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FUNCTIONAL LARGE DEVIATIONS AND MODERATE DEVIATIONS FOR MARKOV-MODULATED RISK MODELS WITH REINSURANCE

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Abstract

We establish a functional large deviation principle and a functional moderate deviation principle for Markov-modulated risk models with reinsurance by constructing an exponential martingale approach. Lundberg's estimate of the ruin time is also presented.

Keywords: Markov-modulated risk model; large deviations; moderate deviations; Lundberg's estimate

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1. Introduction

1.1. Markov-modulated risk model

Firstly, let us introduce the Markov-modulated risk model (cf. [13]). A process $\{N(t), t \geq 0\}$ is said to be a point process if

$$N(t) = \sum_{l=1}^{\infty} \mathbf{1}_{\{T_l \leq t\}},$$

where $\{T_l,\ l \geq 1\}$ is a sequence of stopping times such that $T_1 > 0$ almost surely (a.s.), $T_l < T_{l+1}$ on $\{T_l < \infty\}$ for any $l \geq 1$, and $\lim_{l\to\infty} T_l = \infty$ a.s. A point process $\{N(t),\ t \geq 0\}$ is said to be a Markov-modulated Poisson process if it is a doubly stochastic Poisson process with intensity $\lambda_{J(t)}$, where $J = \{J(t),\ t \geq 0\}$ is an irreducible continuous-time Markov chain with finite state space E and the $\lambda_i,\ i \in E$, are positive numbers, i.e. the conditional characteristic function of $\{N(t),\ t \geq 0\}$ has the following expression:

$$\mathrm{E}(\exp\{\mathrm{i}\theta(N(t) - N(s))\} \mid \mathcal{F}_s) = \exp\biggl\{(\mathrm{e}^{\mathrm{i}\theta} - 1)\int_s^t \lambda_{J(u)}\,\mathrm{d}u\biggr\},$$

where $\mathcal{F}_s = \sigma(N(u),\ u \leq s) \vee \sigma(J(u),\ u \geq 0)$.

Let $\pi_i,\ i \in E$, denote the stationary distribution of the Markov chain J.

Let $\{U_l,\ l \geq 1\}$ be a sequence of positive random variables, and let $G_i,\ i \in E$, be probability distributions with supports in $[0, +\infty)$. Assume that, for all $i \in E$, $\mu_i := \int_0^\infty x G_i(\mathrm{d}x) < \infty$

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and that

- U_l, $l \geq 1$, and $\{N(t),\ t \geq 0\}$ are conditionally independent given J,
- for each $l \geq 1$, the conditional distribution of U_l given J is $G_{J(T_l)}$.

A reinsurance policy is a measurable function from $\mathcal{R}\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ which satisfies $0 \leq R_t(\alpha) \leq \alpha$, where $R_t(\alpha) = \mathcal{R}(t, \alpha)$. The condition $0 \leq R_t(\alpha) \leq \alpha$ represents the part of the claim that the company pays when a claim of size α occurs at time t.

A Markov-modulated risk process with a reinsurance policy $\mathcal{R}$ is defined by

$$X^x_{\mathcal{R}}(t) = x + p_{\mathcal{R}}(t) - S_{\mathcal{R}}(t),$$

where $x > 0$ is the initial capital; $p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t)$ is the deterministic premium, p is a constant premium rate which has the form

$$p = (1 + \kappa) \sum_{i \in E} \pi_i \lambda_i \mu_i$$

for some relative safety loading $\kappa > 0$, and $q_{\mathcal{R}}(t)$ is the premium up to time t paid by the insurer to the reinsurer which has the form

$$q_{\mathcal{R}}(t) = (1 + \eta) \sum_{i \in E} \pi_i \lambda_i \int_0^t \bigg(\mu_i - \int_0^\infty R_s(x) G_i(\mathrm{d}x)\bigg)\,\mathrm{d}s$$

for some relative safety loading $\eta > 0$; and

$$S_{\mathcal{R}}(t) = \sum_{l=1}^{N(t)} R_{T_l}(U_l)$$

is the aggregate claims process, $\{R_{T_l}(U_l),\ l \geq 1\}$ is the sequence of claims, $N(t)$ is the claims number process, which is a doubly stochastic Poisson process with intensity $\lambda_{J(t)}$, and $J = \{J(t),\ t \geq 0\}$ is an irreducible continuous-time Markov chain with finite state space E.

The Markov-modulated risk process with reinsurance is a generalization of the classical case. For example, if $G_i = G$ and $\lambda_i = \lambda$ for all $i \in E$, and $\{U_l,\ l \geq 1\}$, $\{N(t),\ t \geq 0\}$, and J are independent, then $S_{\mathcal{R}}(t)$ is the classic case. Recently, Macci and Stabile [13] studied the large deviations and ruin probability of the Markov-modulated risk process with reinsurance and obtained a functional large deviation principle for the classic case. Large deviations of some risk processes and applications of large deviations to insurance have received much attention in the research literature; see, for example, [1, p. 306], [2], [6], [9, p. 85], [12], [13], and [14]. In this paper we present an exponential martingale method to establish large deviations and moderate deviations for risk processes, and obtain the functional large deviation principle, the functional moderate deviation principle, and Lundberg's estimate of the ruin time for the Markov-modulated risk process with reinsurance.

The rest of the paper is organized as follows. In Subsection 1.2 we introduce some large deviations terminology, the Gärtner–Ellis theorem, and a result on functional large deviations used in this paper. In Section 2 we construct an exponential martingale associated with the Markov-modulated risk model which plays an important role in this paper. The functional large deviation principle is established in Section 3. The moderate deviations are studied in Section 4. In Section 5 we give an estimate for the ruin probability (Lundberg's estimate) using the exponential martingale method.

1.2. Preliminaries

In this subsection we introduce large deviations (cf. [4, Chapter 2] and [5, Chapter 4]). Let S be a metric space with metric d, and let $\{Y_\alpha : \alpha > 0\}$ be a family of S-valued random variables. Denote the law of Y_α by μ_α. Let $\lambda(\alpha)$ be a sequence of positive real numbers satisfying $\lambda(\alpha) \to \infty$ as $\alpha \to \infty$.

(i) A function $I(\cdot)\colon S \to [0, +\infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in S\colon I(x) \le a\}$ is compact for all $a \ge 0$.

(ii) The family of probability measures $\{\mu_\alpha : \alpha > 0\}$ (or the family $\{Y_\alpha : \alpha > 0\}$) is said to satisfy a large deviation principle (LDP) with speed $\lambda(\alpha)$ and rate function $I(\cdot)$ if, for any closed set F and open set G in S,

$$\limsup_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha\{F\} \le -\inf_{x\in F} I(x), \qquad \liminf_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha\{G\} \ge -\inf_{x\in G} I(x).$$

In short form, we say that $(\mu_\alpha, I(\cdot), 1/\lambda(\alpha))$ satisfies an LDP.

(iii) The family $\{\mu_\alpha : \alpha > 0\}$ is exponentially tight with speed $\lambda(\alpha)$ if, for every $L > 0$, there is a compact set K_L in S such that

$$\limsup_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha(K_L^{\mathrm{c}}) \le -L.$$

Lemma 1.1. (Gärtner–Ellis theorem.) *Let $\{Y_\alpha,\ \alpha > 0\}$ be a family of random variables taking values in $\mathbb{R}^d$. Suppose that, for any $y \in \mathbb{R}^d$,*

$$\Lambda(y) := \lim_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mathrm{E}(\exp\{\lambda(\alpha)\langle Y_\alpha, y\rangle\}) \in (-\infty, +\infty]$$

exists and that $\Lambda(\cdot)$ is finite in a neighborhood of 0, where $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ for $x, y \in \mathbb{R}^d$. If Λ is essentially smooth then $\{Y_\alpha,\ \alpha > 0\}$ satisfies the LDP with speed $\lambda(\alpha)$ and rate function Λ^* defined by

$$\Lambda^*(x) = \sup_{y\in\mathbb{R}^d} \{\langle x, y\rangle - \Lambda(y)\}.$$

In particular, if Λ is finite and Gâteaux differentiable then $\{Y_\alpha,\ \alpha > 0\}$ satisfies the LDP with speed $\lambda(\alpha)$ and rate function Λ^*.

Let $D([0, 1])$ be the space of càdlàg functions (i.e. those which are right continuous with left limits) from $[0, 1]$ to $\mathbb{R}$ equipped with the uniform metric $d(x, y) := \sup_{t\in[0,1]} |x(t) - y(t)|$.

Lemma 1.2. (cf. [7].) *Let $\{\mu_\alpha,\ \alpha > 0\}$ be a family of probability measures on $D([0, 1])$. For any finite subset $\{t_1, \dots, t_l\} \subset [0, 1]$, set $\mu_\alpha^{t_1,\dots,t_l} = \mu_\alpha \circ \pi_{t_1,\dots,t_l}^{-1}$, where $\pi_{t_1,\dots,t_l}\colon x \to (x_{t_1}, \dots, x_{t_l})$ denotes the projection from $D([0, 1])$ to $\mathbb{R}^l$. If, for any finite subset $\{t_1, \dots, t_l\} \subset [0, 1]$, $\{\mu_\alpha^{t_1,\dots,t_l}, \alpha > 0\}$ satisfies the LDP with speed $\lambda(\alpha)$ and rate function $I_{t_1,\dots,t_l}(x_1, \dots, x_l)$ in $\mathbb{R}^l$ and for any $\delta > 0$,*

$$\lim_{\varepsilon\to0} \sup_{s\in[0,1]} \limsup_{\alpha\to\infty} \frac{1}{\lambda(\alpha)} \log \mu_\alpha\Big(\sup_{s\le t\le s+\varepsilon} |x(t) - x(s)| \ge \delta\Big) = -\infty,$$

then $\{\mu_\alpha, \alpha > 0\}$ satisfies the LDP on $D([0, 1])$ with speed $\lambda(\alpha)$ and good rate function defined by

$$I(x) := \sup_{\{t_1,\dots,t_l\}\subset[0,1]} I_{t_1,\dots,t_l}(x_{t_1}, \dots, x_{t_l}).$$

2. Exponential martingale and Laplace functional

The main purpose of this section is to construct an exponential martingale associated with the Markov-modulated risk model with reinsurance and to calculate the Laplace functional of the model.

Theorem 2.1. (i) *Set*

$$M_t := S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty R_u(x) G_{J(u)}(\mathrm{d}x) \lambda_{J(u)} \, \mathrm{d}u.$$

Then $\{M_t, \mathcal{G}_t, t \geq 0\}$ *is a martingale, where* $\mathcal{G}_s = \sigma(N(u), u \leq s) \vee \sigma(J(u), u \geq 0) \vee \sigma(U_l, l \leq N(s))$.

(ii) *If, for some* $\delta > 0$,

$$\sup_{i \in E} \int_0^\infty \mathrm{e}^{\delta x} G_i(\mathrm{d}x) < \infty$$

then, for any measurable function $\theta(t)$ *satisfying* $\sup_{t \geq 0} \theta(t) < \delta$,

$$Z_t^\theta := \exp\biggl\{\int_0^t \theta(u) \, \mathrm{d}S_{\mathcal{R}}(u) - \int_0^t \int_0^\infty (\exp\{\theta(u) R_u(x)\} - 1) G_{J(u)}(\mathrm{d}x) \lambda_{J(u)} \, \mathrm{d}u\biggr\}$$

is a $\{\mathcal{G}_t\}$*-martingale. Equivalently,*

$$\mathrm{E}\biggl(\exp\biggl\{\int_0^t \theta(u) \, \mathrm{d}S_{\mathcal{R}}(u)\biggr\} \biggm| J\biggr) = \exp\biggl\{\int_0^t \int_0^\infty (\exp\{\theta(u) R_u(x)\} - 1) G_{J(u)}(\mathrm{d}x) \lambda_{J(u)} \, \mathrm{d}u\biggr\}.$$

Proof. (i) For any $s < t$,

$$\begin{aligned}
\mathrm{E}(S_{\mathcal{R}}(t) \mid \mathcal{G}_s) &= S_{\mathcal{R}}(s) + \mathrm{E}\biggl(\sum_{l=N(s)+1}^{N(t)} R_{T_l}(U_l) \biggm| \mathcal{G}_s\biggr) \\
&= S_{\mathcal{R}}(s) + \mathrm{E}\biggl(\sum_{l=1}^{N(t)-N(s)} R_{T_{N(s)+l}}(U_{N(s)+l}) \biggm| \mathcal{G}_s\biggr)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}\biggl(\sum_{l=1}^{N(t)-N(s)} R_{T_{N(s)+l}}(U_{N(s)+l}) \biggm| \mathcal{G}_s\biggr) \\
&\quad = \sum_{m=1}^\infty \sum_{l=1}^m \mathrm{E}(\mathbf{1}_{\{N(t)-N(s)=m\}} R_{T_{N(s)+l}}(U_{N(s)+l}) \mid \mathcal{G}_s) \\
&\quad = \sum_{l=1}^\infty \mathrm{E}(\mathbf{1}_{\{N(t)-N(s)\geq l\}} R_{T_{N(s)+l}}(U_{N(s)+l}) \mid \mathcal{G}_s) \\
&\quad = \sum_{l=1}^\infty \mathrm{E}\biggl(\mathbf{1}_{\{\tilde{T}_l \leq t-s\}} \int_0^\infty R_{\tilde{T}_l+s}(x) G_{J(\tilde{T}_l+s)}(\mathrm{d}x) \biggm| \mathcal{G}_s\biggr) \\
&\quad = \mathrm{E}\biggl(\int_0^\infty \mathbf{1}_{(0,t-s]}(u) \int_0^\infty R_{s+u}(x) G_{J(s+u)}(\mathrm{d}x) \, \mathrm{d}\tilde{N}(u) \biggm| \mathcal{G}_s\biggr),
\end{aligned}$$

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where

$$\tilde{N}(u) = N(s+u) - N(s), \quad u \geq 0, \quad \text{and} \quad \tilde{T}_l = \inf\{u \geq 0,\ \tilde{N}(u) = l\}.$$

Since $\tilde{N}(u)$ is a doubly stochastic Poisson process with intensity $\lambda_{J(s+u)}$, $\tilde{N}(u) - \lambda_{J(s+u)}$ is an $\{\tilde{\mathcal{F}}_u := \mathcal{F}_{s+u},\ u \geq 0\}$-martingale and so

$$\begin{aligned}
&\mathrm{E}\biggl(\int_0^\infty \mathbf{1}_{(0,t-s]}(u) \int_0^\infty R_{s+u}(x) G_{J(s+u)}(\mathrm{d}x)\, \mathrm{d}\tilde{N}(u) \biggm| \mathcal{G}_s\biggr) \\
&\quad = \mathrm{E}\biggl(\int_0^\infty \mathbf{1}_{(0,t-s]}(u) \int_0^\infty R_{s+u}(x) G_{J(s+u)}(\mathrm{d}x) \lambda_{J(s+u)}\, \mathrm{d}u \biggm| \mathcal{G}_s\biggr) \\
&\quad = \int_s^t \int_0^\infty R_u(x) G_{J(u)}(\mathrm{d}x) \lambda_{J(u)}\, \mathrm{d}u.
\end{aligned}$$

Therefore, $\{S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty R_u(x) G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\, \mathrm{d}u\}$ is a $\{\mathcal{G}_t\}$-martingale.

(ii) Set $L_t = \int_0^t \theta(u)\, \mathrm{d}S_{\mathcal{R}}(u)$. Applying Itô's formula (cf. [11, p. 242]) to $\exp\{L_t\}$, we have

$$\begin{aligned}
\exp\{L_t\} &= 1 + \int_0^t \exp\{L_{u-}\}\theta(u)\, \mathrm{d}S_{\mathcal{R}}(u) + \sum_{0<u\leq t} \exp\{L_{u-}\}(\exp\{\triangle L_u\} - 1 - \triangle L_u) \\
&= 1 + \int_0^t \exp\{L_{u-}\}\theta(u)\, \mathrm{d}S_{\mathcal{R}}(u) \\
&\quad + \sum_{0<u\leq t} \exp\{L_{u-}\}(\exp\{\theta(u) R_u(U_{N(u)})\} - 1 - \theta(u) R_u(U_{N(u)}))\, \mathbf{1}_{\{\triangle N(u)=1\}} \\
&= 1 + \int_0^t \exp\{L_{u-}\}\theta(u)\, \mathrm{d}M_u + \int_0^t \int_0^\infty \exp\{L_{u-}\}\theta(u) R_u(x) G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\, \mathrm{d}u \\
&\quad + \sum_{l=1}^\infty \exp\{L_{T_{l-1}}\}(\exp\{\theta(T_l) R_{T_l}(U_l)\} - 1 - \theta(T_l) R_{T_l}(U_l))\, \mathbf{1}_{\{T_l \leq t\}},
\end{aligned}$$

where $\triangle L_u = L_u - L_{u-}$. Therefore, the conditional expectation $\mathrm{E}(\exp\{L_t\} \mid J)$ of $\exp\{L_t\}$ with respect to $\sigma(J(s),\ s \geq 0)$ satisfies the following equation:

$$\begin{aligned}
\mathrm{E}(\exp\{L_t\} \mid J) &= 1 + \int_0^t \int_0^\infty \mathrm{E}(\exp\{L_{u-}\} \mid J)\theta(u) R_u(x) G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\, \mathrm{d}u \\
&\quad + \sum_{l=1}^\infty \mathrm{E}(\exp\{L_{T_{l-1}}\}(\exp\{\theta(T_l) R_{T_l}(U_l)\} - 1 - \theta(T_l) R_{T_l}(U_l))\, \mathbf{1}_{\{T_l \leq t\}} \mid J).
\end{aligned}$$

Since

$$\sum_{l=1}^{\infty} \mathrm{E}(\exp\{L_{T_{l-1}}\}(\exp\{\theta(T_l)R_{T_l}(U_l)\} - 1 - \theta(T_l)R_{T_l}(U_l))\,\mathbf{1}_{\{T_l \le t\}} \mid J)$$
$$= \sum_{l=1}^{\infty} \int_0^{\infty} \mathrm{E}(\exp\{L_{T_{l-1}}\}(\exp\{\theta(T_l)R_{T_l}(x)\} - 1 - \theta(T_l)R_{T_l}(x))\,\mathbf{1}_{\{T_l \le t\}}\, G_{J(T_l)}(\mathrm{d}x) \mid J)$$
$$= \mathrm{E}\bigg(\int_0^t \int_0^{\infty} \exp\{L_{u-}\}(\exp\{\theta(u)R_u(x)\} - 1 - \theta(u)R_u(x))G_{J(u)}(\mathrm{d}x)\,\mathrm{d}N(u) \;\bigg|\; J\bigg)$$
$$= \mathrm{E}\bigg(\int_0^t \int_0^{\infty} \exp\{L_{u-}\}(\exp\{\theta(u)R_u(x)\} - 1 - \theta(u)R_u(x))G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u \;\bigg|\; J\bigg)$$
$$= \int_0^t \int_0^{\infty} \mathrm{E}(\exp\{L_{u-}\} \mid J)(\exp\{\theta(u)R_u(x)\} - 1 - \theta(u)R_u(x))G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u,$$

we have

$$\mathrm{E}(\exp\{L_t\} \mid J) = 1 + \int_0^t \int_0^{\infty} \mathrm{E}(\exp\{L_{u-}\} \mid J)(\exp\{\theta(u)R_u(x)\} - 1)G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u,$$

which implies that

$$\mathrm{E}(\exp\{L_t\} \mid J) = \exp\bigg\{\int_0^t \int_0^{\infty} (\exp\{\theta(u)R_u(x)\} - 1)G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\bigg\}.$$

Corollary 2.1. *If, for some* $\delta > 0$*,*

$$\sup_{i \in E} \int_0^{\infty} \mathrm{e}^{\delta x} G_i(\mathrm{d}x) < \infty$$

then, for any $m \ge 1$*,* $0 = t_0 < t_1 < \cdots < t_m$*, and* $\theta_1, \ldots, \theta_m \in (-\infty, \delta)$*,*

$$\mathrm{E}\bigg(\exp\bigg\{\sum_{l=1}^{m} \theta_l \sum_{n=N(t_{l-1})+1}^{N(t_l)} R_{T_n}(U_n)\bigg\} \;\bigg|\; J\bigg)$$
$$= \exp\bigg\{\sum_{l=1}^{m} \int_{t_{l-1}}^{t_l} \int_0^{\infty} (\exp\{\theta_l R_u(x)\} - 1)G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\bigg\}. \tag{2.1}$$

Furthermore,

$$\prod_{l=1}^{m} \inf_{i \in E} \mathrm{E}_i\bigg(\exp\bigg\{\int_0^{t_l - t_{l-1}} \int_0^{\infty} (\exp\{\theta_l R_{u+t_{l-1}}(x)\} - 1)G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\bigg\}\bigg)$$
$$\le \mathrm{E}\bigg(\exp\bigg\{\sum_{l=1}^{m} \theta_l \sum_{n=N(t_{l-1})+1}^{N(t_l)} R_{T_n}(U_n)\bigg\}\bigg)$$
$$\le \prod_{l=1}^{m} \sup_{i \in E} \mathrm{E}_i\bigg(\exp\bigg\{\int_0^{t_l - t_{l-1}} \int_0^{\infty} (\exp\{\theta_l R_{u+t_{l-1}}(x)\} - 1)G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\bigg\}\bigg), \tag{2.2}$$

where $\mathrm{E}_i(\cdot) := \mathrm{E}(\cdot \mid J(0) = i)$*.*

Proof. Take $\theta(u)=\sum_{l=1}^{m}\theta_l\,\mathbf{1}_{(t_{l-1},t_l]}(u)$. Then Theorem 2.1(ii) yields (2.1). By the Markov property of J, it is easy to obtain (2.2).

Remark 2.1. Macci and Stabile [13] gave a representation of $\mathrm{E}(\exp\{\theta S_{\mathcal{R}}(t)\})$. Our proof for the general result, (2.1), is based on Itô's formula.

3. Large deviations

In this section we establish a functional LDP for the process $S_{\mathcal{R}}(t)$. In order to obtain the LDP of $S_{\mathcal{R}}(t)$, Macci and Stabile [13] introduced the following two assumptions.

(H1) Let $\tilde{R}\colon [0,\infty)\to[0,\infty)$ be a measurable function. Then, for all $\varepsilon>0$, there exists t_ε such that, for all $t\ge t_\varepsilon$, we have $|R_t(x)-\tilde{R}(x)|\le\varepsilon(x+1)$ for all $x\ge 0$.

(H2) For all $r>0$,

$$\sup_{i\in E}\int_0^\infty \mathrm{e}^{rx}G_i(\mathrm{d}x)<\infty.$$

We will prove that $\{S_{\mathcal{R}}(\alpha t)/\alpha|_{t\in[0,1]},\ \alpha>0\}$ satisfies a functional LDP in $D([0,1])$ under assumptions (H1) and (H2).

Let $Q=(q_{ij})_{i,j\in E}$ be the intensity matrix of the Markov chain $\{J(t),\ t\ge 0\}$. For any vector $v=(v_i)_{i\in E}$, set $Q(v)=(q_{ij}+\delta_{ij}v_i)_{i,j\in E}$ and let $\Lambda(v)$ be the logarithm of the simple and positive eigenvalue of the exponential matrix $\mathrm{e}^{Q(v)}$. By applying the Feymnan–Kac formula we obtain (cf. [3] and [5, Corollary 4.2.27]), for any $j\in E$,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathrm{E}_j\biggl(\exp\biggl\{\int_0^t v_{J(u)}\,\mathrm{d}u\biggr\}\biggr)=\Lambda(v). \tag{3.1}$$

Since, for any $j\in E$,

$$\begin{aligned}\inf_{i\in E}\mathrm{E}_i&\biggl(\exp\biggl\{\int_0^t\lambda_{J(u)}\int_0^\infty(\exp\{\theta R_u(x)\}-1)G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\biggr\}\biggr)\\ &\le \mathrm{E}_j(\exp\{\theta S_{\mathcal{R}}(t)\})\\ &\le\sup_{i\in E}\mathrm{E}_i\biggl(\exp\biggl\{\int_0^t\lambda_{J(u)}\int_0^\infty(\exp\{\theta R_u(x)\}-1)G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\biggr\}\biggr),\end{aligned}$$

under assumptions (H1) and (H2), (3.1) implies that (see [13] for detail), for any $j\in E$ and any $\theta\in\mathbb{R}$,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathrm{E}_j(\exp\{\theta S_{\mathcal{R}}(t)\})=\Lambda\biggl(\biggl(\lambda_i\int_0^\infty(\mathrm{e}^{\theta\tilde{R}(x)}-1)G_i(\mathrm{d}x)\biggr)_{i\in E}\biggr). \tag{3.2}$$

Define

$$\Lambda^*(x)=\sup_{\theta\in\mathbb{R}}\biggl\{\theta x-\Lambda\biggl(\biggl(\lambda_i\int_0^\infty(\mathrm{e}^{\theta\tilde{R}(z)}-1)G_i(\mathrm{d}z)\biggr)_{i\in E}\biggr)\biggr\}.$$

Lemma 3.1. *Let assumptions (H1) and (H2) hold. Then, for any $m\ge 1$ and $0=t_0\le t_1<t_2<\cdots<t_m\le 1$, $\{(S_{\mathcal{R}}(\alpha t_1)/\alpha, S_{\mathcal{R}}(\alpha t_2)/\alpha,\dots,S_{\mathcal{R}}(\alpha t_m)/\alpha),\ \alpha>0\}$ satisfies the LDP speed α and rate function $I^{(ld)}_{t_1,\dots,t_m}$ defined by*

$$I^{(ld)}_{t_1,\dots,t_m}(x_1,\dots,x_m)=\sum_{l=1}^{m}(t_l-t_{l-1})\Lambda^*\biggl(\frac{x_l-x_{l-1}}{t_l-t_{l-1}}\biggr),$$

where $x_0=0$.

Proof. For any $(\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^m$, by (2.2) we have

$$\begin{aligned}
&\sum_{l=1}^{m}(t_l - t_{l-1}) \lim_{\alpha\to\infty} \frac{1}{\alpha(t_l - t_{l-1})} \\
&\quad \times \log \inf_{i\in E} \mathrm{E}_i\biggl(\exp\biggl\{\int_0^{\alpha(t_l-t_{l-1})} \lambda_{J(u)} \int_0^{\infty} (\exp\{\theta_l R_{u+\alpha t_{l-1}}(x)\} - 1) G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\biggr\}\biggr) \\
&\quad \le \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \mathrm{E}\biggl(\exp\biggl\{\sum_{l=1}^{m} \theta_l \sum_{n=N(\alpha t_{l-1})+1}^{N(\alpha t_l)} R_{T_n}(U_n)\biggr\}\biggr) \\
&\quad \le \sum_{l=1}^{m}(t_l - t_{l-1}) \lim_{\alpha\to\infty} \frac{1}{\alpha(t_l - t_{l-1})} \\
&\qquad \times \log \sup_{i\in E} \mathrm{E}_i\biggl(\exp\biggl\{\int_0^{\alpha(t_l-t_{l-1})} \lambda_{J(u)} \int_0^{\infty} (\exp\{\theta_l R_{u+\alpha t_{l-1}}(x)\} - 1) G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\biggr\}\biggr).
\end{aligned}$$

Therefore, (3.2) implies that

$$\begin{aligned}
\Lambda_{t_1,\dots,t_m}(\theta_1, \dots, \theta_m) &:= \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \mathrm{E}\biggl(\exp\biggl\{\sum_{l=1}^{m} \theta_l \sum_{n=N(\alpha t_{l-1})+1}^{N(\alpha t_l)} R_{T_n}(U_n)\biggr\}\biggr) \\
&= \sum_{l=1}^{m}(t_l - t_{l-1}) \Lambda\biggl(\biggl(\lambda_i \int_0^{\infty} (\exp\{\theta_l \tilde{R}(x)\} - 1) G_i(\mathrm{d}x)\biggr)_{i\in E}\biggr).
\end{aligned}$$

By the Gärtner–Ellis theorem (cf. [4, p. 43]),

$$\biggl(\frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_1), \frac{1}{\alpha}(S_{\mathcal{R}}(\alpha t_2) - S_{\mathcal{R}}(\alpha t_1)), \dots, \frac{1}{\alpha}(S_{\mathcal{R}}(\alpha t_m) - S_{\mathcal{R}}(\alpha t_{m-1}))\biggr)$$

satisfies the LDP with speed α and rate function $\Lambda^*_{t_1,\dots,t_m}(\cdot)$ defined by

$$\begin{aligned}
\Lambda^*_{t_1,\dots,t_m}(x_1, \dots, x_m) &= \sup_{(\theta_1,\dots,\theta_m)\in\mathbb{R}^m} \biggl\{\sum_{l=1}^{m} \theta_l x_l - \Lambda_{t_1,\dots,t_m}(\theta_1, \dots, \theta_m)\biggr\} \\
&= \sum_{l=1}^{m}(t_l - t_{l-1}) \Lambda^*\biggl(\frac{x_l}{t_l - t_{l-1}}\biggr).
\end{aligned}$$

Now, since

$$\begin{aligned}
&\biggl(\frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_1), \frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_2), \dots, \frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_m)\biggr) \\
&\quad = \biggl(\frac{1}{\alpha} S_{\mathcal{R}}(\alpha t_1), \frac{1}{\alpha}(S_{\mathcal{R}}(\alpha t_2) - S_{\mathcal{R}}(\alpha t_1)), \dots, \frac{1}{\alpha}(S_{\mathcal{R}}(\alpha t_m) - S_{\mathcal{R}}(\alpha t_{m-1}))\biggr) T,
\end{aligned}$$

where the matrix $T = (t_{lk})_{1\le l,\,k\le m}$ satisfies $t_{lk} = 1$ for $l \le k$ and $t_{lk} = 0$ for $l > k$, we obtain the conclusion of the lemma from the contract principle.

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Theorem 3.1. *Let assumptions (H1) and (H2) hold. Then* $\{\mathrm{P}((S_{\mathcal{R}}(\alpha t)/\alpha)_{t\in[0,1]} \in \cdot),\ \alpha > 0\}$ *satisfies the LDP on* $D([0,1])$ *with speed* α *and rate function* $I^{(ld)}$ *defined by*

$$I^{(ld)}(f) = \begin{cases} \displaystyle\int_0^1 \Lambda^*(\dot{f}(t))\,\mathrm{d}t & \text{if } f(0) = 0 \text{ and } f \text{ is absolutely continuous,} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. Firstly, using the standard argument (cf. [4, p. 180]), we obtain

$$I^{(ld)}(f) = \sup_{m\geq 1} \sup_{0=t_0\leq t_1<t_2<\cdots<t_m\leq 1} \sum_{l=1}^{m} (t_l - t_{l-1})\Lambda^*\biggl(\frac{f(t_l) - f(t_{l-1})}{t_l - t_{l-1}}\biggr).$$

Therefore, by Lemma 1.2 and Lemma 3.1, we only need to prove the exponential tightness, i.e. for any $t \in [0,1]$ and any $\eta > 0$,

$$\lim_{\delta\downarrow 0} \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \mathrm{P}\biggl(\frac{1}{\alpha} \sup_{t\leq s\leq t+\delta} |S_{\mathcal{R}}(\alpha t) - S_{\mathcal{R}}(\alpha s)| \geq \eta\biggr) = -\infty. \tag{3.3}$$

By Theorem 2.1, for any $\beta \in \mathbb{R}$, $(Z_t^\beta)^{-1} Z_{t+s}^\beta$, $s \geq 0$, is a martingale under probability $\mathrm{P}(\cdot \mid J)$, where

$$Z_t^\beta := \exp\biggl\{\beta S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty (\exp\{\beta R_u(x)\} - 1) G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\biggr\}.$$

Then, by the maximum inequality for a martingale, we have, for any $\beta > 0$,

$$\begin{aligned}
&\frac{1}{\alpha} \log \mathrm{P}\biggl(\frac{1}{\alpha} \sup_{t\leq s\leq t+\delta} |S_{\mathcal{R}}(\alpha t) - S_{\mathcal{R}}(\alpha s)| \geq \eta\biggr) \\
&\quad = \frac{1}{\alpha} \log \mathrm{P}\biggl(\frac{1}{\alpha} \sup_{t\leq s\leq t+\delta} (S_{\mathcal{R}}(\alpha t) - S_{\mathcal{R}}(\alpha s)) \geq \eta\biggr) \\
&\quad \leq \frac{1}{\alpha} \log \mathrm{E}\biggl(\mathrm{P}\biggl(\sup_{0\leq s\leq \delta} (Z_{\alpha t}^\beta)^{-1} Z_{\alpha(t+s)}^\beta \geq \mathrm{e}^{\alpha\beta\eta - \alpha\delta C(\beta)} \biggm| J\biggr)\biggr) \\
&\quad \leq \frac{1}{\alpha} \log \mathrm{E}(\mathrm{e}^{-\alpha\beta\eta + \alpha\delta C(\beta)}\, \mathrm{E}((Z_{\alpha t}^\beta)^{-1} Z_{\alpha(t+\delta)}^\beta \mid J)) \\
&\quad = -\beta\eta + \delta C(\beta),
\end{aligned}$$

where

$$C(\beta) := \sup_{i\in E} \lambda_i \int_0^\infty (\mathrm{e}^{\beta x} - 1) G_i(\mathrm{d}x).$$

Now letting $\alpha \to \infty$ firstly, $\delta \downarrow 0$ secondly, and $\beta \to \infty$ finally, we obtain (3.3).

4. Moderate deviations

In this section we establish a functional moderate deviation principle for the process $S_{\mathcal{R}}(t)$. Throughout this section, $\{a(t),\ t \geq 0\}$ denotes a positive function satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t} = 0, \qquad \lim_{t\to\infty} \frac{a(t)}{\sqrt{t}} = \infty.$$

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We introduce the following assumptions.

(H3) There exist two nonnegative measurable functions $\hat{R}(x)$ and $m(u)$ such that, for all $u \geq 0$ and $x > 0$, $|R_u(x) - \hat{R}(x)| \leq m(u)(x+1)$ and

$$\lim_{u\to\infty} m(u) = 0, \qquad \lim_{t\to\infty} \frac{1}{a(t)} \int_0^t m(u)\,\mathrm{d}u = 0.$$

(H4) There exists $\delta > 0$ such that

$$\sup_{i\in E} \int_0^\infty \mathrm{e}^{\delta x} G_i(\mathrm{d}x) < \infty.$$

For example, if $R_t(x) = c(1 - 1/(1+t)^\gamma)x^\tau$, where $c, \tau \in (0, 1]$ and $\gamma > 0$, then (H3) holds for $\hat{R}(x) = cx^\tau$, $m(u) = 1/(1+u)^\gamma$, and $a(t) = t^\beta$, where $\max\{1-\gamma, \frac{1}{2}\} < \beta < 1$.

Since $\{J(t),\ t \geq 0\}$ is a uniformly ergodic Markov process with finite state space E, the following result is known (cf. [10] and [15]).

Lemma 4.1. *Let $P(t) = (p_{ij}(t))_{i,j\in E} = \mathrm{e}^{tQ}$ be the semigroup of the Markov chain J.*

(i) *There exists $c > 0$ such that, for any function f on E,*

$$\sup_{i\in E}\left|\sum_{k\in E} p_{ik}(t) f(k) - \sum_{j\in E} \pi_j f(j)\right| \leq \mathrm{e}^{-ct} \sup_{i\in E} |f(i)|.$$

(ii) *For any $j \in E$ and any function f on E,*

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \mathrm{E}_j\biggl(\int_0^t (f(J(u)) - \mathrm{E}_\pi(f(J(u))))\,\mathrm{d}u\biggr)^2 \\ = 2\int_0^\infty \sum_{i\in E} \pi_i \biggl(f(i) - \sum_{k\in E} \pi_k f(k)\biggr) \sum_{k\in E} p_{ik}(u) f(k)\,\mathrm{d}u.\end{aligned}$$

(iii) *For any $i \in E$ and any function f on E,*

$$\begin{aligned}\lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log E_i\biggl(\exp\biggl\{\frac{a(\alpha)}{\alpha} \int_0^{\alpha t} (f(J(u)) - \mathrm{E}_\pi(f(J(u))))\,\mathrm{d}u\biggr\}\biggr) \\ = t\int_0^\infty \sum_{i\in E} \pi_i \biggl(f(i) - \sum_{k\in E} \pi_k f(k)\biggr) \sum_{k\in E} p_{ik}(u) f(k)\,\mathrm{d}u.\end{aligned}$$

Remark 4.1. Set $\hat{R}_i = \int_0^\infty \hat{R}(x) G_i(\mathrm{d}x)$. Then, by Lemma 4.1,

$$\sigma_1^2 := \lim_{t\to\infty} \frac{1}{t} \mathrm{E}_\pi\biggl(\biggl(\int_0^t (\hat{R}_{J(u)}\lambda_{J(u)} - \mathrm{E}_\pi(\hat{R}_{J(u)}\lambda_{J(u)}))\,\mathrm{d}u\biggr)^2\biggr)$$

exists and

$$\sigma_1^2 = 2\int_0^\infty \sum_{j\in E} \pi_j \biggl(\lambda_j \hat{R}_j - \sum_{i\in E} \pi_i \lambda_i \hat{R}_i\biggr) \sum_{k\in E} p_{jk}(t) \lambda_k \hat{R}_k\,\mathrm{d}t.$$

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Lemma 4.2. *Let assumptions (H3) and (H4) hold. Set* $\bar{S}_{\mathcal{R}}(\cdot) = S_{\mathcal{R}}(\cdot) - \mathrm{E}_\pi(S_{\mathcal{R}}(\cdot))$*. Then, for any* $j \in E$*,* $\theta \in \mathbb{R}$*, and* $t > 0$*,*

$$\lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log \mathrm{E}_j\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right) = \frac{1}{2}\theta^2\sigma^2 t,$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2$ *and*

$$\sigma_2^2 = \sum_{i\in E}\left(\pi_i\lambda_i \int_0^\infty \hat{R}^2(x)G_i(\mathrm{d}x)\right).$$

Proof. By Theorem 2.1, for any $\theta \in \mathbb{R}$, $t > 0$, and $\alpha > 0$,

$$\mathrm{E}_\pi(S_{\mathcal{R}}(\alpha t)) = \mathrm{E}_\pi\left(\int_0^{\alpha t}\lambda_{J(u)}\int_0^\infty R_u(x)G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\right)$$

and, for all $j \in E$,

$$\begin{aligned}
&\mathrm{E}_j\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right)\\
&\quad= \mathrm{E}_j\left(\exp\left\{\int_0^{\alpha t}\left(\lambda_{J(u)}\int_0^\infty\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}R_u(x)\right\}-1\right)G_{J(u)}(\mathrm{d}x)\right.\right.\right.\\
&\qquad\qquad\left.\left.\left. - \mathrm{E}_\pi\left(\theta\frac{a(\alpha)}{\alpha}\lambda_{J(u)}\int_0^\infty R_u(x)G_{J(u)}(\mathrm{d}x)\right)\right)\mathrm{d}u\right\}\right).
\end{aligned}$$

Therefore,

$$\inf_{i\in E}\mathrm{E}_i(\xi\eta\zeta) \le \mathrm{E}_j\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right) \le \sup_{i\in E}\mathrm{E}_i(\xi\eta\zeta),$$

where

$$\begin{aligned}
\xi &= \exp\left\{\int_0^{\alpha t}\theta\frac{a(\alpha)}{\alpha}\left(\lambda_{J(u)}\int_0^\infty R_u(x)G_{J(u)}(\mathrm{d}x)\right.\right.\\
&\qquad\qquad\left.\left. - \mathrm{E}_\pi\left(\lambda_{J(u)}\int_0^\infty R_u(x)G_{J(u)}(\mathrm{d}x)\right)\right)\mathrm{d}u\right\},\\
\eta &= \exp\left\{\int_0^{\alpha t}\lambda_{J(u)}\int_0^\infty\frac{1}{2}\left(\theta\frac{a(\alpha)}{\alpha}R_u(x)\right)^2 G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\right\},\\
\zeta &= \exp\left\{\int_0^{\alpha t}\lambda_{J(u)}\int_0^\infty\sum_{l=3}^\infty\left(\frac{1}{l!}\left(\theta\frac{a(\alpha)}{\alpha}R_u(x)\right)^l\right)G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\right\}.
\end{aligned}$$

By Hölder's inequality,

$$\sup_{i\in E}\mathrm{E}_i(\xi\eta\zeta) \le \left(\sup_{i\in E}\mathrm{E}_i(\xi^p)\right)^{1/p}\left(\sup_{i\in E}\mathrm{E}_i(\eta^q)\right)^{1/q}\left(\sup_{i\in E}\mathrm{E}_i(\zeta^r)\right)^{1/r},$$

where $1/p + 1/q + 1/r = 1$, $p > 1$, $q > 1$, and $r > 1$. From assumption (H3), we have $R_u(x) \le \hat{R}(x) + m(u)(x+1)$, $-R_u(x) \le -\hat{R}(x) + m(u)(x+1)$, and $0 \le R_u(x), \hat{R}(x) \le x$.

Therefore,

$$
\begin{aligned}
&\frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\xi^p)\Big)^{1/p} \\
&\quad \le \frac{\alpha}{pa^2(\alpha)} \log \sup_{i\in E} \mathrm{E}_i\bigg(\exp\bigg\{\frac{p\theta a(\alpha)}{\alpha}\int_0^{\alpha t}(\lambda_{J(u)}\hat{R}_{J(u)} - \mathrm{E}_\pi(\lambda_{J(u)}\hat{R}_{J(u)}))\,\mathrm{d}u\bigg\}\bigg) \\
&\qquad + \frac{2|\theta|A}{a(\alpha)}\int_0^{\alpha t} m(u)\,\mathrm{d}u
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\eta^q)\Big)^{1/q} \\
&\le \frac{\alpha}{qa^2(\alpha)} \log \sup_{i\in E} \mathrm{E}_i\bigg(\exp\bigg\{\frac{q}{2}\bigg(\frac{\theta a(\alpha)}{\alpha}\bigg)^2\int_0^{\alpha t}\lambda_{J(u)}\int_0^\infty \hat{R}^2(x) G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\bigg\}\bigg) \\
&\quad + \frac{\theta^2 A}{\alpha}\int_0^{\alpha t} m(u)\,\mathrm{d}u + \frac{\theta^2 A}{2\alpha}\int_0^{\alpha t} m^2(u)\,\mathrm{d}u \\
&= \frac{\alpha}{qa^2(\alpha)} \log \sup_{i\in E} \mathrm{E}_i\bigg(\exp\bigg\{\frac{q}{2}\bigg(\frac{\theta a(\alpha)}{\alpha}\bigg)^2\int_0^{\alpha t}\bigg(\lambda_{J(u)}\int_0^\infty \hat{R}^2(x) G_{J(u)}(\mathrm{d}x) - \sigma_2^2\bigg)\,\mathrm{d}u\bigg\}\bigg) \\
&\quad + \frac{\theta^2 A}{\alpha}\int_0^{\alpha t} m(u)\,\mathrm{d}u + \frac{\theta^2 A}{2\alpha}\int_0^{\alpha t} m^2(u)\,\mathrm{d}u + \frac{1}{2}\theta^2\sigma_2^2 t,
\end{aligned}
$$

where $A = \sup_{i\in E}\{\lambda_i \int_0^\infty (x+1)^2 G_i(\mathrm{d}x)\}$. Then, applying Lemma 4.1, we have

$$
\limsup_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\xi^p)\Big)^{1/p} \le \frac{1}{2}p\theta^2\sigma_1^2 t
$$

and

$$
\limsup_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\eta^q)\Big)^{1/q} \le \frac{1}{2}\theta^2\sigma_2^2 t.
$$

Since, for any $r, \theta \in \mathbb{R}$, there exists a constant $M > 0$ such that, for all $\alpha \ge M$,

$$
\begin{aligned}
0 < \zeta^r &\le \exp\bigg\{|r|\int_0^{\alpha t}\lambda_{J(u)}\int_0^\infty \sum_{l=3}^\infty \frac{1}{l!}\bigg(\frac{|\theta|xa(\alpha)}{\alpha}\bigg)^l G_{J(u)}(\mathrm{d}x)\,\mathrm{d}u\bigg\} \\
&\le \exp\bigg\{6|r|\alpha t\bigg(\frac{2|\theta|a(\alpha)}{\delta\alpha}\bigg)^3 \sup_{i\in E}\lambda_i\int_0^\infty \mathrm{e}^{\delta x} G_i(\mathrm{d}x)\bigg\},
\end{aligned}
$$

where $\delta > 0$ satisfies assumption (H4), then, for any $r, \theta \in \mathbb{R}$,

$$
\limsup_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\zeta^r)\Big)^{1/r} = 0.
$$

Therefore, for all $p > 1$,

$$
\limsup_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log\Big(\sup_{i\in E} \mathrm{E}_i(\xi\eta\zeta)\Big) \le \frac{1}{2}p\,\theta^2\sigma_1^2 t + \frac{1}{2}\theta^2\sigma_2^2 t,
$$

which implies the following upper bound:

$$\limsup_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\sup_{i\in E}\mathrm{E}_i\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right)\le\frac{1}{2}\theta^2\sigma_1^2t+\frac{1}{2}\theta^2\sigma_2^2t=\frac{1}{2}\theta^2\sigma^2t.$$

Now let us show the lower bound. Again, by Hölder's inequality,

$$\inf_{i\in E}\mathrm{E}_i(\xi)\le\Bigl(\inf_{i\in E}\mathrm{E}_i(\xi\eta\zeta)^q\Bigr)^{1/q}\Bigl(\sup_{i\in E}\mathrm{E}_i(\eta^{-p})\Bigr)^{1/p}\Bigl(\sup_{i\in E}\mathrm{E}_i(\zeta^{-r})\Bigr)^{1/r},$$

where $1/p+1/q+1/r=1$, $p>1$, $q>1$, and $r>1$. From assumption (H3), we also have $R_u(x)+m(u)(x+1)\ge\hat{R}(x)$, and so, for all $u\ge0$, $R_u^2(x)\ge\hat{R}^2(x)-(2m(u)+m(u)^2)(x+1)^2$. Therefore,

$$\begin{aligned}
&\frac{\alpha}{a^2(\alpha)}\log\Bigl(\sup_{i\in E}\mathrm{E}_i(\eta^{-p})\Bigr)^{1/p}\\
&\quad\le\frac{\alpha}{pa^2(\alpha)}\log\sup_{i\in E}\mathrm{E}_i\left(\exp\left\{-\frac{p}{2}\left(\theta\frac{a(\alpha)}{\alpha}\right)^2\int_0^{\alpha t}\left(\lambda_{J(u)}\int_0^\infty\hat{R}^2(x)G_{J(u)}(\mathrm{d}x)-\sigma_2^2\right)\mathrm{d}u\right\}\right)\\
&\qquad+\frac{\theta^2A}{\alpha}\int_0^{\alpha t}m(u)\,\mathrm{d}u+\frac{\theta^2A}{2\alpha}\int_0^{\alpha t}m^2(u)\,\mathrm{d}u-\frac{1}{2}\theta^2\sigma_2^2t
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\alpha}{a^2(\alpha)}\log\Bigl(\inf_{i\in E}\mathrm{E}_i(\xi)\Bigr)\\
&\qquad\ge\frac{\alpha}{a^2(\alpha)}\log\inf_{i\in E}\mathrm{E}_i\left(\exp\left\{\int_0^{\alpha t}\theta\frac{a(\alpha)}{\alpha}(\lambda_{J(u)}\hat{R}_{J(u)}-\mathrm{E}_\pi(\lambda_{J(u)}\hat{R}_{J(u)}))\,\mathrm{d}u\right\}\right)\\
&\qquad\quad-2\theta\sup_{i\in E}(\lambda_i\hat{R}_i)\frac{1}{a(\alpha)}\int_0^{\alpha t}m(u)\,\mathrm{d}u.
\end{aligned}$$

Then by Lemma 4.1 we have

$$\limsup_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\Bigl(\sup_{i\in E}\mathrm{E}_i(\eta^{-p})\Bigr)^{1/p}\le-\frac{1}{2}\theta^2\sigma_2^2t$$

and

$$\liminf_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\Bigl(\inf_{i\in E}\mathrm{E}_i(\xi)\Bigr)\ge\frac{1}{2}\theta^2\sigma_1^2t.$$

Therefore, for any $q>1$,

$$\liminf_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\Bigl(\inf_{i\in E}\mathrm{E}_i(\xi\eta\zeta)^q\Bigr)^{1/q}\ge\frac{1}{2}\theta^2(\sigma_1^2+\sigma_2^2)t=\frac{1}{2}\theta^2\sigma^2t,$$

that is, for any $q>1$,

$$\liminf_{\alpha\to\infty}\frac{\alpha}{qa^2(\alpha)}\log\inf_{i\in E}\mathrm{E}_i\left(\exp\left\{\theta\frac{qa(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right)\ge\frac{1}{2}q\theta^2(\sigma_1^2+\sigma_2^2)t=\frac{q}{2}\theta^2\sigma^2t,$$

which implies the following lower bound:

$$\liminf_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\inf_{i\in E}\mathrm{E}_i\left(\exp\left\{\theta\frac{a(\alpha)}{\alpha}\bar{S}_{\mathcal{R}}(\alpha t)\right\}\right)\ge\frac{1}{2}\theta^2\sigma^2t.$$

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In conclusion, for all $j \in E$, $\theta \in \mathbb{R}$, and $t > 0$, we have

$$\lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log \mathrm{E}_j\left(\exp\left\{\theta \frac{a(\alpha)}{\alpha} \bar{S}_{\mathcal{R}}(\alpha t)\right\}\right) = \frac{1}{2}\theta^2\sigma^2 t.$$

Lemma 4.3. *Let assumptions (H3) and (H4) hold. Then, for any $\alpha > 0$ and $0 = t_0 \le t_1 < t_2 < \cdots < t_m \le 1$, $(\bar{S}_{\mathcal{R}}(\alpha t_1)/a(\alpha), \bar{S}_{\mathcal{R}}(\alpha t_2)/a(\alpha), \ldots, \bar{S}_{\mathcal{R}}(\alpha t_m)/a(\alpha))$ satisfies the LDP with speed $a^2(\alpha)/\alpha$ and rate function $I^{(md)}_{t_1,\ldots,t_m}$ defined by*

$$I^{(md)}_{t_1,\ldots,t_m}(x_1,\ldots,x_m) = \frac{1}{2\sigma^2}\sum_{l=1}^{m} \frac{(x_l - x_{l-1})^2}{t_l - t_{l-1}},$$

where $x_0 = 0$.

Proof. According to (2.2), for any $(\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^m$ and any $\alpha > 0$ satisfying $\max_{1\le l\le m} |\theta_l| a(\alpha)/\alpha < \delta$,

$$\begin{aligned}
&\prod_{l=1}^{m} \inf_{i\in E} \mathrm{E}_i\left(\exp\left\{\int_0^{\alpha(t_l - t_{l-1})}\left(\lambda_{J(u)}\int_0^{\infty}\left(\exp\left\{\theta_l \frac{a(\alpha)}{\alpha} R_{u+\alpha t_{l-1}}(x)\right\} - 1\right) G_{J(u)}(\mathrm{d}x)\right.\right.\right.\\
&\qquad\qquad\qquad \left.\left.\left. - \mathrm{E}_\pi\left(\theta_l \frac{a(\alpha)}{\alpha}\lambda_{J(u)}\int_0^{\infty} R_{u+\alpha t_{l-1}}(x) G_{J(u)}(\mathrm{d}x)\right)\right)\mathrm{d}u\right\}\right)\\
&\quad \le \mathrm{E}_j\left(\exp\left\{\sum_{l=1}^{m} \theta_l \frac{a(\alpha)}{\alpha}(\bar{S}_{\mathcal{R}}(\alpha t_l) - \bar{S}_{\mathcal{R}}(\alpha t_{l-1}))\right\}\right)\\
&\quad \le \prod_{l=1}^{m} \sup_{i\in E} \mathrm{E}_i\left(\exp\left\{\int_0^{\alpha(t_l - t_{l-1})}\left(\lambda_{J(u)}\int_0^{\infty}\left(\exp\left\{\theta_l \frac{a(\alpha)}{\alpha} R_{u+\alpha t_{l-1}}(x)\right\} - 1\right) G_{J(u)}(\mathrm{d}x)\right.\right.\right.\\
&\qquad\qquad\qquad \left.\left.\left. - \mathrm{E}_\pi\left(\theta_l \frac{a(\alpha)}{\alpha}\lambda_{J(u)}\int_0^{\infty} R_{u+\alpha t_{l-1}}(x) G_{J(u)}(\mathrm{d}x)\right)\right)\mathrm{d}u\right\}\right).
\end{aligned}$$

Therefore, by Lemma 4.2 we have

$$\lim_{\alpha\to\infty} \frac{a(\alpha)}{\alpha} \log \mathrm{E}_j\left(\exp\left\{\sum_{l=1}^{m} \theta_l \frac{a(\alpha)}{\alpha}(\bar{S}_{\mathcal{R}}(\alpha t_l) - \bar{S}_{\mathcal{R}}(\alpha t_{l-1}))\right\}\right) = \frac{\sigma^2}{2}\sum_{l=1}^{m}(t_l - t_{l-1})\theta_l^2,$$

and so, by the Gärtner–Ellis theorem,

$$\left(\frac{\bar{S}_{\mathcal{R}}(\alpha t_1)}{a(\alpha)}, \bar{S}_{\mathcal{R}}(\alpha t_2) - \frac{\bar{S}_{\mathcal{R}}(\alpha t_1)}{a(\alpha)}, \ldots, \frac{\bar{S}_{\mathcal{R}}(\alpha t_m) - \bar{S}_{\mathcal{R}}(\alpha t_{m-1})}{a(\alpha)}\right)$$

satisfies the LDP with speed $a^2(\alpha)/\alpha$ and rate function $J^{(md)}_{t_1,\ldots,t_m}$ defined by

$$J^{(md)}_{t_1,\ldots,t_m}(x_1,\ldots,x_m) = \frac{1}{2\sigma^2}\sum_{l=1}^{m} \frac{x_l^2}{(t_l - t_{l-1})},$$

which implies the conclusion of the lemma from the contract principle.

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Theorem 4.1. *Let assumptions (H3) and (H4) hold. Then*

$$\left\{\mathrm{P}\left(\frac{\bar{S}_{\mathcal{R}}(\alpha t)}{a(\alpha)}\bigg|_{t\in[0,1]}\in\cdot\right),\ \alpha>0\right\}$$

and

$$\left\{\mathrm{P}\left(\frac{S_{\mathcal{R}}(\alpha t)-\alpha t\sum_{i\in E}\pi_i\lambda_i\hat{R}_i}{a(\alpha)}\bigg|_{t\in[0,1]}\in\cdot\right),\ \alpha>0\right\}$$

satisfy the LDP on $D([0,1])$ *with speed* $a^2(\alpha)/\alpha$ *and rate function* $I^{(md)}$ *defined by*

$$I^{(md)}(f)=\begin{cases}\dfrac{1}{2\sigma^2}\displaystyle\int_0^1(\dot{f}(t))^2\,\mathrm{d}t & \text{if } f(0)=0 \text{ and } f \text{ is absolutely continuous,}\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. Firstly, it follows from assumption (H3) that

$$\lim_{\alpha\to\infty}\frac{\sup_{t\in[0,1]}|\mathrm{E}(S_{\mathcal{R}}(\alpha t))-\alpha t\sum_{i\in E}\pi_i\lambda_i\hat{R}_i|}{a(\alpha)}=0;$$

therefore, we prove only the first statement. Using the standard argument (cf. [4, p. 180]), it is easy to obtain

$$I^{(md)}(f)=\sup_{m\geq 1}\sup_{0=t_0\leq t_1<t_2<\cdots<t_m\leq 1}\frac{1}{2\sigma^2}\sum_{l=1}^{m}\frac{(f(t_l)-f(t_{l-1}))^2}{t_l-t_{l-1}}.$$

Therefore, by Lemma 1.2 and Lemma 4.3, we only need to prove the exponential tightness, i.e. for any $t\in[0,1]$ and any $\eta>0$,

$$\lim_{\delta\downarrow 0}\lim_{\alpha\to\infty}\frac{\alpha}{a^2(\alpha)}\log\mathrm{P}\left(\frac{1}{a(\alpha)}\sup_{t\leq s\leq t+\delta}|\bar{S}_{\mathcal{R}}(\alpha t)-\bar{S}_{\mathcal{R}}(\alpha s)|\geq\eta\right)=-\infty. \tag{4.1}$$

By Theorem 2.1, for any $\beta\in\mathbb{R}$, $(Z_t^{\beta})^{-1}Z_{t+s}^{\beta}$, $s\geq 0$, is a martingale under probability $\mathrm{P}(\cdot\mid J)$. Then by the maximum inequality for a martingale we have, for any $\beta>0$,

$$\begin{aligned}&\frac{\alpha}{a^2(\alpha)}\log\mathrm{P}\left(\frac{1}{a(\alpha)}\sup_{t\leq s\leq t+\delta}(\bar{S}_{\mathcal{R}}(\alpha t)-\bar{S}_{\mathcal{R}}(\alpha s))\geq\eta\right)\\ &\quad\leq\frac{\alpha}{a^2(\alpha)}\log\mathrm{E}\left(\mathrm{P}\left(\sup_{0\leq s\leq\delta}(Z_{\alpha t}^{a(\alpha)\beta/\alpha})^{-1}Z_{\alpha(t+s)}^{a(\alpha)\beta/\alpha}\geq\exp\left\{\frac{a^2(\alpha)\beta\eta}{\alpha}-\alpha\delta C(\alpha,\beta)\right\}\Bigm| J\right)\right)\\ &\quad\leq\frac{\alpha}{a^2(\alpha)}\log\mathrm{E}\left(\exp\left\{-\frac{a^2(\alpha)\beta\eta}{\alpha}+\alpha\delta C(\alpha,\beta)\right\}\mathrm{E}((Z_{\alpha t}^{\beta})^{-1}Z_{\alpha(t+\delta)}^{\beta}\mid J)\right)\\ &\quad=-\beta\eta+\frac{\alpha^2}{a^2(\alpha)}\delta C(\alpha,\beta),\end{aligned}$$

where

$$C(\alpha,\beta):=\sup_{i\in E}\lambda_i\int_0^{\infty}\left(\mathrm{e}^{a(\alpha)\beta x/\alpha}-1-\frac{a(\alpha)\beta x}{\alpha}\right)G_i(\mathrm{d}x)=O\left(\frac{a^2(\alpha)}{\alpha^2}\right).$$

Now letting $\alpha \to \infty$ firstly, $\delta \downarrow 0$ secondly, and $\beta \to \infty$ finally, we obtain

$$\lim_{\delta\downarrow 0} \lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log \mathrm{P}\bigg(\frac{1}{a(\alpha)} \sup_{t\le s\le t+\delta} (\bar{S}_{\mathcal{R}}(\alpha t) - \bar{S}_{\mathcal{R}}(\alpha s)) \ge \eta\bigg) = -\infty.$$

Similarly, we can obtain

$$\lim_{\delta\downarrow 0} \lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log \mathrm{P}\bigg(\frac{1}{a(\alpha)} \inf_{t\le s\le t+\delta} (\bar{S}_{\mathcal{R}}(\alpha t) - \bar{S}_{\mathcal{R}}(\alpha s)) \le -\eta\bigg) = -\infty.$$

The proof of (4.1) is completed.

By $\inf_{\sup_{0\le t\le 1}|f(t)|\ge r} I^{(md)}(f) = r^2/2\sigma^2$ for any $r > 0$, we have the following corollary, which gives a moderate deviation estimate of the aggregate claims process.

Corollary 4.1. *Let assumptions (H3) and (H4) hold. Then, for any* $r > 0$,

$$\lim_{\alpha\to\infty} \frac{\alpha}{a^2(\alpha)} \log \mathrm{P}\bigg(\sup_{t\in[0,1]} \bigg| S_{\mathcal{R}}(\alpha t) - \alpha t \sum_{i\in E} \pi_i \lambda_i \hat{R}_i \bigg| \ge r a(\alpha)\bigg) = -\frac{r^2}{2\sigma^2}.$$

5. An estimate for the ruin probability

The ruin time and the ruin probability are defined by

$$\tau_x = \inf\{t \ge 0;\ X^x_{\mathcal{R}}(t) < 0\} \quad \text{and} \quad \psi(x) = \mathrm{P}(\tau_x < \infty).$$

Macci and Stabile [13] proved, by the large deviation approach (cf. [8]), that if

$$(1+\eta) \sum_{i\in E} \pi_i \lambda_i \int_0^\infty \tilde{R}(x) G_i(\mathrm{d}x) - (\eta - \kappa) \sum_{i\in E} \pi_i \lambda_i \mu_i > \sum_{i\in E} \pi_i \lambda_i \int_0^\infty \tilde{R}(x) G_i(\mathrm{d}x)$$

holds then there exists $w_{\mathcal{R}} > 0$ such that

$$\lim_{x\to\infty} \frac{1}{x} \log \psi(x) = -w_{\mathcal{R}}.$$

In this section we give an estimate for the ruin probability (Lundberg's estimate) using the exponential martingale method.

Theorem 5.1. *Let assumptions (H1) and (H2) hold. Set*

$$R := \sup\bigg\{r > 0;\ \inf_{t\ge 0}\bigg(r p_{\mathcal{R}}(t) - t \sup_{i\in E} \lambda_i \int_0^\infty (\mathrm{e}^{rx} - 1) G_i(\mathrm{d}x)\bigg) \ge 0\bigg\}.$$

Then

$$\psi(x) \le \mathrm{e}^{-Rx}.$$

Proof. Without loss of generality, we assume that $0 < R < \infty$. By Theorem 2.1, for any $\beta \in \mathbb{R}$,

$$Z^\beta_t := \exp\bigg\{\beta S_{\mathcal{R}}(t) - \int_0^t \int_0^\infty (\mathrm{e}^{\beta R_u(x)} - 1) G_{J(u)}(\mathrm{d}x) \lambda_{J(u)} \,\mathrm{d}u\bigg\}$$

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is a martingale under probability $\mathrm{P}(\cdot \mid J)$. Therefore, by Doob's stopping time theorem, we have, for any $\beta > 0$ and any $t \in [0, \infty)$, $\mathrm{E}(Z^{\beta}_{\tau_x \wedge t}) = 1$, which implies that $\mathrm{E}(Z^{\beta}_{\tau_x} \mathbf{1}_{\{\tau_x < \infty\}}) = 1$. Therefore,

$$\begin{aligned}
\psi(x) &= \mathrm{P}(S_{\mathcal{R}}(\tau_x) \geq x + p_{\mathcal{R}}(\tau_x),\ \tau_x < \infty) \\
&\leq \mathrm{e}^{-Rx}\,\mathrm{E}\biggl(Z^R_{\tau_x} \exp\biggl\{-Rp_{\mathcal{R}}(\tau_x) + \int_0^{\tau_x}\int_0^{\infty} (\exp\{RR_u(x)\} - 1) \\
&\qquad\qquad\qquad\qquad\qquad \times G_{J(u)}(\mathrm{d}x)\lambda_{J(u)}\,\mathrm{d}u\biggr\}\mathbf{1}_{\{\tau_x<\infty\}}\biggr) \\
&\leq \mathrm{e}^{-Rx}\,\mathrm{E}\biggl(Z^R_{\tau_x} \exp\biggl\{-\biggl(Rp_{\mathcal{R}}(\tau_x) - \tau_x \sup_{i\in E}\lambda_i \int_0^{\infty} (\mathrm{e}^{Rx} - 1)G_i(\mathrm{d}x)\biggr)\biggr\}\mathbf{1}_{\{\tau_x<\infty\}}\biggr) \\
&\leq \mathrm{e}^{-Rx}\,\mathrm{E}(Z^R_{\tau_x}\mathbf{1}_{\{\tau_x<\infty\}}) \\
&= \mathrm{e}^{-Rx}.
\end{aligned}$$

Remark 5.1. Assume that $\varrho x \leq R_s(x) \leq x$ for some constant $0 \leq \varrho \leq 1$. Then

$$\begin{aligned}
rp_{\mathcal{R}}(t) &- t\sup_{i\in E}\lambda_i \int_0^{\infty} (\mathrm{e}^{rx} - 1)G_i(\mathrm{d}x) \\
&\geq rt\biggl((\kappa - \eta + (1+\eta)\varrho)\sum_{i\in E}\pi_i\lambda_i\mu_i - \sup_{i\in E}\lambda_i\int_0^{\infty} r^{-1}(\mathrm{e}^{rx} - 1)G_i(\mathrm{d}x)\biggr).
\end{aligned}$$

Since $\sup_{i\in E}\lambda_i\int_0^{\infty} r^{-1}(\mathrm{e}^{rx} - 1)G_i(\mathrm{d}x) \longrightarrow \sup_{i\in E}\lambda_i\mu_i$ as $r \to 0$, we have $R > 0$ if $(\kappa - \eta + (1+\eta)\varrho)\sum_{i\in E}\pi_i\lambda_i\mu_i > \sup_{i\in E}\lambda_i\mu_i$.

Here we present a numerical example in which we calculate R in Theorem 5.1. We consider the proportional policy (see [9, p. 509] and [13]), i.e. $R_t(x) = b_t x$ for some $b_t \in [0, 1]$, and assume that $\lim_{t\to\infty} b_t = b_\infty \in [0, 1]$.

Example 5.1. Let J be a Markov chain with the two state space $E = \{1, 2\}$ with intensity matrix

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $\lambda_1 = 1$ and $\lambda_2 = 2$, and let G_1 and G_2 be the exponential distributions with parameters 1 and 2, respectively. Then the corresponding stationary distribution is $(\pi_1, \pi_2) = (\frac{1}{2}, \frac{1}{2})$. Let $\kappa = 4$ and $\eta = 5$ be the relative safety loading for the insurer and the reinsurer, respectively. Finally, we assume that $b_t \geq \frac{1}{2}$. Then, for any $0 < r < 1$,

$$rp_{\mathcal{R}}(t) - t\sup_{i=1,2}\lambda_i\int_0^{\infty}(\mathrm{e}^{rx} - 1)G_i(\mathrm{d}x) \geq \frac{r(1-2r)t}{1-r}.$$

Therefore, $R \geq \frac{1}{2}$, and corresponding ruin probability $\psi(x) \leq \mathrm{e}^{-x/2}$.

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