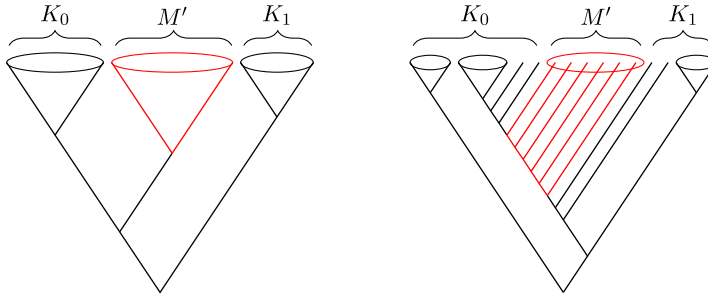


Figure 11. An example of H' satisfying statement (1) and (2).

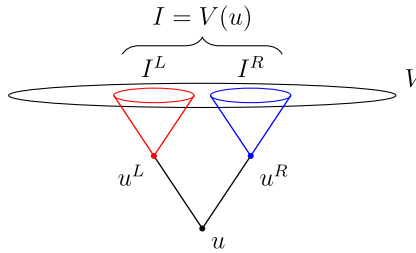


Figure 12. Partition of a closed interval into two other closed intervals.

the left child of u and u^R be the right child. Hence, $I^L = V(u^L)$ and $I^R = V(u^R)$ and consequently $I = I^L \cup I^R$ is a partition of a closed interval in V into two non empty closed intervals in V (see Figure 12).

We will construct our set M' iteratively. This is done in two stages. In the first stage we start with the closed interval $Y_0 = V$ and proceed recursively as follows: For a closed interval $Y_i \subseteq V$, let $Y_i = Y_i^L \cup Y_i^R$ be the partition described above in two closed intervals. The choice of Y_{i+1} is determined by the conditions below

- (P1) Set $Y_{i+1} := Y_i^L$ if $|Y_i^L \cap M| \geq |Y_i^R \cap M|$.
- (P2) Set $Y_{i+1} := Y_i^R$ if $|Y_i^L \cap M| < |Y_i^R \cap M|$.

We stop the process whenever $Y_i \cap K_0 = \emptyset$ or $Y_i \cap K_1 = \emptyset$. Note that since Y_i^L and Y_i^R are non empty, at each iteration of the process the size of $|(K_0 \cup K_1) \cap Y_i|$ reduces at least by one. Thus, in a finite amount of time the process terminates. Let Y be the closed interval obtained in the end. We may assume without loss of generality that $Y \cap K_1 = \emptyset$. Write $Y = K_Y \cup M_Y$, where $K_Y \subseteq K_0$ and $M_Y \subseteq M$. It is not hard to check by the construction that $|M_Y| \geq m/2$.

For the second stage, let $Z_0 = Y$. Given a closed interval $Z_i \subseteq V$, let $Z_i = Z_i^L \cup Z_i^R$ be the partition into two non empty closed intervals. By definition we have that $Z_i^L < Z_i^R$. We say that a partition $Z_i^L \cup Z_i^R$ is of type A if $Z_i^R \cap K_0 = \emptyset$ and of type B if $Z_i^R \cap K_0 \neq \emptyset$. The choice of Z_{i+1} will depend on the type of partition as follows:

Type A: $Z_i^R \cap K_0 = \emptyset$.

- (A1) Set $Z_{i+1} := Z_i^L$ if $|Z_i^R| < \frac{1}{2}k^{-1/2}m^{1/2}$.
- (A2) Set $Z_{i+1} := Z_i^R$ if $|Z_i^R| \geq \frac{1}{2}k^{-1/2}m^{1/2}$ and stop the process.

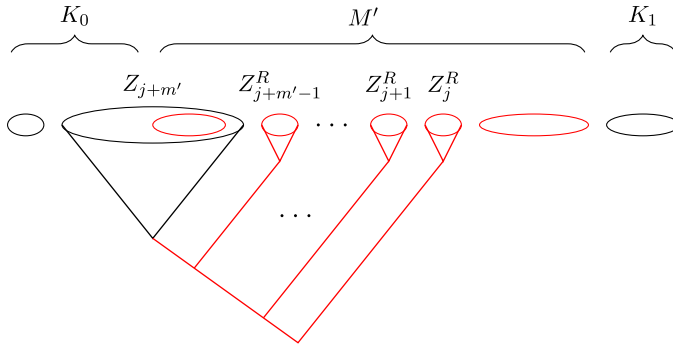


Figure 13. Sequence of closed intervals $Z_j^R, \dots, Z_{j+m'}^R$.

Type B: $Z_i^R \cap K_0 \neq \emptyset$.

(B) Set $Z_{i+1} := Z_i^R$.

We terminate the process if we either reach condition (A2) or if Z_{i+1} is a singleton. Since $|Z_{i+1}| < |Z_i|$, the process is finite. Let Z be the closed interval obtained at the end. We split into two cases.

If the process terminates after some instance of condition (A2), then it means that $Z = Z_i^R$ is a closed interval in V with $|Z| \geq \frac{1}{2}k^{-1/2}m^{1/2}$ for some index i . Because we are in a partition of type A we also obtain that $Z \subseteq M$. Thus, if we set $M' = Z$ the simple subdaisy $H[K_0 \cup M' \cup K_1]$ satisfies condition (1) of the statement.

Now suppose that the process terminates with $|Z| = 1$. Then it means that for every partition of type A we had an instance of condition (A1). If Z_{i+1} is a set obtained after condition (A1), then $|Z_{i+1} \cap M| > |Z_i \cap M| - \frac{1}{2}k^{-1/2}m^{1/2}$ and $|Z_{i+1} \cap K_0| = |Z_i \cap K_0|$. That is, condition (A1) removes less than $\frac{1}{2}k^{-1/2}m^{1/2}$ element of M from Z_i and no elements of K_0 from it. Moreover, if Z_{i+1} is obtained after condition (B), then $|Z_{i+1} \cap M| = |Z_i \cap M|$ and $|Z_{i+1} \cap K_0| < |Z_i \cap K_0|$. That is, M remains unaffected, but K_0 loses at least one element from K_i to K_{i+1} .

Consider the sequence of operations applied to Z_0 in order to obtain Z . Since we start with a set $Z_0 = Y$ with $|Z_0 \cap M| = |M_Y| \geq m/2$, we obtain that during our process we had at least

$$\frac{\frac{m}{2}}{\frac{1}{2}k^{-1/2}m^{1/2}} = k^{1/2}m^{1/2}$$

instances of condition (A1) in the sequence. Similarly, since $|Z_0 \cap K_0| = |K_Y| \leq k$, we obtain that we had at most k instances of condition (B) in the sequence. Hence, by the pigeonhole principle there exists a sequence of consecutive applications of condition (A1) of length at least $m' = k^{1/2}m^{1/2}/(k+1) \geq \frac{1}{2}k^{-1/2}m^{1/2}$.

Let $Z_j, Z_{j+1}, \dots, Z_{j+m'}$ be the closed intervals involved in the sequence. That is, Z_{i+1} is obtained from Z_i by a condition (A1) for every $j \leq i \leq j+m'-1$. By the algorithm, we obtain closed intervals $Z_j^R, \dots, Z_{j+m'-1}^R \subseteq M$ all of them with size less than $\frac{1}{2}k^{-1/2}m^{1/2}$ (Figure 13). For every $j \leq i \leq j+m'-1$, choose a point $z_i \in Z_i^R$.

Set $M' = \{z_j, \dots, z_{j+m'-1}\}$. We claim that M' is a set satisfying condition (2) of the statement. Let $H' = H[K_0 \cup M' \cup K_1]$ and $V' = V(H')$. To see that condition (2) is satisfied we just need to find a maximal comb $I = A \cup B \subseteq V'$ such that $M' \subseteq B$. Let $K' = K_0 \cap Z_{j+m'}$ and consider the interval $I' = K' \cup M'$ in V' . By construction, the intervals K' and $K' \cup \{z_{j+i}, \dots, z_{j+m'-1}\}$ are

closed in V' for every $0 \leq i \leq m' - 1$. Therefore, by condition (a3*) of Definition 2.4, the interval $I' = A' \cup B'$ is a left comb and $M' \subseteq B'$. Since every comb can be extended to a maximal one, there exists a maximal left comb $I = A \cup B$ with $A = A'$ and $B' \subseteq B$ such that $M' \subseteq B$ and we are done. \square

One of the main consequences of our pre-processing is that it allows us to identify certain closed and non-closed intervals in an arbitrary edge of H' . To be more precise, given an edge $X \in E(H')$ with petal P and $V' = V(H')$, let

$$C_{V',M'} = \{I : I \text{ is an interval in } V' \text{ and either } M' \subseteq I \text{ or } M' \cap I = \emptyset\}$$

$$C_{X,P} = \{I : I \text{ is an interval in } X \text{ and either } P \subseteq I \text{ or } P \cap I = \emptyset\}$$

be the set of intervals in V' and X such that the intervals either contain or are disjoint of M' and P , respectively. The next proposition shows that there is a one-to-one correspondence between $C_{V',M'}$ and $C_{X,P}$ preserving the property of being closed.

Proposition 5.2. *For a given edge $X \in E(H')$ with petal P , there exists a bijection $\Psi : C_{V',M'} \rightarrow C_{X,P}$ given by*

$$\Psi(I) = I \cap X$$

such that I is a closed interval in V' if and only if $\Psi(I)$ is a closed interval in X .

Proof. If $I \in C_{V',M'}$ is such that $I \cap M' = \emptyset$, then either $I \subseteq K_0$ or $I \subseteq K_1$. Since $X = K_0 \cup P \cup K_1$ for some $P \in M'^{(r)}$, we obtain that $\Psi(I) = I \cap X = I$. This shows that Ψ is a bijection from the intervals of V' disjoint of M' to the intervals of X disjoint of P .

Now suppose that $I \in C_{V',M'}$ is such that $M' \subseteq I$. Then I can be written as $I = K_I \cup M'$ with $K_I \subseteq K_0 \cup K_1$. Thus $\Psi(I) = I \cap X = K_I \cup P$. Since $K_I \neq K_{I'}$ for $I \neq I'$, we obtain that Ψ is an injection from the intervals of V' containing M' to the intervals of X containing P . To check surjectivity, just notice that $K_I \cup P$ is an interval if and only if $K_I \cup M'$ is an interval.

It remains to prove that I is closed if and only if $\Psi(I)$ is closed. Throughout the rest of the proof, for a set $S \subseteq V$ we define

$$x_S = \min(S), \quad y_S = \max(S), \quad u_S = a(x_S, y_S).$$

Note that the backwards direction is straightforward from the definition of being closed.

Proposition 5.3. *If I is closed in V' , then $I \cap X$ is closed in X .*

Proof. Suppose by contradiction that $I \cap X$ is not closed in X . Then by condition (**) of Definition 2.2, there exists $y \in X \setminus I$ such that $u_{I \cap X}$ is an ancestor of y . Since $I \cap X \subseteq I$, we have that u_I is an ancestor of $u_{I \cap X}$. Therefore, $y \in X \setminus I \subseteq V' \setminus I$ is an ancestor of u_I which contradicts the fact that I is closed in V' . \square

The following observation will be useful for the rest of the proof.

Fact 5.4. *Let $W = V(u_W)$ be a closed interval in V . If x and y are two vertices such that $x \in W$ and $y \notin W$, then $a(x, y) = a(y, u_W)$ (see Figure 14).*

In particular, Fact 5.4 applied to $W = M'$ says that an element $y \notin M'$ have the same common ancestor with any $x \in M'$. We split the proof of the forward implication depending on the structure of H' given by Lemma 5.1.

Case 1: M' is a closed interval in V' .

The proof of Case 1 is slightly different depending on the location of the interval I in V' .

Case 1.1: $I \in C_{V',M'}$ such that $I \cap M' = \emptyset$.

As seen before, we have that $\Psi(I) = I$. By condition (**) of Definition 2.2 there is no vertex $x \in X \setminus I$ such that u_I is an ancestor of x . If there is a descendant of u_I in $V' \setminus I$, then the descendant

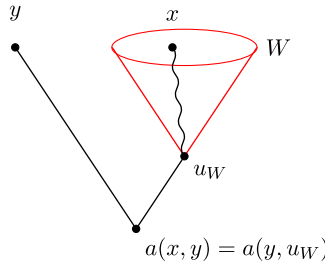


Figure 14. A picture of Fact 5.4.

is in the set $V' \setminus X = M' \setminus P$. Since $I \cap M' = \emptyset$, Fact 5.4, applied to the closed interval M' , implies that for every $y \in I'$ and $x \in M'$ we have $a(x, y) = a(y, u_{M'})$. Thus, if u_I is an ancestor of some $x \in M'$, then u_I is an ancestor of $u_{M'}$. This implies that u_I is an ancestor for the entire set M' and in particular of P , which contradicts the fact that I is closed in X . Therefore, I is a closed interval in V' .

Case 1.2: $I \in \mathcal{C}_{V', M'}$ such that $M' \subseteq I$.

Suppose that $I = K_I \cup M'$ is a interval in V' containing M' . We need to prove that $I = K_I \cup M'$ is closed in V' if $\Psi(I) = I \cap X = K_I \cup P$ is closed in X . If $K_I = \emptyset$, then $I = M'$ which is by assumption closed in V' . Otherwise, we claim that $u_I = u_{\Psi(I)}$. That is I and $\Psi(I)$ have the same common ancestor.

The assumption that $K_I \neq \emptyset$ gives us that either $x_I < \min(M')$ or $y_I > \max(M')$. Assume without loss of generality that $y_I > \max(M')$. Thus, $y_I \in K_I$ and we have that $y_I = y_{\Psi(I)} = \max(K_I)$. If $x_I \notin M'$, then similarly we have $x_I = x_{\Psi(I)}$ and consequently $u_I = a(x_I, y_I) = a(x_{\Psi(I)}, y_{\Psi(I)}) = u_{\Psi(I)}$. Now if $x_I \in M'$, then $x_I \notin K_I$. This implies that $x_{\Psi(I)} \in P \subseteq M'$. Since both $x_I, x_{\Psi(I)} \in M'$ and $y_I = y_{\Psi(I)} \notin M'$, by Fact 5.4 we obtain that $u_I = a(x_I, y_I) = a(u_{M'}, y_I) = a(u_{M'}, y_{\Psi(I)}) = a(x_{\Psi(I)}, y_{\Psi(I)}) = u_{\Psi(I)}$. Hence, $I = K_I \cup M'$ and $\Psi(I) = K_I \cup P$ have the same common ancestor.

To finish the proof note that by condition $(\star\star)$ of Definition 2.2 there are no descendants of $u_{\Psi(I)}$ in $X \setminus \Psi(I)$. Since $u_I = u_{\Psi(I)}$ and $V \setminus I' = K \setminus K_I = X \setminus \Psi(I)$, we conclude that there are no descendants of u_I in $V' \setminus I$ and consequently I is closed in V' .

Case 2: $M' \subseteq Q$ for some maximal comb $Q = A^Q \cup B^Q$ in V' with $M' \subseteq B^Q$.

We may assume without loss of generality that Q is a maximal left comb. Let $K_0^Q = K_0 \cap Q$ and $K_1^Q = K_1 \cap Q$. Clearly, $Q = K_0^Q \cup M' \cup K_1^Q$ with $K_0^Q < M' < K_1^Q$. Moreover, $Q = A^Q \cup B^Q$ with $A^Q < B^Q$ and $M' \subseteq B^Q$ (Figure 15). Thus, $A^Q \subseteq K_0^Q$ and by condition $(a3^*)$ of Definition 2.4, we obtain that K_0^Q is closed in V' . As in the first case, we split into two cases depending on the type of the interval.

Case 2.1: $I \in \mathcal{C}_{V', M'}$ such that $I \cap M' = \emptyset$.

Suppose that I is a closed interval in X . We claim that $V'(u_I) \cap M' = \emptyset$, i.e., the descendants of u_I are disjoint of M' . Applying Proposition 2.3 to the closed interval $V'(u_I)$ and maximal comb Q gives us that either $V'(u_I) \cap Q = \emptyset$, $V'(u_I) \subseteq Q$ or $Q \subseteq V'(u_I)$. If $V'(u_I) \cap Q = \emptyset$, then we immediately obtain that $V'(u_I) \cap M' = \emptyset$, since $M' \subseteq Q$. If $Q \subseteq V'(u_I)$, then $M' \subseteq V'(u_I)$ and consequently $P = M' \cap X \subseteq V'(u_I) \cap X = X(u_I)$. This implies that $X(u_I) \neq I$, which contradicts I being closed in X .

Thus, we may assume that $V'(u_I) \subseteq Q$ and $M' \not\subseteq V'(u_I)$. Then, by Proposition 2.7, we have that $V'(u_I) = A^{V'(u_I)} \cup B^{V'(u_I)}$ where either $V'(u_I) \subseteq A^Q$ or $A^{V'(u_I)} = A^Q$ and $B^{V'(u_I)} \subseteq B^Q$. For the first case, note that $A^Q \cap M' = \emptyset$ and therefore $V'(u_I) \cap M' = \emptyset$. For the second case, note that since $M' \not\subseteq V'(u_I)$, then $M' \not\subseteq B^{V'(u_I)}$. This implies that $V'(u_I) \subseteq K_0 \cup M'$. Together with the

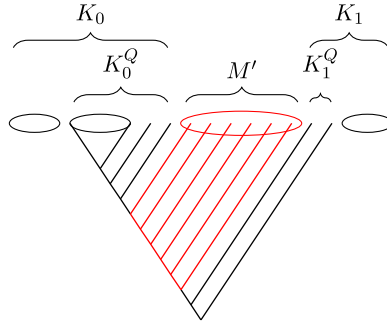


Figure 15. Maximal comb Q and sets K_0^Q and K_1^Q .

fact that $I \cap M' = \emptyset$ and $I \subseteq V'(u_I)$, we obtain that $I \subseteq K_0^Q$. Since K_0^Q is closed in V' , we have that the common ancestor $u_{K_0^Q} = a(\min(K_0^Q), \max(K_0^Q))$ is an ancestor of the entire I and therefore of u_I . Hence, $V'(u_I) \subseteq K_0^Q$, which implies that $V'(u_I) \cap M' = \emptyset$. The fact that I is closed in V' now follows because $V'(u_I) = X(u_I) = I$.

Case 2.2: $I \in \mathcal{C}_{V',M'}$ such that $M' \subseteq I$.

Let $I = K_0^I \cup M' \cup K_1^I$ be an interval in V' containing M' with $K_0^I \subseteq K_0$ and $K_1^I \subseteq K_1$. Suppose that $\psi(I) = K_0^I \cup P \cup K_1^I$ is closed in X . Since $K_0^Q \cup M'$ is closed, by the same argument of Case 1.2 (by considering $K_0^Q \cup M'$ instead of M'), we can show that if $x_I < \min(K_0^Q)$ or $y_I > \max(M')$, then $u_I = a(x_I, y_I) = a(x_{\psi(I)}, y_{\psi(I)}) = u_{\psi(I)}$ and consequently I is closed in V' .

Now suppose that $\min(K_0^Q) \leq x_I \leq \min(M')$ and $y_I = \max(M')$. Since both M' and P are not closed intervals in their respective ground sets, we have that $x_I \neq \min(M')$ and consequently $x_{\psi(I)} = x_I$ and $y_{\psi(I)} = \max(P)$. Hence, in this case, $K_0^I \subseteq K_0^Q$ and $K_I = \emptyset$, which implies that $I = K_0^I \cup M'$ and $\psi(I) = K_0^I \cup P$. Because Q is a maximal left comb with $M' \subseteq Q$, then both sets K_0^Q and $K_0^Q \cup M'$ are closed in V' . Therefore, by Proposition 5.3 the intervals K_0^Q and $K_0^Q \cup P$ are closed in X . Fact 5.4 applied to K_0^Q gives us that $a(z, y_{\psi(I)}) = a(z', y_{\psi(I)})$ for every $z, z' \in K_0^Q$. This implies that $u_{\psi(I)} = a(x_{\psi(I)}, y_{\psi(I)}) = a(\min(K_0^Q), y_{\psi(I)})$, i.e., $K_0^Q \cup P$ and $\psi(I) = K_0^I \cup P$ have $u_{\psi(I)}$ as the same common ancestor. Since $K_0^Q \cup P$ and $K_0^I \cup P$ are both closed in X , we obtain that $K_0^Q = K_0^I$. Thus $I = K_0^Q \cup M'$, which is closed in V' .

The next result shows that we can always find in an edge the location of the maximal comb with colouring data containing $a(P)$. This will be extremely important, since the comb will be the only maximal comb such that colouring data changes while we run through different edges of H' .

Proposition 5.5. *Let H' be a fixed pre-processed daisy obtained by Lemma 5.1. There exists a unique interval $J \subseteq [k+r]$ such that for every edge $X = K_0 \cup P \cup K_1 = \{x_1, \dots, x_{k+r}\}$ in H' , the interval $X_J = \{x_j\}_{j \in J}$ is a maximal comb of type depending only on H' with*

$$a(P) \subseteq F(X_J).$$

Moreover, writing $X_J = A^{X_J} \cup B^{X_J}$ we have one of the following:

1. If H' satisfies statement (1) of Lemma 5.1, then $A^{X_J} \subseteq P$ and X_J is the smallest maximal comb containing P with non-empty colouring data.
2. If H' satisfies statement (2) of Lemma 5.1, then $X_J = I \cap X$, where $I = A \cup B$ is the maximal comb in V' such that $M' \subseteq B$, and X_J satisfies $P \subseteq B^{X_J}$.

Proof. The idea of the proof is to identify certain maximal combs in V' with maximal combs in an edge X . Because the structure of those maximal combs in V' only depends on H' , we will obtain the same for the corresponding combs in X . Proposition 5.2 will be useful here, since by condition (a3*) and (b3*) of Definition 2.4 a comb can be defined by looking at certain closed subintervals. The proof is split into cases depending on the structure of the tree $T_{V'}$

Case 1: M' is a closed interval in V' .

We will construct a maximal comb in X by looking at a maximal comb in V' containing M' . Write $K_0 = \{x_1, \dots, x_{k_0}\}$, $M' = \{y_1, \dots, y_{m'}\}$ and $K_1 = \{z_1, \dots, z_{k_1}\}$. There are two possibilities here:

Case 1.1: Either $M' \cup \{z_1\}$ is a closed interval in V' or $M' \cup \{x_{k_0}\}$ is a closed interval in V' .

Suppose without loss of generality that $M' \cup \{z_1\}$ is closed in V' . In this case $M' \cup \{z_1\}$ is a left comb. Let $M' \cup \{z_1, \dots, z_t\}$ be the maximal left comb obtained by extending $M' \cup \{z_1\}$. We will assume during the entire proof that $t < k_1$. For $t = k_1$, the same proof work by removing any claims and sets involving z_{t+1} . By condition (a3*) of Definition 2.4 and Definition 2.6, $M' \cup \{z_1, \dots, z_t\}$ being a maximal left comb is the same as saying that the intervals M' and $M' \cup \{z_1, \dots, z_i\}$ are closed for every $1 \leq i \leq t$, but the interval $M' \cup \{z_1, \dots, z_{t+1}\}$ is not closed.

Set $J = \{k_0 + 1, \dots, k_0 + r + t\}$. Let X be an edge of H' with petal P . We claim that X_J is a maximal left comb in X with $A^{X_J} \subseteq P$. To see that consider the intervals

$$J_i = \{k_0 + 1, \dots, k_0 + r + i\}, \quad 0 \leq i \leq t + 1.$$

In particular $J_t = J$. Note that $X_{J_0} = M' \cap X = P$ and $X_{J_i} = (M' \cap \{z_1, \dots, z_i\}) \cap X$ for $1 \leq i \leq t + 1$. Thus, by applying Proposition 5.2 with $I = M'$ and $I = M' \cup \{z_1, \dots, z_i\}$, we obtain that X_{J_i} is closed in X for $0 \leq i \leq t$ and $X_{J_{t+1}}$ is not closed in X . Hence, by condition (a3*) of Definition 2.4 and Definition 2.6, we have that $X_J = X_{J_t}$ is a maximal left comb. Since $P = X_{J_0} \subseteq X_{J_1} \subseteq \dots \subseteq X_{J_t} = X_J$ are all closed intervals, we have that $A^{X_J} \subseteq P$. Thus, $|A^{X_J}| \leq |P| = r$ and we have that either X_J is a maximal comb of type 3 or type 4 depending on the size of $|X_J| = r + t$. Because t is a parameter that depends on the size of the maximal comb in V' , i.e., on the structure of H' , we conclude that the type of X_J is independent of our choice of edge X .

It remains to show that $a(P) \subseteq F(X_J)$ and X_J is the smallest maximal comb containing P with non-empty data colouring. For the first, note that $F(X_J) = a(X_J)$ because X_J is of type 3 or 4. Thus, $a(P) \subseteq a(X_J) = F(X_J)$. For the latter, note that the only potential maximal comb smaller than X_J containing P is P itself. However, if P is a maximal comb, then it is a comb of type 2 and therefore $F(P) = \emptyset$. Hence, X_J is the smallest maximal comb containing P with non-empty colouring data.

Case 1.2: Both $M' \cup \{z_1\}$ and $M' \cup \{x_{k_0}\}$ are not closed in V' .

By Definition 2.6, M' is a maximal comb. Set $J = \{k_0 + 1, \dots, k_0 + r\}$. Note that $X_J = P$. By Proposition 5.2, the set $P = M' \cap X$ is closed in X and $P \cup \{z_1\} = (M' \cup \{z_1\}) \cap X$ and $P \cup \{x_{k_0}\} = (M' \cup \{x_{k_0}\}) \cap X$ are not closed in X . Thus, P is a maximal comb in X . It is clear that $A^{X_J} \subseteq X_J = P$. Since $|P| = r$ and $P \cup \{z_1\}$, $P \cup \{x_{k_0}\}$ are not closed, we have that $X_J = P$ is of type 1. Therefore, the type of X_J does not depend on X . Moreover, the fact that X_J is of type 1 gives us that $a(P) = a(X_J) = F(X_J)$. The minimality of X_J is immediate from the fact that all combs with non-empty data has size at least r .

Case 2: $M' \subseteq B$ for a maximal comb $I = A \cup B$ in V' .

Suppose without loss of generality that $I = A \cup B$ is a maximal left comb. Let $A = \{x_{k_0-p-\ell+1}, \dots, x_{k_0-p}\}$, $B \cap K_0 = \{x_{k_0-p+1}, \dots, x_{k_0}\}$ and $B \cap K_1 = \{z_1, \dots, z_t\}$. Set $J = \{k_0 - p - \ell + 1, \dots, k_0 + r + t\}$. Let X be an edge of H' with petal P (see Figure 16). Clearly, $X_J = I \cap X$. We claim that X_J is a maximal left comb with $P \subseteq B^{X_J}$. By Definition 2.4 and 2.6 we have that A is a

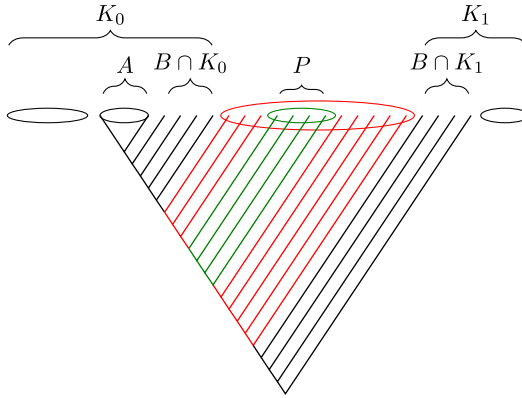


Figure 16. Case 2 of Proposition 5.5.

closed interval in V' , $A \setminus \{x_{k_0-p}\}$ and $I \cup \{z_{t+1}\}$ are not closed in V' and

$$\begin{aligned} \delta(x_{k_0-p}, x_{k_0-p+1}) &> \dots > \delta(x_{k_0-1}, x_{k_0}) > \delta(x_{k_0}, y_1) > \delta(y_1, y_2) > \dots > \delta(y_{m'-1}, y_{m'}) \\ &> \delta(y_{m'}, z_1) > \delta(z_1, z_2) > \dots > \delta(z_{t-1}, z_t). \end{aligned}$$

Let $P = \{y_{i_1}, \dots, y_{i_r}\}$. By Proposition 5.2, the set $A = A \cap X$ is closed in X and the sets $A \setminus \{x_{k_0-p}\} = (A \setminus \{x_{k_0-p}\}) \cap X$ and $X_J \cup \{z_{t_1}\} = (I \cup \{z_{t+1}\}) \cap X$ are not closed in X . Moreover, since $P \subseteq M'$, we have that

$$\begin{aligned} \delta(x_{k_0-p}, x_{k_0-p+1}) &> \dots > \delta(x_{k_0-1}, x_{k_0}) > \delta(x_{k_0}, y_{i_1}) > \delta(y_{i_1}, y_{i_2}) > \dots > \delta(y_{i_r-1}, y_{i_r}) \\ &> \delta(y_{i_r}, z_1) > \delta(z_1, z_2) > \dots > \delta(z_{t-1}, z_t). \end{aligned}$$

Thus, by Definition 2.4 and Definition 2.6, the interval X_J is a maximal left comb in X with $A^{X_J} = A$ and $|B^{X_J}| = r + t + p$. Since $A \subseteq K_0$, we obtain that $P \subseteq B^{X_J}$. Note that to determine the type of X_J we need to know the sizes of X_J , A^{X_J} and B^{X_J} . None of this parameters depends on the choice of X . Hence, the type of X_J is independent of X . Finally, because $|B^{X_J}| \geq |P| \geq r$, we obtain that $|X_J| \geq r + 1$ and consequently the comb X_J is of type 3, 4 or 5. If it is of type 3 or 4, then $a(P) \subseteq a(X_J) = F(X_J)$. If it is of type 5, then $a(P) \subseteq a(\max(A^{X_J}) \cup B^{X_J}) = F(X_J)$. \square

To finish the section we prove that the maximal comb determined by the set J is the comb that essentially determines the colour of the entire edge.

Proposition 5.6. *Let $X = \{x_1, \dots, x_{k+r}\}$, $X' = \{x'_1, \dots, x'_{k+r}\}$ be two edges in H' and let $X_J = \{x_j\}_{j \in J}$, $X'_J = \{x'_j\}_{j \in J}$. If $\chi(X) = \chi(X')$, then $\chi_0(X_J) = \chi_0(X'_J)$.*

Proof. Let $P, P' \subseteq M'$ be the petals of X and X' , respectively. By definition, $\chi(X) = \chi(X')$ implies that

$$\sum_{I \in \mathcal{I}_X} \chi_0(I) = \sum_{I' \in \mathcal{I}_{X'}} \chi_0(I') \pmod{2}.$$

By the definition of colouring data, if $F(I) = \emptyset$, then $\chi_0(I) = 0$. Thus, we may rewrite the equality above as

$$\sum_{\substack{I \in \mathcal{I}_X \\ F(I) \neq \emptyset}} \chi_0(I) = \sum_{\substack{I' \in \mathcal{I}_{X'} \\ F(I') \neq \emptyset}} \chi_0(I') \pmod{2}. \tag{4}$$

We claim that if $I = A \cup B$ is a maximal comb of X with $F(I) \neq \emptyset$, then either $I \cap P = \emptyset$ or $P \subseteq I$. Note that, in the colouring defined in Subsection 3.3, whenever $F(I) = \emptyset$, we have that $|I| \geq |P| = r$. By Lemma 5.1, the daisy H' satisfies one of the following conditions: Either M' is a closed interval in $V' := V(H')$ or there exists a maximal comb $Q = A^Q \cup B^Q$ such that $M' \subseteq B^Q$. If M' is closed in V' , then by Proposition 5.2 the petal $P = M' \cap X$ is a closed interval in X . Thus, Proposition 2.3 applied to the closed intervals I and P gives the desired result that either $I \cap P = \emptyset$ or $P \subseteq I$. Now suppose that we are in Condition (2) of Lemma 5.1. By Proposition 5.5, we have that $P \subseteq B^{X_j}$, where B^{X_j} is the “teeth” part of the comb $X_j = A^{X_j} \cup B^{X_j}$. Thus, Proposition 2.7 applied to the maximal combs I and X_j implies that $I \cap X_j = \emptyset$, $I \subseteq A^{X_j}$ or $X_j \subseteq A \subseteq I$. In the first two cases we obtain $I \cap P = \emptyset$, while in the latter we have $P \subseteq I$.

The idea of the proof of Proposition 5.6 is to show that there exists a bijection between $\{I \in \mathcal{I}_X : F(I) \neq \emptyset\}$ and $\{I' \in \mathcal{I}_{X'} : F(I') \neq \emptyset\}$ such that X_j is sent to X'_j and every $I \neq X_j$ is sent to an $I' \neq X'_j$ with $\chi_0(I) = \chi_0(I')$. Hence, after some cancellation, we obtain from equation (4) that $\chi_0(X_j) = \chi_0(X'_j)$. Based on the last paragraph, we construct such a bijection by splitting $\{I \in \mathcal{I}_X : F(I) \neq \emptyset\}$ into two parts:

Case 1: $I \in \mathcal{I}_X$ is a maximal comb of X with $F(I) \neq \emptyset$ and $I \cap P = \emptyset$.

We claim that $I \in \mathcal{I}_{X'}$ is a maximal comb in X' of the same type and consequently $\chi_0(I)$ is the same in X and X' . Assume without loss of generality that $I \subseteq K_0$. Let x be the element preceding $\min(I)$ in X (In the case that such x does not exist, we simply take $x = \min(I)$). Let y be the element after $\max(I)$ in X . Similarly, define x' as the element before $\min(I)$ in X' and y' as the element after $\max(I)$ in X' . Since $I \subseteq K_0$, clearly $x = x'$. However, y and y' are not necessarily the same. By conditions (a3*) and (b3*) of Definition 2.4 and Definition 2.6, to prove that $I \in \mathcal{I}_{X'}$ is enough to check that $I \cup \{x\}$, $L \subseteq I$, $I \cup \{y\}$ are closed intervals in X if and only if $I \cup \{x'\}$, $L \subseteq I$ and $I \cup \{y'\}$ are closed intervals in X' , respectively. Since $F(I) \neq \emptyset$, we have that I is of type 1, 3, 4 or 5. Note that one can distinguish between these types by determining the size of the “handle” and “teeth” of I . Thus, by checking the properties above, we also obtain that I have the same type in X and X' .

For an interval $L \subseteq I$, by Proposition 5.2 we have that $L = L \cap X$ is a closed interval in X if and only if it is a closed interval in V' . Another application of Proposition 5.2 gives us that $L = L \cap X'$ is a closed interval if it is closed in V' . Hence, L is closed in X if and only if it is closed in X' . Similarly, the same argument works for $I \cup \{x\}$ and $I \cup \{x'\}$, because $x = x' \notin M'$. Moreover, if $y \in K_0$, then $y' = y \notin M'$ and we also obtain that $I \cup \{y\}$ is closed in X if and only if $I \cup \{y'\}$ is closed in X' . Hence, the only case remaining is when $y \notin K_0$, i.e, $y = \min(P)$ and $y' = \min(P')$.

We split the argument into two cases depending on the structure given by Lemma 5.1. Suppose that M' is closed in V' and let $u = a(\min(M'), \max(M'))$ be the common ancestor of M' . By Fact 5.4, we have that $a(z, y) = a(z, y') = a(z, u)$ for every $z \in I$. Therefore, the entire set M' is descendant of the common ancestors of $I \cup \{y\}$ and $I \cup \{y'\}$, which by condition (ii*) of Definition 2.2 implies that both sets are not closed. Now suppose that $M' \subseteq B^Q$ for some maximal comb $Q = A^Q \cup B^Q$. Since I is a closed interval in X , then by Proposition 5.2 it is a closed interval in V' . Thus, by Proposition 2.3, applied to I and the maximal comb Q , one of the following three possibilities holds: $I \cap Q = \emptyset$, $I \subseteq Q$ or $Q \subseteq I$. Clearly, the last possibility cannot hold, since $P \subseteq Q$ and $I \cap P = \emptyset$. Suppose that $I \cap Q = \emptyset$. By Proposition 5.2, the interval $Q \cap X$ is a closed interval in X . Since $(I \cup \{y\}) \cap (Q \cap X) = \{y\} \neq \emptyset$ and $\{y\} \neq Q \cap X$, we obtain by Proposition 2.3 that $I \cup \{y\}$ is not closed in X . Similarly, $I \cup \{y'\}$ is not closed in X' .

Now we handle with the case that $I \subseteq Q$. By Proposition 2.7, either $I \subseteq A^Q$ or $I = A \cup B$ is a comb with $A = A^Q$ and $B \subseteq B^Q$. Since $I \subseteq K_0$, we have that Q is a maximal left comb. Let $K_0^Q = K_0 \cap Q$, $B^Q = \{z_1, \dots, z_b\}$ and let $Q_{z_i} = A^Q \cup \{z_1, \dots, z_i\}$ be the subcomb of Q ending in z_i . By condition (a3*) of Definition 2.4, we have that A^Q and Q_z are closed in V' for every $z \in B^Q$. Moreover, note that $\min(I) \in A^Q$. Let $v = a(\min(A^Q), \max(A^Q))$ and $w = a(\min(I), y)$ be the common ancestor of A^Q and $I \cup \{y\}$, respectively. By Fact 5.4 applied to A^Q , we have that

$a(\min(I), x) = a(v, x) = a(\min(A^Q), x)$ for every $x \in M'$. Thus, $X(w) = V'(w) \cap X = Q_v \cap X = K_0^Q \cup \{y\}$, which implies that $I \cup \{y\}$ is a closed interval in X if and only if $I = K_0^Q$. Similarly, $I \cup \{y'\}$ is a closed interval in X' if and only if $I = K_0^Q$. Hence, $I \cup \{y\}$ is closed in X if and only if $I \cup \{y'\}$ is closed in X' .

Case 2: $I \in \mathcal{I}_X$ is a maximal comb of X with $F(I) \neq \emptyset, P \subseteq I$ and $I \neq X_j$.

In this case, by Proposition 4.2 and Proposition 5.5, we have that $F(I) \cap a(P) = \emptyset$ and consequently $F(I) \neq a(I)$. Thus, by our colouring, we obtain that I is of type 5, i.e., $I = A \cup B$ is a maximal left/right comb with $|A| > r$ and $|B| \geq r$. We may assume that I is a maximal left comb. Hence, $F(I) = a(\{\max(A)\} \cup B)$ and the fact that $F(I) \cap a(P) = \emptyset$ implies that $P \subseteq A$. Write $A = K_A \cup P$ with $K_A \subseteq K_0 \cup K_1$. We claim that $I' = K_A \cup P' \cup B$ is a maximal comb of type 5 with set of “teeth” $B' = B$ and “handle” $A' = K_A \cup P'$.

Write $B = \{y_1, \dots, y_t\}$. Let $x = \max(A)$, $x' = \max(A')$ and let z be the element coming after B in V' (In case that such element does not exist, we take $z = \max(B)$). By condition (a3*) of Definition 2.4 and Definition 2.6 to prove that $I = A' \cup B'$ is a maximal comb of type 5 with $A' = K_A \cup P$ and $B' = B$ it is enough to prove that A' and $A' \cup \{y_1, \dots, y_i\}$ are closed in X' for every $1 \leq i \leq t$, $A' \setminus \{x'\}$ is not closed in X' and $A' \cup \{y_1, \dots, y_t, z\}$ is closed if and only if $A \cup \{y_1, \dots, y_t, z\}$ is closed in X . Applying Proposition 5.2 with X and V' and then V' and X' gives us that $A, A \cup \{y_1, \dots, y_i\}$ and $A \cup \{y_1, \dots, y_t, z\}$ are closed in X if and only if $A', A' \cup \{y_1, \dots, y_i\}$ and $A' \cup \{y_1, \dots, y_t, z\}$ are closed in X' , respectively. Since I is a maximal comb in X , this implies that $A', A' \cup \{y_1, \dots, y_i\}$ are closed in X' for $1 \leq i \leq t$. If $x \notin P$, then $x = x'$ and by the same argument $A' \setminus \{x'\}$ is not closed in X' .

It remains to deal with the case that $x \in P$, i.e., $x = \max(P)$ and $x' = \max(P')$. The proof is split into two cases depending on the structure of H' given by Lemma 5.1. If M' is a closed interval in V' , then by Proposition 5.2 the interval P' is closed in X' . The intersection $A' \setminus \{x'\}$ is proper since $|A'| = |A| > r = |P'|$ and $x' \in P'$. Therefore, by Proposition 2.3, we have that $A' \setminus \{x'\}$ is not closed in X' .

Now suppose that $M' \subseteq B^Q$ for some maximal comb $Q = A^Q \cup B^Q$. By Proposition 5.5, the maximal combs X_j and X'_j satisfies $P \subseteq B^{X_j}$ and $P' \subseteq B^{X'_j}$. Applying Proposition 2.7 to the maximal combs I and X_j gives us that either $I \subseteq A^{X_j}$ or $X_j \subseteq A$. Since $P \cap A^{X_j} = \emptyset$, it follows that $X_j \subseteq A$. Because $x = \max(A) \in P$, we have that $\max(K_A) < \min(P)$. This implies that X_j is a maximal left comb. Hence, by Proposition 5.5 both Q and X'_j are maximal left combs.

We claim that $A' \setminus \{x'\} \cap X'_j$ is a proper intersection. Since $|X'_j| = |X_j| \leq |A| = |A'|$, by Proposition 2.3 applied to A' and X'_j , we have that $X'_j \subseteq A'$. Note that we already proved for $1 \leq i \leq t$ that A' and $A' \cup \{y_1, \dots, y_i\}$ are closed in X' . Hence, by the maximality of X'_j we have that $A' \neq X'_j$ (otherwise we could extend to the left comb $X'_j \cup B$). Thus, X'_j is strictly contained in A' , which implies that $K_A \setminus X'_j \neq \emptyset$. This concludes that $A' \setminus \{x'\} \cap X'_j$ is proper and by Proposition 2.3 the interval $A \setminus \{x\}$ is not closed in X' .

Therefore, the interval $I' = A' \cup B'$ is a maximal left comb in X' of type 5 with $A' = K_A \cup P'$ and $B' = B$. It is not difficult to check (by Proposition 5.2) that the correspondence between I and I' is a bijection. Moreover, since $A = K_A \cup P$ is closed in X , we obtain by Proposition 5.2 that $K_A \cup M'$ is closed in V' . It follows by Fact 5.4 that $a(x, y_1) = a(x', y_1)$ and consequently that $F(I) = a(B \cup \{x\}) = a(B' \cup \{x'\}) = F(I')$. Hence, by Proposition 4.1 we have $\chi_0(I) = \chi_0(I')$. \square

6. Main proof

The proof of Theorem 1.3 follows by a simple induction of the following stepping up theorem.

Theorem 6.1. *Let $m \geq 100kr^2$, $N = \min_{0 \leq j \leq k} \{D_{r-1}^{smp}(\frac{1}{5}k^{-1/2}m^{1/2}, j)\}$ be integers and let $\{\varphi_i\}_{r-1 \leq i \leq k+r-1}$ be a family of colourings $\varphi_i: [N]^{(i)} \rightarrow \{0, 1\}$ without a monochromatic copy*

of a simple $(r - 1, \frac{1}{5}k^{-1/2}m^{1/2}, i - r + 1)$ -daisy. Then, the colouring $\chi : [2^N]^{(k+r)} \rightarrow \{0, 1\}$ described in Subsection 3.3 does not contain a monochromatic simple (r, m, k) -daisy.

Proof. Suppose by contradiction that there exists a monochromatic simple (r, m, k) -daisy H in $[2^N]^{(k+r)}$ with kernel $K = K_0 \cup K_1$ of size k , universe of petals M of size m and $K_0 < M < K_1$. By Lemma 5.1, we obtain a monochromatic simple $(r, \frac{1}{2}k^{-1/2}m^{1/2}, k)$ -daisy H' with same kernel and universe of petals $M' = \{y_1, \dots, y_{m'}\} \subseteq M$ of size $m' = \frac{1}{2}k^{-1/2}m^{1/2}$ satisfying that either M' is a closed interval in $V' = V(H')$ or M' is part of the “teeth” of a maximal comb $I = A \cup B$, i.e., $M' \subseteq B$.

Note that every edge $X \in H'$ can be written in the form $X = K_0 \cup P \cup K_1$ where $P \in (M')^{(r)}$ is a petal of H' . Since H' is monochromatic, we have that

$$\chi(X) = \sum_{I \subseteq \mathcal{I}_X} \chi_0(I) \pmod{2}$$

is constant, for every $X \in H'$. Thus, by Propositions 5.5 and 5.6, there exists a unique interval $J \subseteq [k + r]$ such that for every $X \in E(H')$ the interval $X_J = \{x_j\}_{j \in J}$ is maximal comb with colour $\chi_0(X_J)$ constant.

As in the proof give in Subsection 3.1, our goal is to use the fact that the combs X_J are monochromatic with respect to χ_0 to find a large 1-comb. Let $t = |J| - r$ and let G be the simple $(r, \frac{1}{2}k^{-1/2}m^{1/2}, t)$ -daisy constructed by taking as edges the combs X_J for every edge $X \in H'$. To be more precise, let K_J be the subset of t vertices of $K_0 \cup K_1$ in the interval J . Note that every comb X_J can be partitioned into $X_J = K_J \cup P$, where $P \subseteq M'$ is the petal of X . We define G as the simple $(r, \frac{1}{2}k^{-1/2}m^{1/2}, t)$ -daisy given by

$$\begin{aligned} V(G) &= K_J \cup M' \\ G &= \{X_J : X \in H'\} \end{aligned}$$

As discussed in the last paragraph the $(t + r)$ -graph G is monochromatic under the colouring χ_0 . The following lemma is a variant of Proposition 3.1 for simple daisies.

Proposition 6.2. *If M' is a closed interval in $V(H')$ and G is monochromatic with respect to the colouring χ_0 , then there exists an interval $M'' \subseteq M'$ of size $|M''| \geq (|M'| - r + 6)/2$ such that M'' is a 1-comb in V' .*

Proof. By Proposition 5.5, all the edges X_J of G are combs of the same type. Thus we may assume without loss of generality that X_J is either a broken comb or a ℓ -left comb in X . Since M' is closed, by the same proposition we obtain that $A^{X_J} \subseteq P$ and consequently $K_J \subseteq B^{X_J}$ for every edge $X \in H'$. Therefore, we either have $K_J = \emptyset$ (and X_J is a broken comb) or $P < K_J$ for every X , which implies that $K_J \subseteq K_1$, i.e., $M' < K_J$. Moreover, if X_J is an ℓ -left comb, then $A^{X_J} \subseteq P$ implies that $\ell \leq r$. This implies that X_J is either of type 1, 3 or 4.

We split the proof into two cases according to the size of $t = |K_J|$. Write $M' = \{y_1, \dots, y_{m'}\}$, $K_J = \{y_{m'+1}, \dots, y_{m'+t}\}$ (if $K_J \neq \emptyset$) and $\delta_i^G = \delta(y_i, y_{i+1})$ for $1 \leq i \leq m' + t - 1$.

Case 1: $0 \leq t \leq r - 2$.

Since $|X_J| = r + t \leq 2r - 2$, we obtain that X_J is either of type 1 or 3. The proof follow the same lines of the proof of Proposition 3.1. Write $P = \{y_{i_1}, \dots, y_{i_r}\} \subseteq M'$ for indices $1 \leq i_1 < \dots < i_r \leq m'$. Suppose without loss of generality that G is monochromatic of colour 0, i.e., $\chi_0(X_J) = 0$ for every $X_J \in G$. Thus, by the definition of χ_0 for combs of type 1 and 3, we do not have that $\delta_{i_r-3}^G < \delta_{i_r-2}^G > \delta_{i_r-1}^G$. In particular, because X_J is arbitrary, this implies that there are no indices $r - 3 \leq p < q < s \leq m' - 1$ such that $\delta_p^G < \delta_q^G > \delta_s^G$. That is, the sequence $\{\delta_i^G\}_{i=r-3}^{m'-1}$ has no local maximum.

Now the same argument as in Proposition 3.1 gives that there exists an interval $M'' = \{y_p, \dots, y_q\} \subseteq M'$ such that $\{\delta_i^G\}_{i=p}^{q-1}$ is monotone and $|M''| \geq (|M'| - r + 6)/2$. By the definition given in Example 2.5, it follows that M'' is a 1-comb.

Case 2: $t \geq r - 1$.

In this case X_J is a left comb of type 4 for every $X_J \in G$, since $|X_J| = |P| + |K_J| = r + t \geq 2r - 1$. Suppose without loss of generality that G is monochromatic of colour 0 and that $\varphi_t(\{\delta_{m'}^G, \dots, \delta_{m'+t-1}^G\}) = 0$. Let $u = a(\min(M'), \max(M'))$. Fact 5.4 applied to M' gives us that $\delta(z, y_{m'+1}) = \delta(u, y_{m'+1})$ for every $z \in M'$. In particular, this implies that $\delta(z, y_{m'+1}) = \delta_{m'}^G$ for every $z \in M'$.

Write $P = \{y_{i_1}, \dots, y_{i_r}\} \subseteq M'$ with $1 \leq i_1 < \dots < i_r \leq m'$ and $X_J = P \cup K_J = \{y_{i_1}, \dots, y_{i_r}, y_{m'}, \dots, y_{m'+t-1}\}$. Since $\chi_0(X_J) = 0$, $\delta(y_{i_j}, y_{m'+1}) = \delta_{m'}^G$ for every $1 \leq j \leq r$ and $\varphi_t(\{\delta_{m'}^G, \dots, \delta_{m'+t-1}^G\}) + 1 = 1$ we obtain by the definition of χ_0 for combs of type 4 that the inequality $\delta_{i_{r-3}}^G < \delta_{i_{r-2}}^G > \delta_{i_{r-1}}^G$ cannot hold. Because X_J is arbitrary, we have that there are no indices $r - 3 \leq p < q < s \leq m' - 1$ such that $\delta_p^G < \delta_q^G > \delta_s^G$. Hence, similarly as in Case 1 we find an interval $M'' \subseteq M'$ of size at least $(|M'| - r + 6)/2$ such that M'' is a 1-comb in V' . \square

To finish the proof of Theorem 1.3 we are going to show now that if G is monochromatic with respect to χ_0 , then there exists a monochromatic simple $(r - 1, \frac{1}{5}k^{-1/2}m^{1/2}, j)$ -daisy in $\delta(G) \subseteq [N]$ with respect to some colouring φ_{j+r-1} . The proof is split into several cases depending on the structure of H' given by Lemma 5.1 and on the possible types of X_J .

Case 1: M' is a closed interval in V' .

As usual, we may assume that an edge of G is either a broken comb or a left comb. By Proposition 6.2, there exists an interval $M'' \subseteq M'$ of size $h = (|M'| - r + 6)/2$ such that M'' is a 1-comb. Consider the colouring χ_0 over the monochromatic subdaisy $G' := G[K_J \cup M''] \subseteq G$. As in the proof of Proposition 6.2, we have that either X_J is a broken comb and $t = |K_J| = 0$ or X_J is an ℓ -left comb with $\ell \leq r$ and $M' < K_J$. Write $M'' = \{y_{i_1}, \dots, y_{i_h}\}$ with $1 \leq i_1 < \dots < i_h < m'$, $K_J = \{y_{m'+1}, \dots, y_{m'+t}\}$ (if $K_J \neq \emptyset$) and $\delta_i^G = \delta(y_i, y_{i+1})$.

Let $X_J = P \cup K_J = \{x_1, \dots, x_r\} \cup \{y_{m'+1}, \dots, y_{m'+t}\}$ be an arbitrary edge from G' with $P \subseteq (M'')^{(r)}$. Note that since M'' is a 1-comb, then $\delta(x_{r-3}, x_{r-2}), \delta(x_{r-2}, x_{r-1}), \delta(x_{r-1}, x_r)$ forms a monotone sequence. Moreover, as discussed in Proposition 6.2, the comb X_J is of type 1, 3 or 4. Thus, by the colouring defined in Subsection 3.3, we have $\chi_0(X_J) = \varphi_{r+t-1}(\delta(X_J))$, i.e., the colour of X_J is determined by its full projection on the levels $[N]$.

Let $u = a(\min(M'), \max(M'))$. Note that since M' is closed, by Fact 5.4 we have that $a(x_r, y_{m'+1}) = a(u, y_{m'+1}) = a(y_{m'}, y_{m'+1})$. Consequently, we have that $\delta(x_r, y_{m'+1}) = \delta_{m'}^G$, which implies that $\delta(X_J) = \{\delta(x_1, x_2), \dots, \delta(x_{r-1}, x_r)\} \cup \{\delta_{m'}^G, \dots, \delta_{m'+t-1}^G\}$. Therefore the projection of all the edges of X_J forms a simple $(r - 1, h - 1, t)$ -daisy $D \subseteq [N]$ with universe of petals $\delta(M'')$ an kernel $K_D = \{\delta_{m'}^G, \dots, \delta_{m'+t-1}^G\}$ satisfying $K_D < \delta(M'')$ (see Figure 17). By the fact that G' is monochromatic with respect to χ_0 , we have that D is a monochromatic simple $(r - 1, h - 1, t)$ -daisy with respect to the colouring φ_{r+t-1} . This leads to a contradiction since $h - 1 \geq (m' - r + 4)/2 \geq \frac{1}{5}k^{-1/2}m^{1/2}$ for $m \geq 100kr^2$ and φ_{r+t-1} has no monochromatic simple $(r - 1, \frac{1}{5}k^{-1/2}m^{1/2}, t)$ -daisy.

Case 2: There exists a maximal comb $I = A \cup B$ in $V(H')$ such that $M' \subseteq B$

We may assume without loss of generality that $I = A \cup B$ is a left comb. Write $A = \{y_1, \dots, y_\ell\}$, $B_0 = B \cap K_0 = \{y_{\ell+1}, \dots, y_{\ell+p}\}$, $M' = \{y_{\ell+p+1}, \dots, y_{\ell+p+m'}\}$ and $B_1 = B \cap K_1 = \{y_{\ell+p+m'+1}, \dots, y_{\ell+p+m'+t}\}$ (as in Figure 18). By Proposition 5.5, $V(G) = I$ and all the edges $X_J = A^{X_J} \cup B^{X_J} \in G$ are maximal left comb of same type with $A^{X_J} = A$, $B^{X_J} = B_0 \cup P \cup B_1$ and

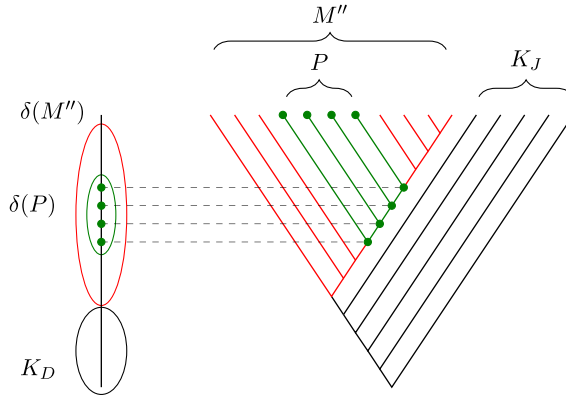


Figure 17. Case 1 of Theorem 6.1.

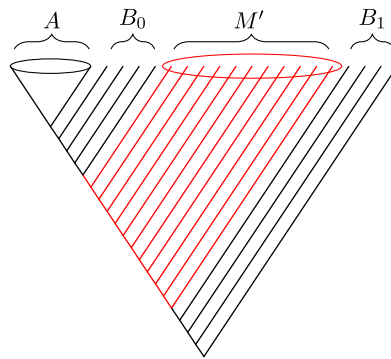


Figure 18. Auxiliary tree of G in Case 2.

$A < B_0 < P < B_1$. In particular, this implies that $|B^{X_J}| \geq r$ and X_J is of type 3, 4 or 5. We split the cases depending on the type of X_J . Let $\delta_i^G = \delta(y_i, y_{i+1})$ for $1 \leq i \leq \ell + p + m' + t - 1$. For an arbitrary edge $X_J \in G$, write $X_J = \{x_1, \dots, x_{\ell+p+r+t}\}$ with $x_i = y_i$ for $1 \leq i \leq \ell + p$, $P = \{x_{\ell+p+1}, \dots, x_{\ell+p+r}\} \subseteq M'$ and $x_{\ell+p+r+i} = y_{\ell+p+m'+i}$ for $1 \leq i \leq t$ and let $\delta_i^{X_J} = \delta(x_i, x_{i+1})$ for $1 \leq i \leq \ell + p + r + t - 1$.

Case 2.1: X_J is of type 3.

Recall that if X_J is of type 3, then $|A^{X_J}| \leq r$ and $r \leq |X_J| = |A^{X_J}| + |B^{X_J}| \leq 2r - 2$. Because $|B^{X_J}| \geq r$, we obtain that $|A^{X_J}| \leq r - 2$. This implies that $\{x_{r-1}, x_r\} \subseteq B^{X_J}$ and consequently $\delta_{r-3}^{X_J} > \delta_{r-2}^{X_J} > \delta_{r-1}^{X_J}$. Therefore, by the fact that $|\delta(X_J)| \geq |\delta(\{x_\ell, \dots, x_{\ell+p+r+t}\})| = p + r + t \geq r$, we obtain that $\chi_0(X_J) = \varphi_{|\delta(X_J)|}(\delta(X_J))$.

Note that

$$\begin{aligned} \delta(X_J) &= \{\delta_1^G, \dots, \delta_{\ell+p-1}^G\} \cup \{\delta_{\ell+p}^{X_J}, \dots, \delta_{\ell+p+r-1}^{X_J}\} \cup \{\delta_{\ell+p+m'}^G, \dots, \delta_{\ell+p+m'+t-1}^G\} \\ &= \delta(A \cup B_0) \cup \{\delta_{\ell+p}^{X_J}, \dots, \delta_{\ell+p+r-1}^{X_J}\} \cup \delta(\{y_{\ell+p+m'}\} \cup B_1). \end{aligned}$$

Hence, the projection of the edges of G is a simple $(r, m', |\delta(A \cup B_0)| + t)$ -daisy with kernel $\delta(A \cup B_0) \cup \delta(\{y_{\ell+p+m'}\} \cup B_1)$ (as in Figure 19). Since G is monochromatic with respect to χ_0 , we obtain that $\delta(G) \subseteq [N]$ is monochromatic with respect to the projection colouring, which is

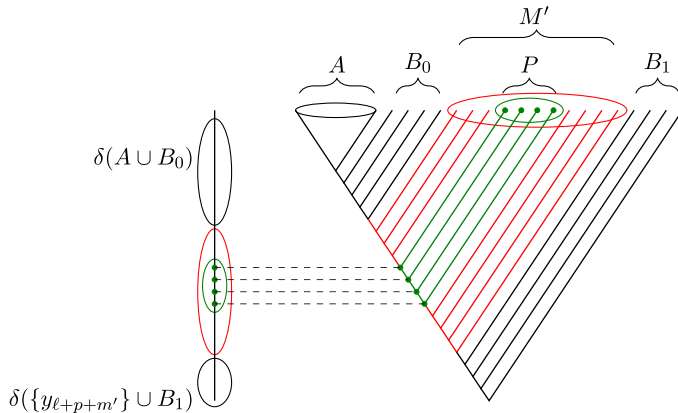


Figure 19. Case 2.1 of Theorem 6.1.

a contradiction because any simple $(r, m', |\delta(A \cup B_0)| + t)$ -daisy contains a simple $(r - 1, m' - 1, |\delta(A \cup B_0)| + t + 1)$ -subdaisy and $m' - 1 \geq \frac{1}{5}k^{-1/2}m^{1/2}$.

Case 2.2: X_j is of type 4.

If X_j is of type 4, then $|A^{X_j}| \leq r$ and $|X_j| = |A^{X_j}| + |B^{X_j}| \geq 2r - 1$. We split the proof into two subcases depending on the sequence formed by $\{\delta_{r-3}^{X_j}, \delta_{r-2}^{X_j}, \delta_{r-1}^{X_j}\}$:

Case 2.2.a: Either $\delta_{r-3}^{X_j} > \delta_{r-2}^{X_j} < \delta_{r-1}^{X_j}$ or $\delta_{r-3}^{X_j} < \delta_{r-2}^{X_j} > \delta_{r-1}^{X_j}$.

Suppose without loss of generality that $\delta_{r-3}^{X_j} > \delta_{r-2}^{X_j} < \delta_{r-1}^{X_j}$. Hence, by the colouring definition, we have $\chi_0(X_j) = \varphi_{\ell+p+t}(\{\delta_r^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\})$. Thus, we just need to look at the projection $\{\delta_r^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\}$ for every $X_j \in G$. Note that $\delta_{r-3}^{X_j} > \delta_{r-2}^{X_j} < \delta_{r-1}^{X_j}$ implies that $|A^{X_j}| \geq r - 1$. Indeed, by the same argument made in Case 2.1, if $|A^{X_j}| \leq r - 2$, then $\delta_{r-3}^{X_j} > \delta_{r-2}^{X_j} > \delta_{r-1}^{X_j}$, which is a contradiction. So, it follows that $r - 1 \leq |A^{X_j}| = \ell \leq r$.

Suppose that $|A^{X_j}| = r - 1$ and $B_0 = \emptyset$, i.e., $\ell = r - 1, p = 0$ and $M' = \{y_r, \dots, y_{r+m'-1}\}$. Then the projection of the relevant part of an edge X_j can be written as

$$\begin{aligned} \{\delta_r^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\} &= \{\delta_r^{X_j}, \dots, \delta_{2r-2}^{X_j}\} \cup \{\delta_{r+m'-1}^G, \dots, \delta_{r+m'+t-2}^G\} \\ &= \delta(P) \cup \{\delta_{r+m'-1}^G, \dots, \delta_{r+m'+t-2}^G\}, \end{aligned}$$

since $\delta_{2r-1+i}^{X_j} = \delta_{\ell+p+r+i}^{X_j} = \delta_{\ell+p+m'+i}^G = \delta_{2r+m'-1+i}^G$ for $0 \leq i \leq t - 1$. Therefore, the projection of the edges X_j is a simple $(r - 1, m' - 1, t)$ -daisy with kernel $\{\delta_{r+m'-1}^G, \dots, \delta_{r+m'+t-2}^G\}$ (see Figure 20). Because G is monochromatic under χ_0 , the projection is also monochromatic under φ_{r+t-1} , which is a contradiction.

Now suppose that $|A^{X_j} \cup B_0| = \ell + p \geq r$. The relevant projection of X_j in this case would be

$$\begin{aligned} \{\delta_r^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\} &= \{\delta_r^G, \dots, \delta_{\ell+p-1}^G\} \cup \{\delta_{\ell+p}^{X_j}, \dots, \delta_{\ell+p+r-1}^{X_j}\} \\ &\quad \cup \{\delta_{\ell+p+m'}^G, \dots, \delta_{\ell+p+m'+t-1}^G\}, \end{aligned}$$

where the set $\{\delta_r^G, \dots, \delta_{\ell+p-1}^G\}$ is empty for $\ell + p = r$. Since $\{\delta_{\ell+p}^{X_j}, \dots, \delta_{\ell+p+r-1}^{X_j}\} = \delta(\{y_{\ell+p}\} \cup P)$, we obtain that the projection of all edges X_j is a simple $(r, m', \ell + p + t - r)$ -daisy with kernel $\{\delta_r^G, \dots, \delta_{\ell+p-1}^G\} \cup \{\delta_{\ell+p+m'}^G, \dots, \delta_{\ell+p+m'+t-1}^G\}$. Because every simple $(r, m', \ell + p + t - r)$ -daisy contains an $(r - 1, m' - 1, \ell + p + t - r + 1)$ -daisy and the projection is monochromatic with

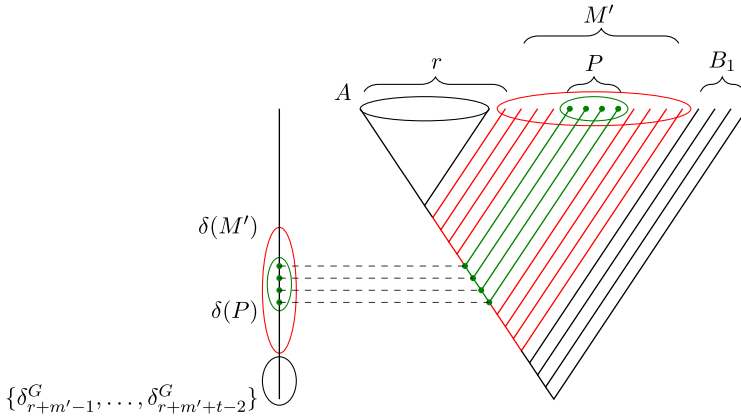


Figure 20. Case 2.2 of Theorem 6.1 when $|A| = r - 1$.

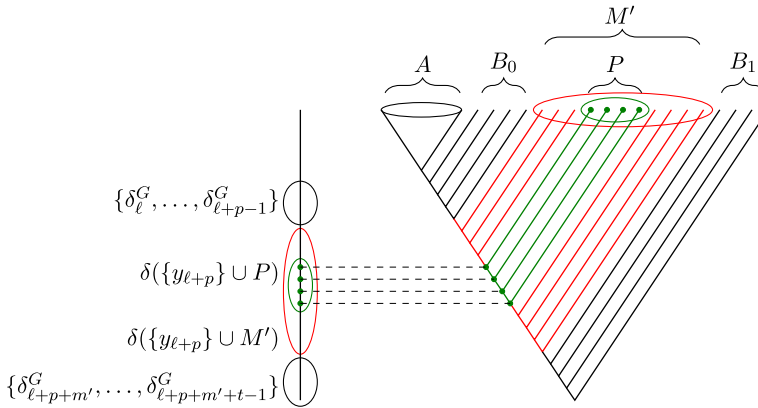


Figure 21. Case 2.3 of Theorem 6.1.

respect to $\varphi_{\ell+p+t-1}$, we obtain a monochromatic simple $(r - 1, \frac{1}{5}k^{-1/2}m^{1/2}, \ell + p + t - r + 1)$ -daisy, which is a contradiction.

Case 2.2.b: Either $\delta_{r-3}^{X_j} < \delta_{r-2}^{X_j} < \delta_{r-1}^{X_j}$ or $\delta_{r-3}^{X_j} > \delta_{r-2}^{X_j} > \delta_{r-1}^{X_j}$.

In this case we obtain that $\chi_0(X_j) = \varphi_{|\delta(X_j)|}(\delta(X_j))$, i.e., the colouring of χ_0 is just the colouring of the projection of X_j . The proof now follows similarly as in Case 2.1.

Case 2.3: X_j is of type 5.

If X_j is of type 5, then $|A^{X_j}| > r$ and $|B^{X_j}| = p + r + t \geq r$. By the colouring definition, we have $\chi_0(X_j) = \varphi_{p+r+t}(\{\delta_\ell^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\})$. The projection here can be rewritten as

$$\begin{aligned} \{\delta_\ell^{X_j}, \dots, \delta_{\ell+p+r+t-1}^{X_j}\} &= \{\delta_\ell^G, \dots, \delta_{\ell+p-1}^G\} \cup \{\delta(\{y_{\ell+p}\} \cup P) \\ &\cup \{\delta_{\ell+p+m'}^G, \dots, \delta_{\ell+p+m'+t-1}^G\}. \end{aligned}$$

Thus, the relevant projection over all edges X_j is a simple $(r, m', p + t)$ -daisy with kernel $\{\delta_\ell^G, \dots, \delta_{\ell+p-1}^G\} \cup \{\delta_{\ell+p+m'}^G, \dots, \delta_{\ell+p+m'+t-1}^G\}$ (see Figure 21). Therefore, by the same argument did in the previous cases, we reach a contradiction since there is no monochromatic simple $(r - 1, \frac{1}{5}k^{-1/2}m^{1/2}, p + t + 1)$ -daisy in the colouring φ_{p+r+t} . \square

Proof of Theorem 1.3. We will prove by induction on the size of r that there exists an absolute positive constants c and c' not depending on k and r such that

$$D_r^{\text{smp}}(m, k) = t_{r-2}(c'(5\sqrt{k})^{2^{5-r}-4}m^{2^{4-r}}) \geq t_{r-2}(ck^{2^{4-r}-2}m^{2^{4-r}})$$

holds for $k \geq 1$ and $m \geq (25k)^{2^r-1}$. For $r = 3$, the result follows by the next proposition given in [5].

Proposition 6.3 ([5], Proposition 1.2). *There exists a positive constant c' not depending on k such that*

$$D_3(m, k) \geq 2^{c'm^2}$$

holds for $m > 3$.

Now suppose that $r \geq 4$ and that for any integer $\ell < r$ the induction hypothesis is satisfied, i.e.,

$$D_\ell^{\text{smp}}(m, k) \geq t_{\ell-2}(c'(5\sqrt{k})^{2^{5-\ell}-4}m^{2^{4-\ell}})$$

for $m \geq (25k)^{2^\ell-1}$ and $k \geq 1$. Let $N = \min_{0 \leq i \leq k-1} D_{r-1}^{\text{smp}}(\frac{1}{5}k^{-1/2}m^{1/2}, i)$. For $i = 0$, by equation (2) we have that

$$D_{r-1}^{\text{smp}}\left(\frac{1}{5}k^{-1/2}m^{1/2}, 0\right) \geq R_{r-1}\left(\frac{1}{5}k^{-1/2}m^{1/2}\right) \geq t_{r-2}(c_1k^{-1}m)$$

for a positive constant c_1 . Since $m \geq (25k)^{2^r-1}$, we obtain that $\frac{1}{5}k^{-1/2}m^{1/2} \geq (25k)^{2^{r-1}-1}$. Thus, by induction hypothesis we also have that

$$D_{r-1}^{\text{smp}}\left(\frac{1}{5}k^{-1/2}m^{1/2}, i\right) \geq t_{r-3}\left(c'(5\sqrt{i})^{2^{6-r}-4}\left(\frac{1}{5}k^{-1/2}m^{1/2}\right)^{2^{5-r}}\right),$$

for $i \geq 1$. Therefore,

$$\begin{aligned} N &\geq \min \left\{ t_{r-2}(c_1k^{-1}m), \min_{1 \leq i \leq k} \left\{ t_{r-3}\left(c'(5\sqrt{i})^{2^{6-r}-4}\left(\frac{1}{5}k^{-1/2}m^{1/2}\right)^{2^{5-r}}\right) \right\} \right\} \\ &\geq t_{r-3}(c'(5\sqrt{k})^{2^{5-r}-4}m^{2^{4-r}}). \end{aligned}$$

Finally, Theorem 6.1, applied to $m \geq (25k)^{2^r-1} \geq 100kr^2$, gives us that

$$D_r^{\text{smp}}(m, k) \geq 2^N \geq t^{r-2}(c'(5\sqrt{k})^{2^{5-r}-4}m^{2^{4-r}}).$$

□

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