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ON THE CENTRALIZER ALGEBRA OF THE UNITARY REFLECTION GROUP $G(m, p, n)$

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Abstract. The imprimitive unitary reflection group $G(m, p, n)$ acts on the vector space $V = \mathbf{C}^n$ naturally. The symmetric group S_k acts on $\otimes^k V$ by permuting the tensor product factors. We show that the algebra of all matrices on $\otimes^k V$ commuting with $G(m, p, n)$ is generated by S_k and three other elements. This is a generalization of Jones's results for the symmetric group case [J].

§1. Introduction

In 1937, Brauer considered the centralizer algebra of the orthogonal group $O(n)$

$$\mathrm{End}_{O(n)}(\otimes^k \mathbf{C}^n) := \left\{ f \in \mathrm{End}_{\mathbf{C}}(\otimes^k \mathbf{C}^n) \;\middle|\; fg = gf \text{ for any } g \in O(n) \right\},$$

in relation to the decomposition of $\otimes^k \mathbf{C}^n$ into irreducible representations of $O(n)$. He defined the Brauer algebra B_k and showed that $\mathrm{End}_{O(n)}(\otimes^k \mathbf{C}^n)$ is always a quotient of B_k. B_k has simple generators as a $\mathbf{C}$-algebra [B].

Recently, Jones considered the centralizer algebra of the symmetric group S_n in relation to a certain model of statistical mechanics, where we identify S_n with the set of all permutation matrices. He showed that this algebra is generated by a quotient of a subalgebra of B_k and the action of the symmetric group by permuting the tensor product factors. This algebra has also simple generators [J].

We are interested in the generalization of Jones's results. Therefore we study the centlarizer algebra of $G(m, p, n)$ in Shephard-Todd [ST], because S_n is equivalent to $G(1, 1, n)$. We will show that this algebra is generated by the action of the symmetric group by permuting the tensor product factors and three other elements, where in the case of S_n, these generators are those found by Jones.

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§2. Preliminaries

2.1. A basis of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$

We denote the set of all nonnegative integers by $\mathbf{N}$ and the set of all complex numbers by $\mathbf{C}$. For $c_1, \ldots, c_n \in \mathbf{C}$, we denote by $\mathrm{diag}(c_1, \ldots, c_n)$ the $n \times n$-diagonal matrix whose (i,i)-th element is c_i $(1 \le i \le n)$.

Throughout this paper let m, p, and n be positive integers, where p is a divisor of m, $d := m/p$, and let ξ be a primitive m-th root of unity. We define the imprimitive unitary reflection group $G(m,p,n)$ as follows:

DEFINITION 2.1. (cf. [C], [ST]) $G(m,p,n)$ is the subgroup of $GL(n,\mathbf{C})$ generated by the set of all permutation matrices S_n and $\mathrm{Diag}(m,p,n)$, where

$$\mathrm{Diag}(m,p,n) := \left\{ \mathrm{diag}(\xi^{i_1}, \ldots, \xi^{i_n}) \;\middle|\; \begin{array}{l} i_1, \ldots, i_n \in \mathbf{N}, \\ i_1 + \cdots + i_n \equiv 0 \pmod{p}. \end{array} \right\}.$$

Since $S_n \cap \mathrm{Diag}(m,p,n) = 1$ and $\mathrm{Diag}(m,p,n)$ is a normal subgroup of $G(m,p,n)$, $G(m,p,n)$ is a semidirect product of S_n and $\mathrm{Diag}(m,p,n)$. $G(m,p,n)$ is a unitary reflection group of order $m^n n!/p$.

For convenience we denote the vector space $\mathbf{C}^n$ by V and the set $\{1, 2, \ldots, n\}$ by A. Let v_a be the vector in V whose a-th entry is 1 and whose other entries are all 0 $(1 \le a \le n)$. $G(m,p,n)$ acts on V naturally. Thus for each k, $\otimes^k V$ is a $G(m,p,n)$-module. For $X \in \mathrm{End}(\otimes^k V)$ we denote by $X^{b_1 \cdots b_k}_{a_1 \cdots a_k}$ the matrix coefficients of X with respect to the basis $\{v_{a_1} \otimes \cdots \otimes v_{a_k} \mid a_1, \ldots, a_k \in A\}$.

The purpose of this paper is to find the generators of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$. Let π be the representation of the symmetric group S_k on $\otimes^k V$ obtained by permuting the tensor product factors, i.e.,

$$\pi(\alpha)(u_1 \otimes \cdots \otimes u_k) := u_{\alpha^{-1}(1)} \otimes \cdots \otimes u_{\alpha^{-1}(k)}, \; u_1, \ldots, u_k \in V \text{ and } \alpha \in S_k.$$

$\pi(S_k)$ is clearly included in $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$. We also have

$$\mathrm{End}_{G(m,1,n)}(\otimes^k V) \subset \mathrm{End}_{G(m,p,n)}(\otimes^k V) \subset \mathrm{End}_{S_n}(\otimes^k V).$$

We will determine a basis of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$. For a positive integer N we denote by Π_N the set of all partitions of $\{1, 2, \ldots, N\}$ into subsets and introduce the following partial order on Π_N. For $B = \{B_1, \ldots, B_s\}$ and $C = \{C_1, \ldots, C_t\} \in \Pi_N$, $C \le B$ if and only if for any i $(1 \le i \le s)$ there exists j $(1 \le j \le t)$ such that $B_i \subset C_j$.

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As there is a one to one correspondence between the set of all equivalence relations on $\{1, 2, \ldots, N\}$ and those on Π_N, we denote by $\sim B$ the equivalence relation corresponding to $B \in \Pi_N$ and define the partial order on the set of all equivalence relations induced from that on Π_N.

Let us consider orbits for the action of S_n on A^{2k}. It is easy to see that each orbit for this S_n action is given by an element of Π_{2k} whose size is at most n. That is, if $\sim$ denotes the equivalence relation defined by such a partition, then the corresponding orbit of A^{2k} consists of $2k$-tuples $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_{2k})$ for which $a_i = a_j$ if and only if $i \sim j$.

For $X \in \mathrm{End}(\otimes^k V)$ and $\sigma \in S_n$,

$$\begin{aligned} &\sigma^{-1} X \sigma \, (v_{a_1} \otimes \cdots \otimes v_{a_k}) \\ &= \sum_{b_1, \ldots, b_k \in A} X^{\sigma(b_1) \cdots \sigma(b_k)}_{\sigma(a_1) \cdots \sigma(a_k)} v_{b_1} \otimes \cdots \otimes v_{b_k}. \end{aligned}$$

Hence we have the basis of $\mathrm{End}_{S_n}(\otimes^k V)$

$$\left\{ T_\sim \;\middle|\; \begin{array}{l} \sim \text{ is an equivalence relation on } \{1, \ldots, 2k\} \\ \text{whose number of classes is lesser than or equal to } n. \end{array} \right\},$$

where for the equivalence relation $\sim$,

$$(T_\sim)^{a_{k+1} \cdots a_{2k}}_{a_1 \cdots a_k} := \begin{cases} 1, & \text{if } (a_i = a_j \text{ if and only if } i \sim j), \\ 0, & \text{otherwise}, \end{cases}$$

setting $a_{k+i} := b_i$ $(1 \leq i \leq k)$. Note that $T_\sim$ is zero if the number of classes for $\sim$ is more than n.

For $B = \{B_1, \ldots, B_s\} \in \Pi_{2k}$, let $N(B_i) := \#\left(B_i \cap \{1, \ldots, k\}\right)$ and $M(B_i) := \#\left(B_i \cap \{k+1, \ldots, 2k\}\right)$, $(1 \leq i \leq s)$. We define the following three sets:

$$\Pi_{2k}(m) := \left\{ B = \{B_1, \ldots, B_s\} \in \Pi_{2k} \;\middle|\; \begin{array}{l} s \geq 1 \text{ and} \\ N(B_i) \equiv M(B_i) \pmod{m} \\ \qquad (1 \leq i \leq s). \end{array} \right\},$$

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$$\Lambda_{2k}(m,p,n) := \left\{ B = \{B_1, \ldots, B_n\} \in \Pi_{2k} \;\middle|\; \begin{array}{l} N(B_i) \equiv M(B_i) \pmod d, \\ N(B_i) \not\equiv M(B_i) \pmod m, \\ \qquad (1 \le i \le n), \\ \text{and} \\ N(B_i) - M(B_i) \\ \quad \equiv N(B_j) - M(B_j) \pmod m, \\ \qquad (1 \le i, j \le n). \end{array} \right\},$$

$$\Pi_{2k}(m,p,n) := \{B = \{B_1, \ldots, B_s\} \in \Pi_{2k}(m) \mid 1 \le s \le n\} \cup \Lambda_{2k}(m,p,n).$$

Note that $\Pi_{2k}(m,1,n) = \{B = \{B_1, \ldots, B_s\} \in \Pi_{2k}(m) \mid 1 \le s \le n\}$, and $\Pi_{2k} = \Pi_{2k}(1)$.

LEMMA 2.1. $\{ T_{\sim B} \mid B \in \Pi_{2k}(m,p,n) \}$ is a basis of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$.

Proof. $T_{\sim B}$ is a non-zero element in $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$ since any $B \in \Pi_{2k}(m,p,n)$ and $\{ T_{\sim B} \mid B \in \Pi_{2k}(m,p,n) \}$ are linearly independent. Then for $X \in \mathrm{End}_{G(m,p,n)}(\otimes^k V)$, we have $X = \sum_{B \in \Pi_{2k}} \alpha_B T_{\sim B}$ ($\alpha_B \in \mathbf{C}$) since $\mathrm{End}_{G(m,p,n)}(\otimes^k V) \subset \mathrm{End}_{S_n}(\otimes^k V)$. Let $B \in \Pi_{2k}$ such that $\alpha_B \neq 0$, and let $(a_1, \ldots, a_{2k}) \in A^{2k}$ such that $(T_{\sim B})^{a_{k+1}, \ldots, a_{2k}}_{a_1, \ldots, a_k} = 1$. We define $B_i := \{j \in \{1, \ldots, 2k\} \mid a_j = i\}$ $(1 \le i \le n)$. Then we have $B = \{B_1, \ldots, B_n\}$ (some B_i may be empty).

We define the following elements of $\mathrm{Diag}(m,p,n)$:

$$g_i := \mathrm{diag}(1, \ldots, 1, \underset{i}{\xi^p}, 1, \ldots, 1), \quad 1 \le i \le k,$$

$$h_{i,j} := \mathrm{diag}(1, \ldots, 1, \underset{i}{\xi}, 1, \ldots, 1, \underset{j}{\xi^{-1}}, 1, \ldots, 1), \quad 1 \le i < j \le k.$$

It is easily seen that g_i $(1 \le i \le n)$ and $h_{i,j}$ $(1 \le i < j \le n)$ generate $\mathrm{Diag}(m,p,n)$. We have

$$g_i^{-1} X g_i \, (v_{a_1} \otimes \cdots \otimes v_{a_k}) = \sum_{b_1, \ldots, b_k \in A} \xi^{p(\#\{\, l \mid a_l = i \,\} - \#\{\, l \mid b_l = i \,\})} X^{b_1 \cdots b_k}_{a_1 \cdots a_k} v_{b_1} \otimes \cdots \otimes v_{b_k}, \quad 1 \le i \le n,$$

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$$h_{i,j}^{-1} X h_{i,j} \ (v_{a_1} \otimes \cdots \otimes v_{a_k})$$
$$= \sum_{b_1,\ldots,b_k \in A} \xi^{\#\{ l \mid a_l=i \}-\#\{ l \mid b_l=i \}-\#\{ l \mid a_l=j \}+\#\{ l \mid b_l=j \}}$$
$$\times X_{a_1\cdots a_k}^{b_1\cdots b_k} v_{b_1} \otimes \cdots \otimes v_{b_k}, \quad 1 \leq i,j \leq n.$$

As $g_i^{-1}Xg_i = X$ $(1 \leq i \leq n)$, and $h_{i,j}^{-1}Xh_{i,j} = X$ $(1 \leq i,j \leq n)$, we also have

$$N(B_i) \equiv M(B_i) \pmod d, \quad (1 \leq i \leq n), \quad \text{and}$$
$$N(B_i) - M(B_i) \equiv N(B_j) - M(B_j) \pmod m, \quad (1 \leq i,j \leq n).$$

If $N(B)_1 \equiv M(B)_1 \pmod m$, then $N(B_i) \equiv M(B_i) \pmod m$ $(1 \leq i \leq n)$. Thus $B \in \Pi_{2k}(m)$. If $N(B)_1 \not\equiv M(B)_1 \pmod m$, then $N(B_i) \not\equiv M(B_i) \pmod m$ $(1 \leq i \leq n)$. We have $\#B_i = N(B_i) + M(B_i) \neq 0$, and therefore $B_i \neq \phi$ $(1 \leq i \leq n)$. Thus we obtain $B \in \Lambda_{2k}(m,p,n)$. □

For each equivalence relation $\sim$, we define $L_\sim$ by

$$L_\sim := \sum_{B \in \Pi_{2k} \ ; \ \sim_B \leq \sim} T_{\sim_B}.$$

LEMMA 2.2. (cf. [J]) *For* $B \in \Pi_{2k}(m)$, $L_{\sim B} \in \mathrm{End}_{G(m,1,n)}(\otimes^k V)$, *and*

$$\sum_{B \in \Pi_{2k}(m)} \mathbf{C} L_{\sim B} = \mathrm{End}_{G(m,1,n)}(\otimes^k V).$$

Proof. Let $B \in \Pi_{2k}(m)$ and $C = \{C_1, \ldots, C_t\} \in \Pi_{2k}$. If $C \leq B$, it is easy to check the condition $N(C)_i \equiv M(C)_i \pmod m$ $(1 \leq i \leq t)$, so $C \in \Pi_{2k}(m)$. Hence

$$L_{\sim B} = \sum_{C \in \Pi_{2k}(m); \ C \leq B} T_{\sim C} \ \in \mathrm{End}_{G(m,1,n)}(\otimes^k V).$$

Then by the Möbius inversion ([S], p.116), the $T_{\sim B}$ can be expressed by a linear combination of $\{ L_{\sim C} \mid C \in \Pi_{2k}(m) \}$. Thus they also span $\mathrm{End}_{G(m,1,n)}(\otimes^k V)$. □

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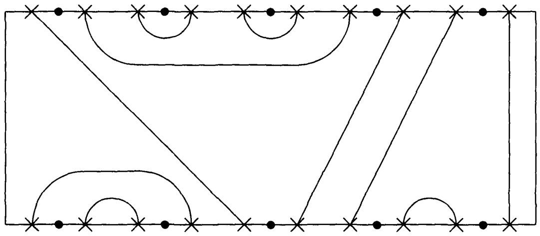
2.2. "Planar " form

Consider a rectangle with k marked points on the bottom and k symmetrically placed marked points on the top, as shown in the figure (where $k = 5$).



Close to and on either side of each of the marked points on both the top and bottom identify a single point (marked with an $\times$ in the figure). Now join the points marked with "$\times$'s " to each other within the rectangle, using any system of non-intersecting curves. The regions inside the rectangle can then be shaded black and white (with the regions touching the left and right sides of the rectangle shaded white). Any such diagram defines a partition of the original $2k$ marked points. We will call such a partition (or its equivalence relation) "planar." We denote by $\mathrm{P}\Pi_{2k}$ the set of all planar partitions in Π_{2k} and define $\mathrm{P}\Pi_{2k}(m)$ by $\mathrm{P}\Pi_{2k}(m) := \Pi_{2k}(m) \cap \mathrm{P}\Pi_{2k}$. For $B = \{B_1, \ldots, B_s\} \in \Pi_{2k}$, α_1, $\alpha_2 \in S_k$, we define

$$\begin{aligned} &\alpha_2 B_u \alpha_1 \\ &:= \{\alpha_1^{-1}(i) \mid i \in B_u \text{ and } 1 \leq i \leq k\} \\ &\quad \cup \{k + \alpha_2^{-1}(j-k) \mid j \in B_u \text{ and } k+1 \leq j \leq 2k\}, \quad 1 \leq u \leq s. \end{aligned}$$

and $\alpha_2 B \alpha_1 := \{\alpha_2 B_1 \alpha_1, \ldots, \alpha_2 B_s \alpha_1\}$. Namely, α_1 permutes the bottom k points of B, α_2 permutes the top k points of B, and we obtain $\alpha_2 B \alpha_1$ from B as a result of these permutations. Note that if $B \in \Pi_{2k}(m)$ (resp. $\Lambda_{2k}(m,p,n)$), then $\alpha_2 B \alpha_1 \in \Pi_{2k}(m)$ (resp. $\Lambda_{2k}(m,p,n)$) for any α_1, $\alpha_2 \in S_k$, and

$$T_{\sim \alpha_2 B \alpha_1} = \pi(\alpha_2) T_{\sim B} \pi(\alpha_1).$$

The following lemma is almost obvious.

LEMMA 2.3. ([J], Lemma 2.) *Let* $C = \{C_1, \ldots, C_s\} \in \Pi_{2k}$, $t := \#\{i \mid N(C_i) \neq 0\}$, *and* $u := \#\{i \mid N(C_i) \neq 0$ *and* $M(C_i) \neq 0\}$. *Then*

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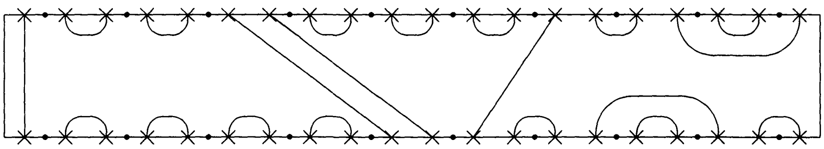
there exist α_1 *and* $\alpha_2 \in S_k$ *such that* $\alpha_2 C \alpha_1 =: \{B_1, \dots, B_s\}$ (*renumbering indices*), *satisfies*

$$B_i = \begin{cases} \left\{ \sum_{j=1}^{i-1} N(B_j) + 1, \dots, \sum_{j=1}^{i} N(B_j) \right\} \\ \cup \left\{ k + \sum_{j=1}^{i-1} M(B_j) + 1, \dots, k + \sum_{j=1}^{i} M(B_j) \right\}, & \textit{if } 1 \le i \le u, \\ \left\{ \sum_{j=1}^{i-1} N(B_j) + 1, \dots, \sum_{j=1}^{i} N(B_j) \right\}, & \textit{if } u+1 \le i \le t, \\ \left\{ k + \sum_{j=1}^{i-1} M(B_j) + 1, \dots, k + \sum_{j=1}^{i} M(B_j) \right\}, & \textit{if } t+1 \le i \le s. \end{cases}$$

Note that $\alpha_2 C \alpha_1$ *is planar in this case.*

Perhaps the following partition helps in understanding the above lemma:



Let δ be an nonzero element of $\mathbf{C}$. We define the $\mathbf{C}$-algebra $K(2k, \delta)$ with a basis $\mathrm{P}\Pi_{2k}$. Multiplication is defined by a 2-step procedure:

Step 1. Stack the two rectangles on top of each other, lining up the $\times$'s.

Step 2. Remove the middle edges and middle $\times$'s. You then have a new diagram, possibly containing some closed loops. If there are r closed loops, the product is then the resulting diagram, with the closed loops removed, times the scalar δ^r.

We illustrate multiplication in $K(8, \delta)$ below.

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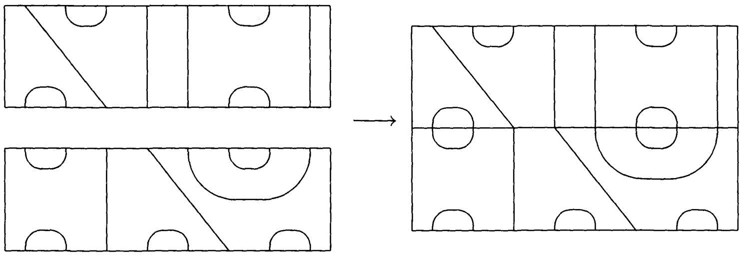
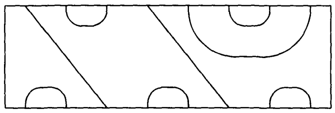
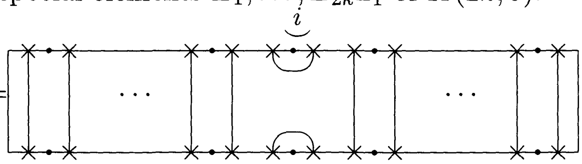
$B =$

$C =$

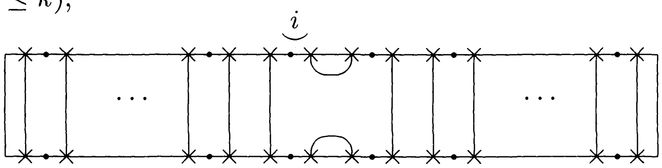


so $BC = \delta^2$ 

It is clear that the above defined multiplication is associative and that the identity element of $K(2k,\delta)$ is $\{\ \{i,\ i+k\} \mid 1 \leq i \leq k\ \} \in \mathrm{P\Pi}_{2k}$. We now define special elements $E_1, \ldots, E_{2k-1}$ of $K(2k,\delta)$.

$E_{2i-1} :=$ 

$(1 \leq i \leq k)$,

$E_{2i} :=$ 

$(1 \leq i \leq k-1)$.

It is clear that the E_i, together with 1, generate $K(2k,\delta)$. Note that for $a_1, \ldots, a_k \in A$,

$$L_{\sim E_{2i-1}}(v_{a_1} \otimes \cdots \otimes v_{a_k})$$
$$= \sum_{j=1}^{n} v_{a_1} \otimes \cdots \otimes v_{a_{i-1}} \otimes v_j \otimes v_{a_{i+1}} \otimes \cdots \otimes v_{a_k}, \ (1 \leq i \leq k),$$

$$L_{\sim E_{2i}}(v_{a_1} \otimes \cdots \otimes v_{a_k}) = \delta_{a_i,\ a_{i+1}}\ v_{a_1} \otimes \cdots \otimes v_{a_k}, \ (1 \leq i \leq k-1).$$

LEMMA 2.4. ([J], Lemma 3.) *The map*

$$\varphi(E_i) := \begin{cases} \frac{1}{\sqrt{n}} L_{\sim E_i}, & \textit{if } i \textit{ is odd}, \\ \sqrt{n} L_{\sim E_i}, & \textit{if } i \textit{ is even}. \end{cases}$$

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extends to an algebra homomorphism from $K(2k, \sqrt{n})$ *to* $\mathrm{End}_{S_n}(\otimes^k V)$ *so that* $\varphi(B)$ *is a non-zero multiple of* $L_{\sim B}$ *for* $B \in \mathrm{P}\Pi_{2k}$.

§3. Generators of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$

First, we consider the case $p = 1$. We denote by $K(2k, \delta)_m$ the subspace of $K(2k, \delta)$ spanned by $\mathrm{P}\Pi_{2k}(m)$. As $K(2k, \delta)_m$ is closed under multiplication, $K(2k, \delta)_m$ is a subalgebra of $K(2k, \delta)$.

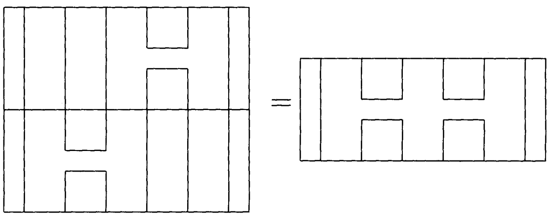
From Lemmas 2.2, 2.3, and 2.4, it follows that $\varphi(K(2k, \sqrt{n})_m)$ and $\pi(S_k)$ generate $\mathrm{End}_{G(m,1,n)}(\otimes^k V)$.

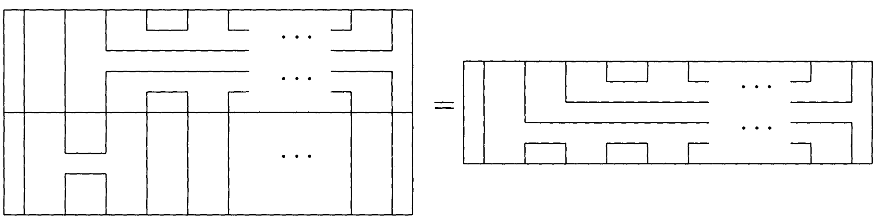
LEMMA 3.1. *Let*

$$F_i^m := \Big\{\{i, \ldots, i+m-1\},\ \{k+i, \ldots, k+m+i-1\}\Big\} \cup \Big\{\{j,\ k+j\}\ \Big|\ 1 \le j \le i-1 \text{ or } m+i \le j \le k\Big\} \in \mathrm{P}\Pi_{2k}(m), \quad 1 \le i \le k-m.$$

Then $K(2k, \delta)_m$ *is generated by* F_i^m $(1 \le i \le k-m)$ *and* E_{2j} $(1 \le j \le k-1)$ *as a* $\mathbf{C}$*-algebra. In particular,* $\mathrm{End}_{G(m,1,n)}(\otimes^k V)$ *is generated by* $\varphi(E_2)$, $\varphi(F_1^m)$, *and* $\pi(S_k)$ *as a* $\mathbf{C}$*-algebra, where if* $m > k$ *we define* $\varphi(F_1^m) := 0$.

Proof. The assertion is clear from the following calculations:





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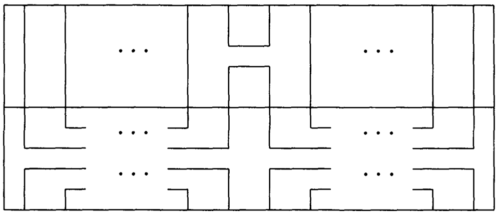

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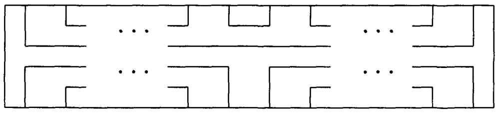

∎

We consider $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$ for general p. Let $\Lambda'_{2k}(m,p,n)$ be the set of all elements $B := \{B_1, \ldots, B_n\} \in \Lambda_{2k}(m,p,n)$ which have the following form:

$$(*) \quad B_i = \begin{cases} \{N_i+1, \ldots, N_{i+1}\} \cup \{M_i+1, \ldots, M_{i+1}\}, & \text{if } 1 \leq i \leq u, \\ \{N_i+1, \ldots, N_{i+1}\}, & \text{if } u+1 \leq i \leq t, \\ \{M_{i-t+u}+1, \ldots, M_{i-t+u+1}\}, & \text{if } t+1 \leq i \leq n, \end{cases}$$

for some $u, t \in \mathbf{N}$ $(u \leq t)$ and $N_1, N_2, \ldots, N_t,\ M_1, M_2, \ldots, M_{n-t+u} \in \mathbf{N}$ such that

$$\begin{aligned} 0 = N_1 < N_2 < \cdots < N_{t+1} = k < k+1 \\ = M_1 < N_2 < \cdots < M_{n-t+u+1} = 2k. \end{aligned}$$

We define

$$H_{m,p,n} := \sum_{B \in \Lambda'_{2k}(m,p,n)} T_{\sim B}.$$

Note that for any $B \in \Lambda_{2k}(m,p,n)$, there exist α_1 and $\alpha_2 \in S_k$ such that $\alpha_2 B \alpha_1 \in \Lambda'_{2k}(m,p,n)$ from Lemma 2.3. We determine the condition that $\Lambda'_{2k}(m,p,n)$ is not empty, namely, that $H_{m,p,n} \neq 0$.

Note that for $B = \{B_1, \ldots, B_n\} \in \Lambda_{2k}(m,p,n)$, there is a w $(1 \leq w \leq p-1)$ such that $N(B_i) - M(B_i) \equiv wd \pmod{m}$ $(1 \leq i \leq n)$ by the definition of $\Lambda_{2k}(m,p,n)$. For $C \subset \mathbf{N}$ and $i \in \mathbf{N}$, we define $C + i := \{x + i \mid x \in C\}$.

LEMMA 3.2. *Let* $B = \{B_1, \dots, B_n\} \in \Lambda'_{2k}(m,p,n)$, *where the ordering of indices satisfies* $(*)$, $t := \#\{i \mid N(B_i) \neq 0a\}$, *and* $u := \#\{i \mid N(B_i) \neq 0$ *and* $M(B_i) \neq 0\}$. *Let* $w \in \{1, \dots, p-1\}$ *such that* $N(B_i) - M(B_i) \equiv wd \pmod m$ $(1 \leq i \leq n)$.

(1) *Let*

$$\begin{cases} C_1^1 & := & \{1, k+2\} \cup \Big((B_1 \cap \{1, \dots, k\}) + 1\Big) \\ & & \cup \Big((B_1 \cap \{k+1, \dots, 2k\}) + 2\Big), \\ C_i^1 & := & \Big((B_i \cap \{1, \dots, k\}) + 1\Big) \cup \Big((B_i \cap \{k+1, \dots, 2k\}) + 2\Big), \\ & & \qquad\qquad 2 \leq i \leq n. \end{cases}$$

Then $C^1 := \{C_1^1, \dots, C_n^1\} \in \Lambda'_{2(k+1)}(m,p,n)$.

(2) *Assume* $N(B_i) \neq 0$ *and* $M(B_i) \neq 0$ *for some* i $(1 \leq i \leq n)$. *Let*

$$C_i^2 := \left\{ \sum_{j=1}^{i-1} N(B_j) + 1, \dots, \sum_{j=1}^{i} N(B_j) - 1 \right\}$$
$$\cup \left\{ k + \sum_{j=1}^{i-1} M(B_j) + 1, \dots, k + \sum_{j=1}^{i} M(B_j) - 1 \right\},$$

$$C_j^2 := \begin{cases} (B_j \cap \{1, \dots, k\}) \cup \Big((B_j \cap \{k+1, \dots, 2k\}) - 1\Big), \\ \qquad\qquad \text{if } 1 \leq j < i, \\ \Big((B_j \cap \{1, \dots, k\}) - 1\Big) \\ \cup \Big((B_j \cap \{k+1, \dots, 2k\}) - 2\Big), \quad \text{if } i < j \leq n. \end{cases}$$

Then $C^2 := \{C_1^2, \dots, C_n^2\} \in \Lambda'_{2(k-1)}(m,p,n)$.

(3) *Assume*

$$B_i \subset \begin{cases} \{1, \dots, k\}, & \text{if } 1 \leq i \leq t, \\ \{k+1, \dots, 2k\}, & \text{if } t+1 \leq i \leq n. \end{cases}$$

And assume $\#B_i > wd$ *for some* i $(1 \leq i \leq t)$ *or* $\#B_j > (p-w)d$ *for some* j $(t+1 \leq j \leq n)$. *Then there is a* $C^3 = \{C_1^3, \dots, C_n^3\} \in \Lambda'_{2k}(m,p,n)$ *such that* $N(C_1^3) \neq 0$ *and* $M(C_1^3) \neq 0$.

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Proof. (1) and (2) are clear. We will show (3). We assume $\#B_i > wd$ for some $i \in \{1, \ldots, t\}$. We may also assume $\#B_1 > wd$ using the S_k action and renumbering indices. Then there is a non-zero integer α such that $\#B_1 = wd + \alpha m$. We define

$$\begin{cases} C_1^3 := \{1, \ldots, \alpha m\} \cup B_{t+1}, \\ C_2^3 := \{\alpha m + 1, \ldots, \alpha m + wd\}, \\ C_i^3 := \begin{cases} B_{i-1}, & \text{if } 3 \leq i \leq t, \\ B_i, & \text{if } t+2 \leq i \leq n. \end{cases} \end{cases}$$

Then $C^3 := \{C_1^3, \ldots, C_n^3\} \in \Lambda'_{2k}(m,p,n)$, $N(C_1^3) \neq 0$, and $M(C_1^3) \neq 0$. The other cases can be shown similarly. □

For positive integers i and j, let (i,j) be their greatest common measure. Let $p =: a_p(p,n)$, $n =: a_n(p,n)$ $((a_p, a_n) = 1)$, and $k_{m,p,n} := (n - a_n)a_p d$. If $(p,n) \geq 2$, we define the following partition:

$$B := \left\{ \{1, \ldots, a_p d\} + (i-1)(a_p d) \;\middle|\; 1 \leq i \leq (n - a_n)a_p d \right\}$$
$$\cup \left\{ \begin{array}{l} \{1, \ldots, ((p,n)-1)\, a_p d\} + k_{m,p,n} \\ +(i-1)\,((p,n)-1)\, a_p d \end{array} \;\middle|\; 1 \leq i \leq a_n \right\}.$$

It is easy to see that $B \in \Lambda'_{2k_{m,p,n}}(m,p,n)$, and thus $\Lambda'_{2k_{m,p,n}}(m,p,n)$ is not empty.

Lemma 3.3. *$\Lambda'_{2k}(m,p,n)$ is not empty if and only if $(p,n) \neq 1$ and $k \geq k_{m,p,n}$.*

Proof. We assume $(p,n) \neq 1$ and $k \geq k_{m,p,n}$. Note that if $\Lambda'_{2k}(m,p,n)$ is not empty, then $\Lambda'_{2(k+1)}(m,p,n)$ too is not empty by Lemma 3.2 (1). So $\Lambda'_{2k}(m,p,n)$ is not empty since $\Lambda'_{2k_{m,p,n}}(m,p,n)$ is not empty. Conversely, we assume $\Lambda'_{2k}(m,p,n)$ is not empty. Let k be the minimum integer in this set. From Lemma 3.2 (2) and (3), we may assume that there is a $B = \{B_1, \ldots, B_n\} \in \Lambda'_{2k}(m,p,n)$ such that

$$\begin{cases} B_1 & := & \{1, \ldots, wd\}, \\ B_{t+1} & := & \{k+1, \ldots, k + (p-w)d\}, \\ B_i & := & \begin{cases} B_1 + (i-1)wd, & 2 \leq i \leq t, \\ B_{t+1} + (i-t-1)(p-w)d, & t+2 \leq i \leq n, \end{cases} \end{cases}$$

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for some $t \in \mathbf{N}$ and w $(1 \leq w \leq p-1)$. We have

$$k = \sum_{i=1}^{t} \#B_i = twd,$$
$$k = \sum_{i=t+1}^{n} \#B_i = (n-t)(p-w)d.$$

Hence $nw = (n-t)p$. We have the condition that if $(p,n) = 1$, then p is a divisor of w. But this is a contradiction since $1 \leq w \leq p-1$. We have $w = \beta a_p$ for some positive integer β from the relations $a_n w = (n-t)a_p$ and $(a_p, a_n) = 1$. We also have $k = k_{m,p,n}$ when $\beta = 1$. Assume $\beta > 1$. If $w \leq p - w$, then $n - t \leq t$ from $k = twd = (n-t)(p-w)d$, so $k = twd = (n/2)\beta a_p d \geq n a_p d > k_{m,p,n}$. This is a contradiction since k is minimal. Similarly, we have $k > k_{m,p,n}$ when $w > p - w$. ∎

THEOREM 3.1. *$H_{m,p,n} \neq 0$ if and only if $(n,p) \neq 1$ and $k \geq k_{m,p,n}$. $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$ is generated by $\varphi(E_2)$, $\varphi(F_1^m)$, $H_{m,p,n}$, and $\pi(S_k)$ as a $\mathbf{C}$-algebra.*

Proof. The first assertion is clear from Lemma 3.3. Let Γ be the subalgebra of $\mathrm{End}_{G(m,p,n)}(\otimes^k V)$ generated by $\varphi(E_2)$, $\varphi(F_1^m)$, $H_{m,p,n}$, and $\pi(S_k)$. From Lemma 3.1 we have $\mathrm{End}_{G(m,1,n)}(\otimes^k V) \subset \Gamma$. Let $B = \{B_1, \dots, B_n\} \in \Lambda'_{2k}(m,p,n)$, where the ordering of indices satisfies $(*)$, $t := \#\{i \mid N(B_i) \neq 0\}$, $s := \#\{i \mid M(B_i) \neq 0\}$, and $u := \#\{i \mid N(B_i) \neq 0 \text{ and } M(B_i) \neq 0\}$. Note that $t, s \leq n$.

We define the following elements of Π_k:

$$C := \{B_i \cap \{1, \dots, k\} \mid 1 \leq i \leq n\},$$
$$D := \{B_i \cap \{k+1, \dots, 2k\} \mid 1 \leq i \leq n\}.$$

We then denote $C = \{C_1, \dots, C_t\}$ and $D = \{D_1, \dots, D_s\}$ and define

$$\hat{C} := \{C_1 \cup (C_1 + k), \dots, C_s \cup (C_s + k)\} \in \Pi_{2k}(m),$$
$$\hat{D} := \{D_1 \cup (D_1 + k), \dots, D_t \cup (D_t + k)\} \in \Pi_{2k}(m).$$

Clearly $T_{\sim\hat{D}} T_{\sim B} T_{\sim\hat{C}} = T_{\sim B}$. By the definitions of C and D,

$$B_j = \begin{cases} C_j \cup D_j, & \text{if } 1 \leq j \leq u, \\ C_l \text{ or } D_l \text{ for some } l > u, & \text{otherwise.} \end{cases}$$

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Comparing the number of classes of B, C and D, we have $n = u + (s - u) + (t - u)$. Thus $u = s + t - n$. Hence if $(C, D) \in \Pi_k \times \Pi_k$ are given, we can reconstruct the original $B \in \Lambda'_{2k}(m,p,n)$ from (C, D). So for $B' \in \Lambda'_{2k}(m,p,n)$ $(B' \neq B)$, we have

$$C \neq \{B'_i \cap \{1, \ldots, k\} \mid 1 \leq i \leq n\},$$

or

$$D \neq \{B'_i \cap \{k+1, \ldots, 2k\} \mid 1 \leq i \leq n\}.$$

This implies that $T_{\sim\hat{D}} T_{\sim B'} T_{\sim\hat{C}} = 0$. From the above results we have $T_{\sim\hat{D}} H_{m,p,n} T_{\sim\hat{C}} = T_{\sim B}$. Hence $T_{\sim B} \in \Gamma$ for any $B \in \Lambda'_{2k}(m,p,n)$, and $\Gamma = \mathrm{End}_{G(m,p,n)}(\otimes^k V)$. □

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