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PROPERTIES OF THE BEREZIN TRANSFORM OF BOUNDED FUNCTIONS

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We find the spectrum of the Berezin operator T on $L^\infty(B_n)$, then we show that if $f \in L^\infty(B_n)$ satisfies $Sf = rf$ for some r in the unit circle, where S is any convex combination of the iterations of T, then f is M-harmonic.

Finally we decompose the subspace of $L^\infty(B_n)$ where $\lim T^k f$ exists into the direct sum of two subspaces of $L^\infty(B_n)$.

1. INTRODUCTION

Let B_n be the unit ball of $\mathbf{C}^n$ and ν be Lebesgue measure on $\mathbf{C}^n$ normalised to $\nu(B_n) = 1$. For $f \in L^1(B_n, \nu)$, Tf (the Berezin transform of f) is by definition,

$$(Tf)(z) = \int_{B_n} f\big(\varphi_z(w)\big)\, d\nu(w)$$

where $\varphi_a \in \operatorname{Aut}(B_n)$ is the canonical automorphism given by

$$\varphi_a(z) = \frac{a - Pz - (1-|a|^2)^{1/2} Qz}{1 - \langle z, a\rangle}$$

where P is the projection into the space spanned by $a \in B_n$, $Q_z = z - Pz$. Equivalently we can write

$$(Tf)(z) = \int_{B_n} f(w) \frac{\big(1-|z|^2\big)^{n+1}}{\big|1 - \langle z, w\rangle\big|^{2n+2}}\, d\nu(w).$$

The invariant Laplacian $\widetilde{\Delta}$ is defined for $f \in C^2(B_n)$ by

$$(\widetilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0).$$

The M-harmonic functions in B_n are those for which $\widetilde{\Delta} f = 0$. τ is the measure on B_n defined by

$$d\tau(z) = \big(1-|z|^2\big)^{-n-1}\, d\nu(z)$$

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and satisfies

$$\int_{B_n} f\,d\tau = \int_{B_n} (f\circ\phi)\,d\tau$$

for every $f \in L^1(\tau)$ and $\phi \in \mathrm{Aut}(B_n)$.

We denote by $L^p_R(\tau)$ the subspace of $L^p(\tau)$ which consists of radial functions. That is, $f \in L^p_R(\tau)$ if and only if $f \in L^p(\tau)$ and $f(z) = f(|z|)$ for all $z \in B_n$. Throughout the paper, we follow notations in [**1**] and [**7**]. [**1**] is our main reference, and we've got motivations from it. One of the main theorems of [**1**] is that *if f $L^\infty(B_n)$ satisfies $Tf = f$, then f is M-harmonic.* Here we generalise that result and investigate further properties of the operator T on $L^\infty(B_n)$ and $L^1(\tau)$ by finding the spectrum of T, which gives us an essential connection to our investigation of the iteration of the Berezin transform. The theorem of Katznelson and Tzafriri [5] on the spectrum of contractions plays an important role.

We start from basic properties of T on $L^p(\tau)$. The operator T is not bounded in $L^1(\nu)$ [**1**, 2.2], but the next lemma shows that T has nice behaviour on $L^p(\tau)$ for $1 \leqslant p \leqslant \infty$.

LEMMA 1.1. *For $1 \leqslant p \leqslant \infty$, $1/p + 1/q = 1$ ($p = \infty$ means $q = 1$):*

(a) T is a linear contraction on $L^p(\tau)$.

(b) For $f \in L^p(\tau)$ and $g \in L^q(\tau)$

$$\int_{B_n} (Tf)g\,d\tau = \int_{B_n} f(Tg)\,d\tau.$$

PROOF: (a) Let $f \in L^1(\tau)$. Then

$$\begin{aligned}\int_{B_n} |Tf(z)|\,d\tau(z) &= \int_{B_n} \left| \int_{B_n} f(w)\,\frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}\,d\nu(w) \right| d\tau(z)\\ &\leqslant \int_{B_n} |f(w)| \int_{B_n} \frac{(1-|w|^2)^{n+1}}{|1-\langle w,z\rangle|^{2n+2}}\,d\nu(z)\,d\tau(w)\\ &= \int_{B_n} |f|\,d\tau. \end{aligned} \tag{1}$$

Let $f \in L^\infty(B_n)$. Then

$$\begin{aligned}\| Tf \|_\infty &\leqslant \|f\|_\infty \sup_{z\in B_n} \left| \int_{B_n} \frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}\,d\nu(w) \right|\\ &= \|f\|_\infty. \end{aligned} \tag{2}$$

By (1), (2) and the Riesz-Thorin interpolation theorem, we get (a).

(b)

$$\begin{aligned}\int_{B_n} (\,T|f|\,)|g|\,d\tau &\leqslant \|\,T|f|\,\|_p \|g\|_q\\ &\leqslant \|f\|_p\,\|g\|_q < \infty \quad \text{by } (a).\end{aligned}$$

Hence by using Fubini's theorem and simple calculation, we get the proof of (b). ∎

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2. The spectrum of T

From Lemma 1.1, the operator T on $L^\infty(B_n)$ is the adjoint of T on $L^1(\tau)$, and since $L^\infty(B_n) = L^1(\tau)^*$, the spectrum of T on $L^\infty(B_n)$ is the same as the spectrum of T on $L^1(B_n,\tau)$. We get the following theorem.

Theorem 2.1. *The spectrum of T on $L^\infty(B_n)$ is*

$$\left\{\frac{\Gamma(z+1)\Gamma(n+1-z)}{\Gamma(n+1)} \,\Big|\, 0 \leqslant \operatorname{Re} z \leqslant n\right\}.$$

Before proving Theorem 2.1, we need some preliminaries. Since Tf is radial for a radial f, by Lemma 1.1 T is a contraction on $L^1_R(\tau)$, which is a commutative Banach algebra under the convolution

$$(f * g)(z) = \int_{B_n} f(\varphi_z(w))g(w)\,d\tau(w)$$

for $f, g \in L^1_R(\tau)$. Hence if $f \in L^1_R(\tau)$, we can write $Tf = f * h$ where

$$h(z) = (1-|z|^2)^{n+1} \in L^1_R(\tau).$$

From this, we get the following Lemma.

Lemma 2.2. *The spectrum of T on $L^1_R(\tau)$ is*

$$\left\{\frac{\Gamma(z+1)\Gamma(n+1-z)}{\Gamma(n+1)} \,\Big|\, 0 \leqslant \operatorname{Re} z \leqslant n\right\}.$$

Proof: For $f \in L^1_R(\tau)$, the Gelfand(or spherical) transform of f is defined by (see [**3**, **4**])

$$\widehat{f}(\alpha) = \int_{B_n} f(z)\, g_\alpha(z)\, d\tau(z) \tag{1}$$

where g_α is a spherical function defined by [**7**, 4.2.2]

$$g_\alpha(z) = \int_S P^\alpha(z,\xi)\, d\sigma(\xi).$$

$\widehat{f}(\alpha)$ exists if α lies in the vertical strip

$$\Sigma_\infty = \{0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$$

which is the maximal ideal space of $L^1_R(\tau)$, and satisfies

$$(f*g)\widehat{\ }(\alpha) = \widehat{f}(\alpha)\widehat{g}(\alpha), \ \|\widehat{f}\|_\infty \leqslant \|f\|_1.$$

(Note that g_α is bounded if and only if $\alpha \in \Sigma_\infty$ by [**7**, 1.4.10].) Since $Tf = f * h$ where $h(z) = \left(1-|z|^2\right)^{n+1}$, the spectrum of T on $L^1_R(\tau)$ is the same as the spectrum of h in the commutative Banach algebra $L^1_R(\tau)$, which is $\left\{\widehat{h}(\alpha) \mid \alpha \in \Sigma_\infty\right\}$. From (1),

$$\widehat{h}(\alpha) = \int_{B_n} h g_\alpha \, d\tau = \int_{B_n} g_\alpha \, d\nu.$$

By [**1**, Proposition 3.4 and 3.5]

$$\int_{B_n} g_\alpha \, d\nu = \frac{\Gamma(1+n\alpha)\Gamma(n+1-n\alpha)}{\Gamma(n+1)}.$$

This completes the proof of the lemma. ∎

PROOF OF THEOREM 2.1: Since the operator T on $L^\infty_R(B_n)$ is the adjoint of T on $L^1_R(\tau)$, the spectrum of T on $L^\infty_R(B_n)$ is

$$\left\{\frac{\Gamma(1+z)\Gamma(n+1-z)}{\Gamma(n+1)} \;\middle|\; 0 \leqslant \operatorname{Re} z \leqslant n\right\}. \tag{1}$$

Now let λ be in the spectrum of T on $L^\infty(B_n)$. Then there exists a sequence $\{f_k\}$ in $L^\infty(B_n)$, $\|f_k\|_\infty = 1$, for which

$$\lim_{k\to\infty} \|Tf_k - \lambda f_k\|_\infty = 0.$$

Let $\phi_k \in \operatorname{Aut}(B_n)$ satisfy $\left\|R(f_k \circ \phi_k)\right\|_\infty = 1$ where Rf is the radialisation [**7**, 4.2.1] of f. Since T and R are contractions on $L^\infty(B_n)$,

$$\begin{aligned}\left\|T\big(R(f_k\circ\phi_k)\big) - \lambda R(f_k\circ\phi_k)\right\|_\infty &= \left\|R\big(T(f_k\circ\phi_k)\big) - R(\lambda f_k\circ\phi_k)\right\|_\infty \\ &\leqslant \left\|T(f_k\circ\phi_k) - \lambda f_k\circ\phi_k\right\|_\infty \\ &= \left\|(Tf_k)\circ\phi_k - \lambda f_k\circ\phi_k\right\|_\infty \\ &\quad \text{(by Proposition 2.3 of [1])} \\ &= \|Tf_k - \lambda f_k\|_\infty \to 0 \quad \text{as} \quad k\to\infty.\end{aligned}$$

Hence λ is in the spectrum of T on $L^\infty_R(B_n)$.
Thus from (1), we complete the proof. ∎

The next corollary plays an important role in this paper.

COROLLARY 2.3. *Let $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. Then*

$$\lim_{k\to\infty}\left\|T^k(f-Tf)\right\|_1 = 0 \quad \textit{and} \quad \lim_{k\to\infty}\left\|T^k(g-Tg)\right\|_\infty = 0.$$

PROOF: By [**1**, Proposition 3.7(b)], $\widehat{h}(\alpha) < 1$ when $\alpha \in \Sigma_\infty \setminus \{0,1\}$ and $\widehat{h}(0) = \widehat{h}(1) = 1$. Hence the spectrum of T on $L^1(\tau)$ (or $L^\infty(B_n)$) intersects the unit circle only at one point $z = 1$. Hence by [**5**, Theorem 1],

$$\lim_{k\to\infty}\left\|T^k(I-T)\right\| = 0 \text{ on } L^1(\tau) \ \big(\text{or } L^\infty(B_n)\big).$$

This completes the proof. ∎

3. The iteration of T

Here we use the results of the previous section to get the behaviour of functions in $L^\infty(B_n)$ or its predual $L^1(\tau)$ under the infinite iteration of T. First we generalise one of the Main Theorems of [**1**].

Lemma 3.1. *Let $f \in L^1_R(\tau)$. Then*

$$\lim_{k\to\infty}\int_{B_n} |T^k f|\,d\tau = 0 \quad \textit{if and only if} \quad \int_{B_n} f\,d\tau = 0.$$

Proof: Since

$$\int_{B_n} T^k f\,d\tau = \int_{B_n} f\,d\tau \quad \text{for every } k \geqslant 0,$$

$$\lim_{k\to\infty}\int_{B_n} |T^k f|\,d\tau = 0 \quad \text{implies} \quad \int_{B_n} f\,d\tau = 0.$$

On the other hand, if we define

$$LO^1_R = \left\{ f \in L^1_R(\tau) \;\middle|\; \int_{B_n} f\,d\tau = 0 \right\},$$

then

$$(I - T)L^1_R(\tau) \subset LO^1_R.$$

Now let $\ell \in L^\infty_R(B_n)$ satisfy

$$\int_{B_n} (f - Tf)\ell\,d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).$$

Then by Lemma 1.1

$$\int_{B_n} f(\ell - T\ell)\,d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).$$

Hence $T\ell = \ell$, which means ℓ is radial M-harmonic by [**1**]. Thus ℓ is a constant. Hence we get

$$\int_{B_n} g \cdot \ell\,d\tau = 0 \quad \text{for every } g \in LO^1_R.$$

By the Hahn-Banach theorem, this means $(I - T)L^1_R(\tau)$ is dense in LO^1_R. Hence from Corollary 2.3,

$$\lim_{k\to\infty}\int_{B_n} |T^k g|\,d\tau = 0 \quad \text{for every } g \in LO^1_R.$$

∎

Theorem 3.2. *Let $0 \leqslant \alpha_k \leqslant 1$ satisfy $\sum_{k=1}^{N} \alpha_k = 1$ and let m_k be positive numbers for $k = 1, 2, \cdots, N$. If $f \in L^\infty(B_n)$ satisfies*

$$\left(\sum_{k=1}^{N} \alpha_k T^{m_k}\right) f = rf \quad \textit{for some } r \textit{ with } |r| = 1,$$

then f is M-harmonic.

PROOF: Let

$$S = \sum_{k=1}^{N} \alpha_k T^{m_k}$$

and

$$X = \{f \in L^\infty(B_n) \mid Sf = rf\}.$$

Fix j which satisfies $0 < \alpha_j < 1$, and define U on $L^\infty(B_n)$ by

$$U = \frac{1}{1-\alpha_j} \sum_{k \neq j} \alpha_k T^{m_k}.$$

If $f \in X$, then

$$ST^{m_j} f = T^{m_j} Sf = rT^{m_j} f.$$

Hence $T^{m_j} f \in X$ and in the same way $Uf \in X$. Thus by Lemma 1.1, T^{m_j} and U are contractions on the Banach space X. And on $L^\infty(B_n)$

(1) $$S = \alpha_j T^{m_j} + (1-\alpha_j)U.$$

Let P be the operator on X defined by

(2) $$P = \alpha_j T^{m_j} - \alpha_j r I.$$

Now let q be an extreme point of A^*, the closed unit ball of X^*. Then from (1)

(3) $$rq = \alpha_j (T^{m_j})^* q + (1-\alpha_j)(U^* q).$$

Since $(T^{m_j})^*, U^*$ are contractions on X^*, (3) forces

$$q = \frac{(T^{m_j})^* q}{r} = \frac{U^* q}{r}.$$

Therefore on X^*,

$$P^* q = \alpha_j \ (T^{m_j})^* q - \alpha_j r q = 0.$$

But by the Krein-Milman theorem, A^* is the closed convex hull of the set of its extreme points. It follows that $P^* \equiv 0$ on A^*. That is, $P \equiv 0$. Hence $T^{m_j} = rI$ on X. ∎

Now pick $\ell \in L^\infty_R(B_n) \cap X$ and $g \in L^1_R(\tau)$ with

$$\int_{B_n} g d\tau = 0.$$

Then

$$\lim_{k\to\infty} \left| \int_{B_n} T^{m_j k} g \cdot \ell \, d\tau \right| \leqslant \|\ell\|_\infty \lim_{k\to\infty} \int_{B_n} \left| T^{m_j k} g \right| d\tau = 0 \quad \text{by Lemma 3.2.}$$

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But for all $k \geqslant 0$,

$$\int_{B_n} T^{m_j k} g \cdot \ell \, d\tau = \int_{B_n} g \cdot T^{m_j k} \ell \, d\tau$$
$$= r^k \int_{B_n} g \cdot \ell \, d\tau.$$

Hence

$$\int_{B_n} g \cdot \ell \, d\tau = 0,$$

which implies that ℓ is a constant. For an arbitrary $f \in X$, consider the radialisation of $R(f \circ \varphi_z)$.

$$T^{m_j}\big(R(f \circ \varphi_z)\big) = R\big(T^{m_j}(f \circ \varphi_z)\big) = R\big(T^{m_j} f \circ \varphi_z\big)$$
$$\text{(by Proposition 2.3 of [1])}$$
$$= rR(f \circ \varphi_z).$$

Hence

$$R(f \circ \varphi_z) \in X \cap L_R^\infty(B_n),$$

which means $R(f \circ \varphi_z)$ is a constant. Hence for any $w \in B_n$

$$R(f \circ \varphi_z)(w) = R(f \circ \varphi_z)(0) = f\big(\varphi_z(0)\big) = f(z).$$

By [7, 4.2.4], f is M-harmonic. This proves the theorem. ∎

Corollary 3.3. *If $f \in L^1(\nu)$ satisfies $Tf = rf$ for some r with $|r| = 1$ and $R(f \circ \phi) \in L^\infty(B_n)$ for every $\phi \in \mathrm{Aut}(B_n)$, then f is M-harmonic.*

Proof:

$$T\big(R(f \circ \phi)\big) = R\big(T(f \circ \phi)\big) = R\big(Tf \circ \phi\big) = rR(f \circ \phi).$$

Thus $R(f \circ \phi)$ is a constant by Theorem 3.2. Then by the same argument as in the proof of Theorem 3.2, we can see that f is M-harmonic. ∎

The next two propositions are about the iteration of T on $L^1(\tau)$. We need the following lemma first.

Lemma 3.4. *If $f \in L^1(\tau)$ satisfies $Tf = rf$ for some $|r| = 1$, then $f \equiv 0$.*

Proof:

$$|f(z)| = |Tf(z)| \leqslant \int_{B_n} |f(w)| \frac{\big(1 - |z|^2\big)^{n+1}}{\big|1 - \langle z, w\rangle\big|^{2n+2}} \, d\mu(w)$$
$$\leqslant \sup_{z \in B_n} \left(\frac{\big(1 - |z|^2\big)\big(1 - |w|^2\big)}{\big|1 - \langle z, w\rangle\big|^2} \right)^{n+1} \int_{B_n} |f| \, d\tau$$
$$= \int_{B_n} |f| \, d\tau \quad < \quad \infty.$$

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So f is bounded, thus f is M-harmonic by Theorem 3.2. But the constant zero is the only M-harmonic function which belongs to $L^1(\tau)$. ∎

PROPOSITION 3.5. *For $f \in L^1(\tau)$, if $\{T^k f\}$ has a subsequence that converges weakly, then*

$$\lim_{k\to\infty} \|T^k f\|_1 = 0.$$

PROOF: Let $\{T^{m_k} f\}$ be a subsequence that converges weakly to some $g \in L^1(\tau)$. Then for any $\ell \in L^\infty(B_n)$ we get

$$\begin{aligned}\int_{B_n} (g - Tg)\cdot \ell\, d\tau = &\left(\int_{B_n} g\cdot \ell\, d\tau - \int_{B_n} (T^{m_k} f)\cdot \ell\, d\tau\right)\\ &+\left(\int_{B_n} (T^{m_k} f)\cdot \ell\, d\tau - \int_{B_n} (T^{m_k+1} f)\cdot \ell\, d\tau\right)\\ &+\left(\int_{B_n} (T^{m_k+1} f)\cdot \ell\, d\tau - \int_{B_n} (Tg)\cdot \ell\, d\tau\right).\end{aligned}$$

As $k \to \infty$ the first and the third terms of the right hand side converge to zero since $T^{m_k} \to g$ weakly and the second term converges to zero by Corollary 2.3. Hence $Tg = g$, which means $g \equiv 0$ by Lemma 3.4. Thus by Masur's Theorem, for any $\varepsilon > 0$ there exists an operator

$$S = \sum_{j=1}^{N} \alpha_j T^{m_{k_j}} \quad \left(0 \leqslant \alpha_j \leqslant 1, \sum \alpha_j = 1\right)$$

on $L^1(\tau)$ such that $\|Sf\|_1 < \varepsilon$. For $k \geqslant 0$,

$$\|T^k f\|_1 \leqslant \|T^k S f\|_1 + \|T^k f - T^k S f\|_1$$

where

$$\|T^k S f\|_1 \leqslant \|Sf\|_1 < \varepsilon$$

and

$$\lim_{k\to\infty} \|T^k f - T^k S f\|_1 = 0$$

by Corollary 2.3. Therefore,

$$\lim_{k\to\infty} \|T^k f\|_1 = 0$$

and this completes the proof. ∎

However $\lim \int_{B_n} |T^k f|\, d\tau$ exists for all $f \in L^1(\tau)$ since T is a contraction on $L^1(\tau)$. When f is radial we get a similar result to [2].

PROPOSITION 3.6. *If $f \in L^1_R(\tau)$, then*

$$\lim_{k\to\infty} \|T^k f\|_1 = \left|\int_{B_n} f\, d\tau\right|.$$

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PROOF: Let

$$A = \{\ell \in L_R^\infty(B_n) \mid \|\ell\|_\infty \leqslant 1\}$$

and $E_k = T^k A$,

$$E = \bigcap_{k=1}^{\infty} E_k.$$

Then for $f \in L_R^1(\tau)$,

$$\begin{aligned}\int_{B_n} |T^k f|\, d\tau &= \sup\left\{\left|\int_{B_n} (T^k f)\cdot \ell\, d\tau\right| \,\middle|\, \ell \in A\right\}\\ &= \sup\left\{\left|\int_{B_n} f\cdot (T^k \ell)\, d\tau\right| \,\middle|\, \ell \in A\right\}.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \|T^k f\|_1 \geqslant \sup\left\{\left|\int_{B_n} f\cdot h d\tau\right| \,\middle|\, h \in E\right\}. \tag{1}$$

On the other hand, for any $\varepsilon > 0$ and $k \geqslant 1$ there exists $h_k \in A$ with

$$\begin{aligned}\|T^k f\|_1 &\leqslant \left|\int_{B_n} \left(T^k f\right)\cdot h_k\, d\tau\right| + \varepsilon\\ &\leqslant \left|\int_{B_n} f\cdot \left(T^k h_k\right) d\tau\right| + \varepsilon.\end{aligned}$$

Since E_k is weak $*$ compact and $E_k \downarrow E$, E is weak $*$ compact. Thus if g is a weak $*$ limit of a subsequence $\{T^{k_j} h_{k_j}\}$ of $\{T^k h_k\}$, then $g \in E$ and

$$\begin{aligned}\left|\int_{B_n} f\cdot g\, d\tau\right| &= \lim_{j\to\infty}\left|\int_{B_n} f\left(T^{k_j} h_{k_j}\right) d\tau\right|\\ &\geqslant \lim_{j\to\infty} \|T^{k_j} f\|_1 - \varepsilon.\end{aligned}$$

Hence

$$\lim_{k\to\infty} \|T^k f\|_1 \leqslant \sup\left\{\left|\int_{B_n} f\cdot h\, d\tau\right| \,\middle|\, h \in E\right\}. \tag{2}$$

From (1), (2) we get

$$\lim_{k\to\infty} \|T^k f\|_1 = \sup\left\{\left|\int_{B_n} f\cdot h\, d\tau\right| \,\middle|\, h \in E\right\}. \tag{3}$$

From (3) and Lemma 3.1, if $f \in L_R^1(\tau)$ then

$$\int_{B_n} f\, d\tau = 0 \quad \text{if and only if} \quad \int_{B_n} f\cdot h\, d\tau = 0$$

for every $h \in E$. Hence

$$E = \{c \in \mathbb{C} \mid |c| \leqslant 1\}$$

and we can rewrite (3) as

$$\begin{aligned}\lim_{k\to\infty} \|T^k f\|_1 &= \sup\left\{\left|\int_{B_n} cf\, d\tau\right| \;\middle|\; |c| \leqslant 1\right\}\\ &= \left|\int_{B_n} f\, d\tau\right|.\end{aligned}$$

∎

Using Proposition 3.5 and 3.6, it follows that there exists $f \in L^\infty(B_n)$ for which $\lim T^k f$ does not exist even pointwise. To see this, assume that $\lim T^k \ell$ exists for every $\ell \in L^\infty(B_n)$. Then for any $g \in L^1(\tau)$,

$$\lim_{k\to\infty} \int_{B_n} (T^k g)\ell\, d\tau = \lim_{k\to\infty} \int_{B_n} g(T^k \ell)\, d\tau$$

exists. This means $\{T^k g\}$ converges weakly since $L^1(\tau)$ is weak complete, which implies that, by Proposition 3.5,

$$\lim_{k\to\infty} \|T^k g\|_1 = 0$$

for any $g \in L^1(\tau)$, which is not true by Proposition 3.6.

The next theorem is about the subspace of $L^\infty(B_n)$ for which $\lim T^k f$ exists with the L^∞-norm, which is the unit ball analogue of [**6**, Theorem 2.7] in the polydisc.

THEOREM 3.7. *Let X be the subspace of $L^\infty(B_n)$ defined by*

$$X = \{f \in L^\infty(B_n) \mid \lim_{k\to\infty} T^k f \text{ exists}\}.$$

Then

$$X = H \oplus N$$

where

$$\begin{aligned} H &= \{f \in L^\infty(B_n) \mid f \text{ is } M\text{-harmonic}\} \quad \text{and}\\ N &= \overline{(I-T)L^\infty(B_n)}.\end{aligned}$$

PROOF: By Corollary 2.3, we get

$$\lim_{k\to\infty} \|T^k g\|_\infty = 0 \quad \text{for all } g \in N$$

and for $h \in H$, $Th = h$. Hence

$$H \cap N = \{0\}.$$

Let P be the operator on X defined by

$$(Pf)(z) = \lim_{k\to\infty} (T^k f)(z) \quad \text{for } f \in X.$$

Since T and P commute on X by the Dominated Convergence Theorem, if $f \in X$ then

$$T(Pf) = P(Tf) = \lim_{k\to\infty} T^{k+1} f = Pf.$$

Hence by Theorem 3.2, $Pf \in H$. Since $f = Pf + (f - Pf)$, it remains to show that $f - Pf \in N$ when $f \in X$. Let $d \in L^\infty(B_n)^*$ satisfy $d(g - Tg) = 0$ for all $g \in L^\infty(B_n)$, then we get $T^* d = d$. Hence

$$d(f - Pf) = d\big(T^k(f - Pf)\big) \quad \text{for all } k \geqslant 0 \qquad (1)$$

But

$$\lim_{k\to\infty} \left\| T^k(f - Pf) \right\|_\infty = 0.$$

By taking the limit as $k \to \infty$ in (1), we get $d(f - Pf) = 0$. Hence by the Hahn-Banach theorem, $f - Pf \in N$. This completes the proof of Theorem 3.7. ∎

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