ANZIAM J. 61(2019), 382–397
doi:10.1017/S1446181119000142

PRICING HOLDER-EXTENDABLE CALL OPTIONS WITH MEAN-REVERTING STOCHASTIC VOLATILITY

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(Received 9 May, 2017; accepted 8 August, 2019; first published online 14 October 2019)

Abstract

Options with extendable features have many applications in finance and these provide the motivation for this study. The pricing of extendable options when the underlying asset follows a geometric Brownian motion with constant volatility has appeared in the literature. In this paper, we consider holder-extendable call options when the underlying asset follows a mean-reverting stochastic volatility. The option price is expressed in integral forms which have known closed-form characteristic functions. We price these options using a fast Fourier transform, a finite difference method and Monte Carlo simulation, and we determine the efficiency and accuracy of the Fourier method in pricing holder-extendable call options for Heston parameters calibrated from the subprime crisis. We show that the fast Fourier transform reduces the computational time required to produce a range of holder-extendable call option prices by at least an order of magnitude. Numerical results also demonstrate that when the Heston correlation is negative, the Black–Scholes model under-prices in-the-money and over-prices out-of-the-money holder-extendable call options compared with the Heston model, which is analogous to the behaviour for vanilla calls.

2010 *Mathematics subject classification*: 91G60.

Keywords and phrases: extendable options, Heston model, fast Fourier transform, finite difference method, Monte Carlo simulation.

1. Introduction

As Longstaff [23] has described, "Any financial contract that could involve a rescheduling of payments, a renegotiation of terms, an early call or exercise provision

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or some similar type of flexibility over the timing of cash flows could be viewed generally as including an option with an extendable maturity". One such option is the extendable option which can either be a holder-extendable option or a writer-extendable option, in addition to being classified as a call or a put. At the initial maturity time, the holder-extendable option can be extended to another maturity time for an additional premium, whereas the writer-extendable option can be extended to another maturity time if the option is out-of-the-money. The pricing for the writer-extendable option is more straightforward than the holder-extendable option; hence it is not included in this study.

An extendable option is an example of a dual-expiry exotic option [4], and its framework has been used in other financial contracts such as extendable warrants [13, 16] and extendable bonds [26]. In addition, Longstaff [23] discussed other applications of extendable options to real-estate and shared-equity mortgages. Hauser and Lauterbach [13] suggested that investors are in favour of extendable call warrants because they produce lower absolute pricing errors than the standard call warrants. Extendable options are also used for commodity markets, for instance, price extendable oil options as discussed by Dias and Rocha [9]. Neftci and Santos [26] noted that extendable bonds have stabilizing properties and "the embedded options work as a cushion and replicate the trading gains from hedging long-term bonds with interest rate derivatives". Recently, Koussis et al. [22] considered the problem of product development, which inherently contained extendable features, within the real-option framework that generalized the results of Longstaff [23]. Indeed, the application of extendable features within the real-option framework deserves further investigation.

The earliest examples of financial derivatives that have this feature appear in the work of Ananthanarayanan and Schwartz [1] and Brennan and Schwartz [3], which displayed theoretical pricing for retractable and extendable bonds. Longstaff [23] discussed extendable options extensively and provided a closed-form solution for extendable options under the Black–Scholes model [2]. In recent years, Chung and Johnson [6] extended the work of Longstaff [23] to a general case where the holder or the writer can extend the option more than once, and they derived a closed-form solution for n-extensions. While these studies are within the Black–Scholes framework [2], Ibrahim et al. [19] derived an analytical pricing formula for holder-extendable call options in the Schöbel–Zhu model. On the other hand, Gukhal [12] provided closed-form solutions for the extendable option with a jump feature based on the Merton jump-diffusion model [24] and demonstrated that a compound option is a special case of the extendable option. Additionally, Peng and Peng [27] extended the study of Chung and Johnson [6] by deriving a value for an n-time extendable option with jumps, where the underlying asset price followed a fractional process, while Shevchenko [31] derived the price for an extendable option in the case of an underlying asset that follows a geometric Brownian motion with time-dependence and volatility.

Given the underlying asset price S and the initial strike price K_1, at a given initial maturity time T_1, the call option can be extended to time T_2 for a new strike price K_2 by paying an additional premium A. The payoff of the holder-extendable call option

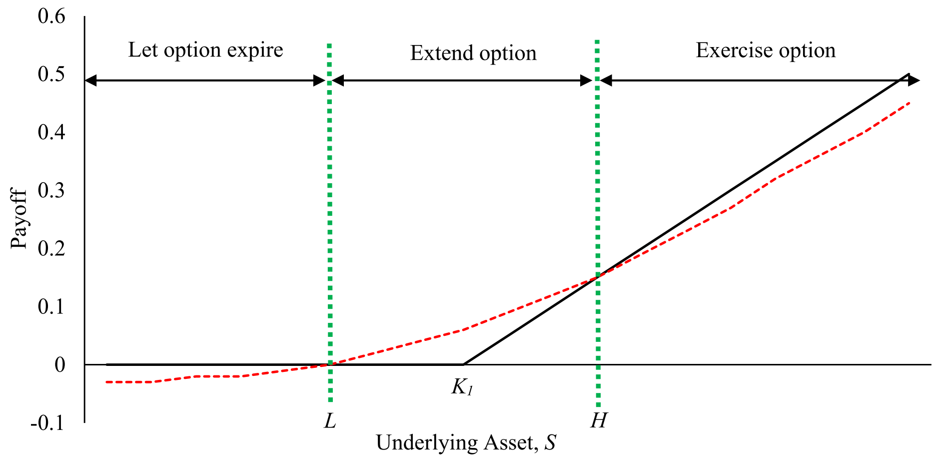

FIGURE 1. Holder-extendable call payoff at time T_1 when H is finite and unique.

can be represented by

$$\max[0,\ S_{T_1} - K_1,\ C(S_{T_1}, K_2, T_2 - T_1) - A], \tag{1.1}$$

or, similarly, by

$$\max[\max(S_{T_1} - K_1, 0), \max(C(S_{T_1}, K_2, T_2 - T_1) - A, 0)],$$

which indicates that, at time T_1, the holder has to compare two risky payoffs and choose the largest payoff, where $C(S, K, t)$ is the price of a vanilla call option.

The holder of a holder-extendable call option has the right, but not the obligation, to do the following: let the option expire worthless; exercise the option; or extend the option's maturity time. The choice region where the option is either exercised or extended is determined by solving for the critical asset values s^* and s^{**}, which may be obtained from the equations

$$C(s^*, K_2, T_2 - T_1) - A = 0, \tag{1.2}$$

$$s^{**} - K_1 = C(s^{**}, K_2, T_2 - T_1) - A. \tag{1.3}$$

Equation (1.2) has a unique solution $s^* = L$ that is bounded by the relation $A \leq L \leq A + K_2 e^{-r(T_2 - T_1)}$. If $L \geq K_1$, then the call option is never extended, and hence the holder receives $C(S_{T_1}, K_1, T_1)$. If $L < K_1$, then equation (1.3) has a finite unique solution $s^{**} = H$ when $A > K_1 - K_2 e^{-r(T_2 - T_1)}$, where the call option is extended when $L < S_{T_1} < H$, exercised when $S_{T_1} \geq H$ and is worthless when $S_{T_1} \leq L$. This is the usual case for a holder-extendable call option and is depicted in Figure 1, where the solid line represents the payoff of $\max(S_{T_1} - K_1, 0)$ and the dashed line represents the payoff of $C(S_{T_1}, K_2, T_2 - T_1) - A$. However, when $A \leq K_1 - K_2 e^{-r(T_2 - T_1)}$, equation (1.3) has no solution if $L < K_1$, where the call option is extended when $L < S_{T_1}$ and is worthless

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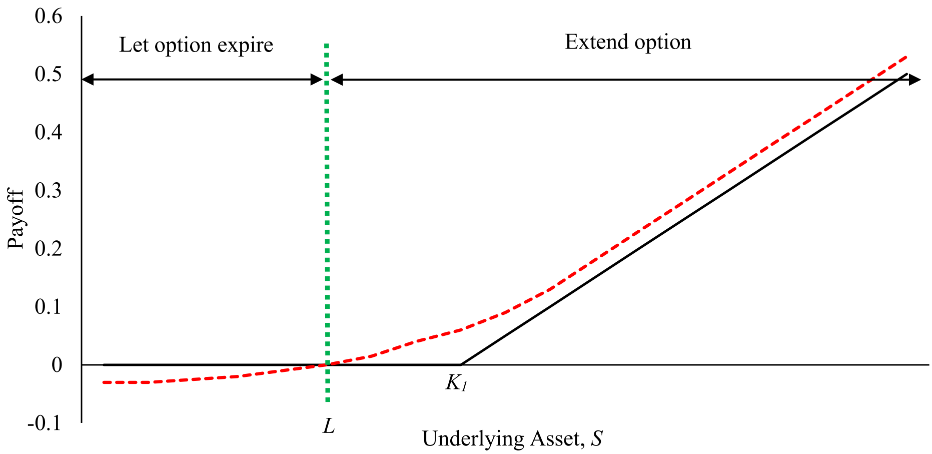

FIGURE 2. Holder-extendable call payoff at time T_1 when H does not exist.

when $L \geq S_{T_1}$. An analysis of these conditions was also given by Gukhal [12] and Shevchenko [31]; Figure 2 illustrates this.

The fair price of an option whose price depends on its underlying asset price can be determined under the risk-neutral probability measure $\mathbb{Q}$, where the expected return on the risky asset is the same as that on a risk-free investment in cash. Therefore, at maturity time T, the price of a holder-extendable call option EC is computed as the discounted risk-neutral conditional expectation of its payoff (1.1) at a risk-free rate r defined as

$$EC = e^{-r(T_1-t)}\mathbb{E}^{\mathbb{Q}} \max[0,\ S_{T_1} - K_1,\ C(S_{T_1}, K_2, T_2 - T_1) - A], \tag{1.4}$$

where C is calculated in either the Black–Scholes framework [2], C_{BS}, or the Heston framework [14], C_H. (Note that the Heston vanilla call has a semianalytic solution which is used in the Monte Carlo simulation (MCS) under the Heston framework.) Hence, the analytical pricing solution for a holder-extendable call option in the Black–Scholes framework [2] is given as follows [23].

THEOREM 1.1. *Given underlying asset price S, initial maturity date T_1 and strike price K_1, the price of a holder-extendable call option whose maturity time may be extended to T_2 for an additional payment A with a new strike price K_2, is given by*

$$\begin{aligned} EC_{BS}&(S_t, K_1, T_1, K_2, T_2, A)\\ &= C_BS(S_t, K_1, T_1)\\ &\quad + [S_t M^{(2)}(a_1, b_1, -\infty, c_1; \rho) - K_2 e^{-r(T_2-t)} M^{(2)}(a_2, b_2, -\infty, c_2; \rho)]\\ &\quad - [S_t M(a_1, d_1) - K_1 e^{-r(T_1-t)} M(a_2, d_2)] - A e^{-r(T_1-t)} M(a_2, b_2), \end{aligned} \tag{1.5}$$

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where

$$a_1 = \frac{\ln(S_t/H) + (r + \sigma^2/2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \quad a_2 = a_1 - \sigma\sqrt{T_1 - t},$$
$$b_1 = \frac{\ln(S_t/L) + (r + \sigma^2/2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \quad b_2 = b_1 - \sigma\sqrt{T_1 - t},$$
$$c_1 = \frac{\ln(S_t/K_2) + (r + \sigma^2/2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \quad c_2 = c_1 - \sigma\sqrt{T_2 - t},$$
$$d_1 = \frac{\ln(S_t/K_1) + (r + \sigma^2/2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \quad d_2 = d_1 - \sigma\sqrt{T_1 - t},$$
$$\rho = \sqrt{\frac{T_1 - t}{T_2 - t}}$$

and $M(a, b)$ is the cumulative probability of the standard normal density in the interval $[a, b]$, while $M^{(2)}(a, b, c, d; \rho)$ is the cumulative probability of the standard bivariate normal density with correlation ρ for the region $[a, b] \times [c, d]$.

The price of a holder-extendable call option (1.5) in Theorem 1.1 can be represented in terms of fewer univariate normal distributions by using the identities

$$M^{(2)}(a, b, c, d; \rho) = N^{(2)}(b, d; \rho) - N^{(2)}(a, d; \rho) - N^{(2)}(b, c; \rho) + N^{(2)}(a, c; \rho),$$
$$M^{(2)}(a, b, -\infty, d; \rho) = N^{(2)}(b, d; \rho) - N^{(2)}(a, d; \rho),$$
$$M(a, b) = N(b) - N(a),$$

where $N(\cdot)$ is the standard normal distribution and $N^{(2)}(\cdot, \cdot; \rho)$ is the standard bivariate normal distribution with correlation ρ. This yields the following corollary.

Corollary 1.2. *The price of a holder-extendible call option with maturity T_1 and strike price K_1, whose maturity may be extended to T_2 with a new strike price K_2 by making an additional payment A, is given by*

$$\begin{aligned} &EC_{BS}(S_t, K_1, T_1, K_2, T_2, A) \\ &\quad = C_{BS}(S_t, K_1, T_1) + S_t N^{(2)}(b_1, c_1; \rho) - K_2 e^{-r(T_2 - t)} N^{(2)}(b_2, c_2; \rho) \\ &\qquad - [S_t N^{(2)}(a_1, c_1; \rho) - K_2 e^{-r(T_2 - t)} N^{(2)}(a_2, c_2; \rho)] - A e^{-r(T_1 - t)}[N(b_2) - N(a_2)], \end{aligned}$$

where a_1, a_2, b_1, b_2, c_1, and c_2 are as defined in Theorem 1.1.

Equation (1.5) represents the price of a vanilla call option with strike price K_1 plus the nonnegative value of the extension privilege. Hence, the holder-extendable call option is worth at least as much as its corresponding vanilla call option, and the holder-extendable call option is worthless when letting $S = 0$. Moreover, letting $L = 0$ and $H = \infty$ reduces the holder-extendable call option to a vanilla call option that is always extended and which yields $C(S_t, K_2, T_2)$. Additionally, in equation (1.2), imposing $A = 0$ yields $L = 0$, and taking $H = K_1$ reduces the holder-extendable call option pricing formula (1.5) to a writer-extendable call option pricing formula.

In this study, we consider the problem of pricing holder-extendable call options under the Heston model [14], which is characterized by the dynamics

$$dS_t = rS_t\,dt + \sqrt{v_t}S_t\,dW_{t,1},$$
$$dv_t = \kappa(\theta - v_t)\,dt + \sigma_0\sqrt{v_t}\,dW_{t,2},$$

where $\langle dW_{t,1}, dW_{t,2}\rangle = \rho\,dt$, $\kappa \geq 0$ is the speed of mean reversion, $\theta \geq 0$ is the mean level of variance, $\sigma_0 > 0$ is the volatility of the volatility and v_t follows a mean-reverting square-root process [8]. It is convenient to write the above equations in terms of two independent Brownian motions $(\tilde{W}_{t,1}, \tilde{W}_{t,2})$ such that

$$dS_t = rS_t dt + \sqrt{v_t}S_t\Big(\sqrt{1-\rho^2}d\tilde{W}_{t,1} + \rho d\tilde{W}_{t,2}\Big)$$
$$dv_t = \kappa(\theta - v_t)\,dt + \sigma_0\sqrt{v_t}d\tilde{W}_{t,2},$$

or

$$d\begin{pmatrix} S_t \\ v_t \end{pmatrix} = \begin{pmatrix} rS_t \\ \kappa(\theta - v_t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{v_t(1-\rho^2)}S_t & \rho\sqrt{v_t}S_t \\ 0 & \sigma_0\sqrt{v_t} \end{pmatrix}\begin{pmatrix} d\tilde{W}_{t,1} \\ d\tilde{W}_{t,2} \end{pmatrix},$$

where $E[d\tilde{W}_{t,1}d\tilde{W}_{t,2}] = 0$, that is, $d\tilde{W}_{t,1}$ is uncorrelated with $d\tilde{W}_{t,2}$. In order to compare this with the Black–Scholes formulation, the expected variance (given an initial variance v_0) over the life of an option of maturity T is required [14]. Under the Heston dynamics, this was given by Rouah [29] as

$$\overline{\upsilon}(T) = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^T v_t\,dt\Big|v_0\Big] = (v_0 - \theta)\Big(\frac{1 - e^{-\kappa T}}{\kappa}\Big) + \theta T. \tag{1.6}$$

According to Sophocleous et al. [32], the complexity of the model increases when stochastic volatility is incorporated. Hence, a numerical technique is used to price options with this additional feature. Numerical techniques may include MCS and finite difference methods (FDMs) [7, 10]. The fast Fourier transform (FFT) technique in option pricing was introduced by Carr and Madan [5] and has since gained popularity in option pricing because its algorithm offers computational efficiency by employing the characteristic function of the log price, which is known in closed-form for many models discussed in the literature [17, 28, 33–35]. Ibrahim et al. [20] applied the FFT technique to price the holder-extendable call options in the Black–Scholes environment [2], while, in this study, we aim to apply the FFT technique to price the holder-extendable call options under the Heston model [14].

The remainder of the paper is organized as follows. Section 2 provides the characteristic functions and numerical solutions of extendable options using the FFT technique. The extendable option is expressed as expectations of indicator functions, and the inverse Fourier transform (IFT) is obtained for one- and two-dimensional FFTs. We employ known closed-form representations of characteristic functions in the implementation of the FFT. The numerical results in Section 3 document the effectiveness and efficiency of the proposed model against two benchmarks: MCS and FDMs. Section 4 concludes the paper.

https://doi.org/10.1017/S1446181119000142 Published online by Cambridge University Press

2. The FFT

In this section, we implement the FFT technique to price an extendable option by expressing the payoff function as a difference of its expectations of indicator functions [4]. The FFT approach utilizes the characteristic function of the underlying asset price process. For the extendable option, the implementation involves univariate and bivariate characteristic functions under the risk-neutral measure $\mathbb{Q}$. The characteristic function is defined as follows.

DEFINITION 2.1. Given two stochastic processes X_t and Y_t for $0 \leq t \leq T$, with density functions $q_T(X_T)$ and $q_T(Y_T)$, the characteristic function is the Fourier transform of its density function such that

$$\varphi(u_1) = \mathbb{E}^{\mathbb{Q}}(e^{iu_1 X_T})$$

for a one-dimensional stochastic process and

$$\varphi(u_1, u_2) = \mathbb{E}^{\mathbb{Q}}(e^{iu_1 X_T} + e^{iu_2 Y_T})$$

for a two-dimensional stochastic process, where u_1 and u_2 are arbitrary real numbers and $i = \sqrt{-1}$ is the imaginary unit.

The following lemmas present the univariate characteristic function as provided by Heston [14] and the bivariate characteristic function that is obtained from the arguments presented by Griebsch and Wystup [11], under the Heston model.

LEMMA 2.2. *Under the Heston model, a univariate characteristic function is given by*

$$\begin{aligned}\varphi_{x_{T_1}}(u_1) = \exp\Big[iu_1\Big\{x_t + r(T_1 - t) + \frac{\rho}{\sigma_0}\{-v_0 - \kappa\theta(T_1 - t)\}\Big\}\Big] \\ \times \exp[A(T_1 - t, a(u_1), b(u_1))v_0 + B(T_1 - t, a(u_1), b(u_1))],\end{aligned}$$

where

$$\begin{aligned}A(\tau, a, b) &= \frac{da(u)(1 + e^{-d\tau}) - [\kappa a(u) + 2b(u)][1 - e^{-d\tau}]}{2de^{-d\tau} + [\sigma_0^2 a(u) - \kappa - d][e^{-d\tau} - 1]}, \\ B(\tau, a, b) &= \frac{\kappa\theta}{\sigma_0^2}(\kappa - d)\tau + \frac{2\kappa\theta}{\sigma_0^2}\ln\Big[\frac{2d}{2de^{-d\tau} + \{\kappa + d - \sigma_0^2 a(u)\}[1 - e^{-d\tau}]}\Big], \\ a(u) &= iu\frac{\rho}{\sigma_0}, \quad b(u) = iu\Big[-\frac{1}{2} + \kappa\frac{\rho}{\sigma_0} + \frac{1}{2}iu(1 - \rho^2)\Big], \quad d = \sqrt{\kappa^2 + 2\sigma_0^2 b(u)}.\end{aligned}$$

LEMMA 2.3. *Under the Heston model, a bivariate characteristic function is given by*

$$\begin{aligned}\varphi_{x_{T_1}, x_{T_2}}(u_1, u_2) = \exp\Big\{iu_1\Big(x_t + r(T_1 - t) + \frac{\rho}{\sigma_0}[-v_0 - \kappa\theta(T_1 - t)]\Big)\Big\} \\ \times \exp\Big\{iu_2\Big(x_t + r(T_2 - t) + \frac{\rho}{\sigma_0}[-v_0 - \kappa\theta(T_2 - t)]\Big)\Big\} \\ \times \exp\{B(T_2 - T_1, a(u_2), b(u_2)) + A(T_1 - t, C(\tau, a, b), b(u_1 + u_2))v_0\} \\ \times \exp\{B(T_1 - t, C(\tau, a, b), b(u_1 + u_2))\},\end{aligned}$$

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where

$$C(\tau, a, b) = a(u_1) + A(T_2 - T_1, a(u_2), b(u_2)),$$

and $A(\tau, a, b)$, $B(\tau, a, b)$, $a(u)$, $b(u)$, d *are as defined in Lemma* 2.2.

By the martingale property under the risk-neutral measure $\mathbb{Q}$, the holder-extendible call option as shown in equation (1.4) can be valued at time t using expectations of indicator functions

$$\begin{aligned} &EC_H(S_t, K_1, T_1, K_2, T_2, A) \\ &\quad = e^{-r(T_1-t)} \underbrace{\mathbb{E}^{\mathbb{Q}}[(e^{x_{T_1}} - e^{k_1})\mathbf{1}_{\{x_{T_1} > h\}}]}_{I} + e^{-r(T_2-t)} \underbrace{\mathbb{E}^{\mathbb{Q}}[(e^{x_{T_2}} - e^{k_2})\mathbf{1}_{\{x_{T_1} \geq l, x_{T_2} \geq k_2\}}]}_{II} \\ &\qquad - e^{-r(T_2-t)} \underbrace{\mathbb{E}^{\mathbb{Q}}[(e^{x_{T_2}} - e^{k_2})\mathbf{1}_{\{x_{T_1} \geq h, x_{T_2} \geq k_2\}}]}_{III} - e^{-r(T_1-t)} \underbrace{\mathbb{E}^{\mathbb{Q}}[e^{a}\mathbf{1}_{\{x_{T_1} \geq l\}} - e^{a}\mathbf{1}_{\{x_{T_1} \geq h\}}]}_{IV}, \end{aligned} \tag{2.1}$$

where $x_t = \ln S_t$, $k_1 = \ln K_1$, $k_2 = \ln K_2$, $l = \ln L$, $h = \ln H$ and $a = \ln A$. In integral form, equation (2.1) can be written as

$$\begin{aligned} &EC_H(S_t, K_1, T_1, K_2, T_2, A) \\ &\quad = e^{-r(T_1-t)} \underbrace{\int_h^\infty (e^{x_{T_1}} - e^{k_1}) q(x_{T_1})\, dx_{T_1}}_{I} \\ &\qquad + e^{-r(T_2-t)} \underbrace{\int_l^\infty \int_{k_2}^\infty (e^{x_{T_2}} - e^{k_2}) q(x_{T_1}, x_{T_2})\, dx_{T_2} dx_{T_1}}_{II} \\ &\qquad - e^{-r(T_2-t)} \underbrace{\int_h^\infty \int_{k_2}^\infty (e^{x_{T_2}} - e^{k_2}) q(x_{T_1}, x_{T_2})\, dx_{T_2} dx_{T_1}}_{III} \\ &\qquad - e^{-r(T_1-t)} \Big[\underbrace{e^{a} \int_l^\infty q(x_{T_1}) dx_{T_1}}_{IV} - \underbrace{e^{a} \int_h^\infty q(x_{T_1})\, dx_{T_1}}_{V} \Big], \end{aligned}$$

where $q(\cdot)$ is the conditional density function of the random value x_{T_1} and $q(\cdot, \cdot)$ is the joint conditional density function of the random variables x_{T_1} and x_{T_2} for a given value x_t.

Employing a similar approach to that of Carr and Madan [5], we implement a FFT on terms $I - V$. To avoid repetition, we only consider term V for the univariate case and term III for the bivariate case. First, we multiply terms V and III by an exponentially decaying term so that it is square-integrable, and we define the damped integral

$$\begin{aligned} V^D(h) &= e^{\alpha_1 h} V(h), \\ III^D(h, k_2) &= e^{\alpha_1 h + \alpha_2 k_2} III(h, k_2), \end{aligned}$$

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for $\alpha_1, \alpha_2 > 0$. Then we apply the Fourier transform

$$\psi(u_1) = \int_{-\infty}^{\infty} e^{iu_1 h} V^D(h)\, dh,$$
$$\psi(u_1, u_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iu_1 h + iu_2 k_2} III^D(h, k_2)\, dk_2\, dh,$$

where the Fourier transform ψ is available in closed-form in terms of the characteristic function φ as

$$\psi(u_1) = \frac{\varphi_{x_{T_1}}(u_1 - i\alpha_1)}{iu_1 + \alpha_1},$$
$$\psi(u_1, u_2) = \frac{\varphi_{x_{T_1}, x_{T_2}}(u_1 - i\alpha_1, u_2 - i(\alpha_2 + 1))}{(iu_1 + \alpha_1)(iu_2 + \alpha_2)(1 + iu_2 + \alpha_2)}.$$

Using an IFT, we recover terms V and III as

$$V(h) = \frac{e^{-\alpha_1 h}}{2\pi} \int_{-\infty}^{\infty} e^{-iu_1 h} \psi(u_1)\, du_1, \tag{2.2}$$
$$III(h, k_2) = \frac{e^{-\alpha_1 h - \alpha_2 k_2}}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iu_1 h - iu_2 k_2} \psi(u_1, u_2)\, du_2\, du_1. \tag{2.3}$$

The integrals in equations (2.2) and (2.3) are evaluated by numerical approximation using the trapezium rule and a FFT, and are given by

$$V(h) \approx \frac{e^{-\alpha_1 h}}{2\pi} \sum_{j=0}^{N-1} e^{-iu_{1,j} h} \psi(u_{1,j}) \Delta_1,$$
$$III(h, k_2) \approx \frac{e^{-\alpha_1 h - \alpha_2 k_2}}{(2\pi)^2} \sum_{j=0}^{N-1}\sum_{m=0}^{N-1} e^{-iu_{1,j} h - iu_{2,m} k_2} \psi(u_{1,j}, u_{2,m}) \Delta_2\, \Delta_1,$$

where Δ_1 and Δ_2 denote the distances between the points of the integration grid, and $u_{1,j} = (j - N/2)\Delta_1$, $\quad u_{2,m} = (m - N/2)\Delta_2$ for $j, m = 0, \ldots, N-1$ (where $N = 2^n$, $n \in \mathbb{N}$). We define a grid of size $N \times N$ by $H_2 = \{(h_u, k_{2,p}) \mid 0 \le u, p \le N-1\}$, with $\omega_1, \omega_2 > 0$ denoting the distances between the logarithmic critical prices and the logarithmic strike prices, where

$$h_u = (u - N/2)\omega_1, \quad k_{2,p} = (p - N/2)\omega_2,$$

and then we evaluate using the sum

$$Z(h) = \sum_{j=0}^{N-1} e^{-iu_{1,j} h} \psi(u_{1,j}),$$
$$Z(h, k_2) = \sum_{j=0}^{N-1}\sum_{m=0}^{N-1} e^{-iu_{1,j} h - iu_{2,m} k_2} \psi(u_{1,j}, u_{2,m}).$$

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Choosing $\omega_1 \Delta_1 = 2\pi/N$ and $\omega_2 \Delta_2 = 2\pi/N$, yields the values of the sum on H_2

$$Z(h_u) = \sum_{j=0}^{N-1} e^{-iu_{1,j}h_u}\psi(u_{1,j}) = (-1)^u \sum_{j=0}^{N-1} e^{-i(2\pi/N)ju}[(-1)^j\psi(u_{1,j})], \tag{2.4}$$

$$\begin{aligned} Z(h_u, k_{2,p}) &= \sum_{j=0}^{N-1}\sum_{m=0}^{N-1} e^{-iu_{1,j}h_u - iu_{2,m}k_{2,p}}\psi(u_{1,j}, u_{2,m}) \\ &= (-1)^{u+p}\sum_{j=0}^{N-1}\sum_{m=0}^{N-1} e^{-i(2\pi/N)ju - i(2\pi/N)mp}[(-1)^{j+m}\psi(u_{1,j}, u_{2,m})]. \end{aligned} \tag{2.5}$$

On that account, equations (2.4) and (2.5) are computed via FFT by taking the input arrays, respectively, as

$$X[j] = (-1)^j\psi(u_{1,j}),$$
$$X[j,m] = (-1)^{j+m}\psi(u_{1,j}, u_{2,m})$$

for $j, m = 0, \ldots, N-1$. Therefore, the result is an approximation of term $V(h)$ at $N \times 1$ different logarithmic critical prices h and of term $III(h, k_2)$ at $N \times N$ different logarithmic critical prices h and logarithmic strike prices k_2, specified by

$$V(h_u) \approx \frac{e^{-\alpha_1 h_u}}{2\pi} Z(h_u)\Delta_1,$$
$$III(h_u, k_{2,p}) \approx \frac{e^{-\alpha_1 h_u - \alpha_2 k_{2,p}}}{(2\pi)^2} Z(h_u, k_{2,p})\Delta_2 \Delta_1$$

for $0 \le u,\ p \le N-1$. Following similar procedures to those shown above, analogous results are obtained for terms I and IV in the univariate case and for term II in the bivariate case.

3. Numerical example

In this section, we analyse the pricing of extendable options using the model from Section 2. We first evaluate option prices using the FFT. Then we compare the accuracy and computational time of the pricing under the Heston model with two benchmark prices determined via MCS and FDMs. We adopt two commonly employed calibration errors: the absolute relative error (ARE) and the root-mean-squared error (RMSE). The ARE for each initial stock price j and the RMSE are defined, respectively, as

$$\mathrm{ARE}_j = \left|\left(\frac{\widehat{EC}_j}{EC_j} - 1\right)\right|, \quad \mathrm{RMSE} = \sqrt{\frac{1}{n_S}\sum_{j=1}^{n_S} |\widehat{EC}_j - EC_j|^2},$$

where the sum is over the number $n_S (= 5)$ of initial stock values, $\widehat{EC}$ is the estimate price obtained via a FFT and EC is the exact price determined by MCS or FDMs. Note

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TABLE 1. Inputs to price the extendable options.

Input	Value
Initial stock prices, S	$0.8, 0.85, 0.9, 0.95, 1.0$
Initial strike price, K_1	0.9
Initial expiration time, T_1	1
Extended strike price, K_2	0.95
Extended expiration time T_2	2
Risk-free rate, r	0.02
Premium, A	0.03

that the computations were implemented in MATLAB and conducted on an Intel (R) Core(TM) i7-7700 CPU @ 3.60 GHz machine running under Windows 10 with 12GB RAM and a 64-bit operating system. For the implementation of the FFT technique, it is convenient to allow the outer and the inner sum of equation (2.5) to have different $N = N_1$ and N_2 ($< 2^{12}$), respectively. The FFT prices are sensitive to the choice of Δ_1, Δ_2, α_1, α_2, N_1 and N_2 [15]. Hence, to determine an appropriate choice of these parameters, we adapt the methodology employed by Hurd and Zhou [18], where the average of the absolute value of the log price differences, as in equation (3.1), was minimized as

$$Err = \frac{1}{n_S}\sum_{j=1}^{n_S} |\log(\widehat{EC}_j) - \log(EC_j)|. \tag{3.1}$$

A two-step approach is taken to optimize the FFT parameters. First, α_1 and Δ_1 are fixed at 0.75 and 0.1, respectively, which are reasonable parameters [15]. The outputs for α_2 and Δ_2 are then employed (together with α_1 and Δ_1) as initial inputs to minimize Err. This is repeated for different N_1 and N_2. In all that follows, these parameters are determined to have these values: $N_1 = 2^6$, $N_2 = 2^4$, $\Delta_1 = 0.1637$, $\Delta_2 = 0.0166$, $\alpha_1 = 0.7496$ and $\alpha_2 = 0.7502$. In the MCS approach, we use $100,000$ simulations each of 1000 steps (following Hurd and Zhou [18], no variance reduction was employed). Moreover, the FDM is applied over a three-dimensional grid of size $50 \times 50 \times 7000$ (stock price, variance and time to maturity).

Table 1 lists the other parameter values used in the computation. The Heston parameters, tabulated in Table 2, are sourced from Table 5 of Moyaert and Petitjean [25], where these values are calibrated from the market prices of Eurostoxx 50 index options during the subprime crisis. With the parameter values from Tables 1 and 2, we solve for the critical prices using a root-search algorithm such as the Newton–Raphson method [21] and obtain $L = 0.7946$ and $H = 1.0753$. A comparison of holder-extendable call prices is tabulated in Table 3.

Our numerical examples demonstrate that the computational time difference is significant as the FFT takes 50.03s to produce five holder-extendable call prices including optimizing the FFT parameters, while MCS and the FDM take 892.68s and 126.49s, respectively, to produce five holder-extendable call prices.

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TABLE 2. Heston parameters.

Input	Value
Instantaneous volatility, $\sqrt{v}$	0.33
Long run volatility, $\sqrt{\theta}$	0.28
Mean reversion rate, κ	3.15
Volatility of variance, σ_0	0.76
Correlation parameter, ρ	−0.81

TABLE 3. Holder-extendable call option prices under the Heston model: FFT, MCS and FDM.

S	FFT	FDM	MCS (95% confidence interval)
0.8	0.0646	0.0648	0.0646 (0.0641, 0.0651)
0.85	0.0916	0.0910	0.0911 (0.0904, 0.0918)
0.9	0.1219	0.1208	0.1216 (0.1208, 0.1224)
0.95	0.1550	0.1545	0.1550 (0.1540, 0.1560)
1.0	0.1902	0.1909	0.1913 (0.1902, 0.1924)

TABLE 4. ARE and RMSE (in %) for pricing holder-extendable call options under the Heston model: FFT vs MCS and FDMs.

S	MCS	FDM
0.8	0.00	0.31
0.85	0.55	0.66
0.9	0.25	0.91
0.95	0.00	0.32
1.0	0.58	0.37
RMSE	0.05568	0.06856

In Table 4, the ARE and the RMSE for the holder-extendable call option priced using the FFT under the Heston model are compared with MCS and FDMs. The ARE and the RMSE indicate the better performance of the FFT model compared with MCS and FDMs, and the errors obtained for the FFT are generally close to those for nonextendable options [15].

TABLE 5. Vanilla and holder-extendable call prices under the Black–Scholes model and the Heston model.

S	Black–Scholes			Heston		
	Vanilla	Holder-extendable	Extension privilege	Vanilla	Holder-extendable	Extension privilege
0.8	0.0632	0.0721	0.0089	0.0494	0.0656	0.0162
0.85	0.0866	0.0962	0.0096	0.0755	0.0918	0.0163
0.9	0.1140	0.1238	0.0098	0.1071	0.1216	0.0145
0.95	0.1451	0.1548	0.0097	0.1416	0.1545	0.0129
1.0	0.1794	0.1888	0.0094	0.1796	0.1908	0.0112

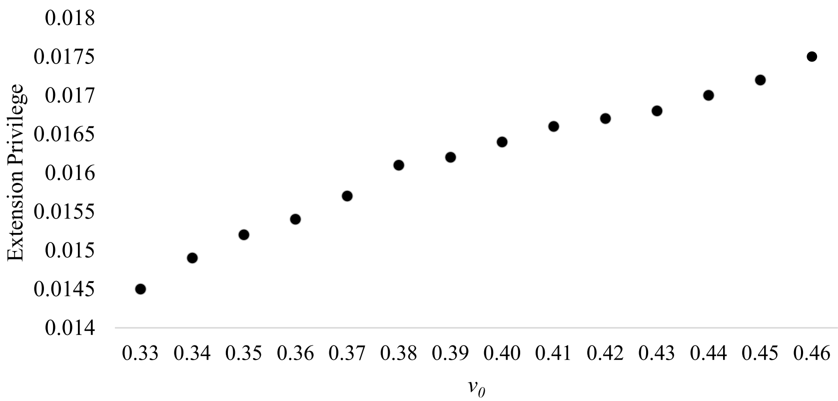

FIGURE 3. Extension privilege at time T_1 with different v_0.

Finally, in Table 5, we document the prices of a vanilla call option and a holder-extendable call option under the Black–Scholes model [2] and the Heston model [14]. The volatility input for the Black–Scholes model is given by $\sqrt{\bar{v}(T)}$ in equation (1.6), where $T = T_1$ for the vanilla call and $T = T_2$ for the holder-extendable call, leading to Black–Scholes annualized volatilities of 29.61% and 28.85%, respectively. Note that, for out-of-the money options, the prices for both vanilla calls and holder-extendable calls are greater under the Black–Scholes environment [2] than the prices under the Heston environment [14]. This is also true for in-the-money options for both vanilla calls and holder-extendable calls. This is well known for vanilla calls when $\rho < 0$ [29], and it is also the case for holder-extendable calls (whether priced by FFT, MCS or FDMs) because the distribution of the logarithmic asset prices is negatively skewed when $\rho < 0$, producing a heavier left tail of the distribution. Moreover, the Black–Scholes implied volatility exhibits larger curvature than the Heston implied volatility. We also observe from Table 5 that the extension privilege is higher under the Heston model [14] than under the Black–Scholes model [2]. Figures 3, 4 and 5 illustrate the changes in the values of the extension privilege under the Heston model when v_0, σ

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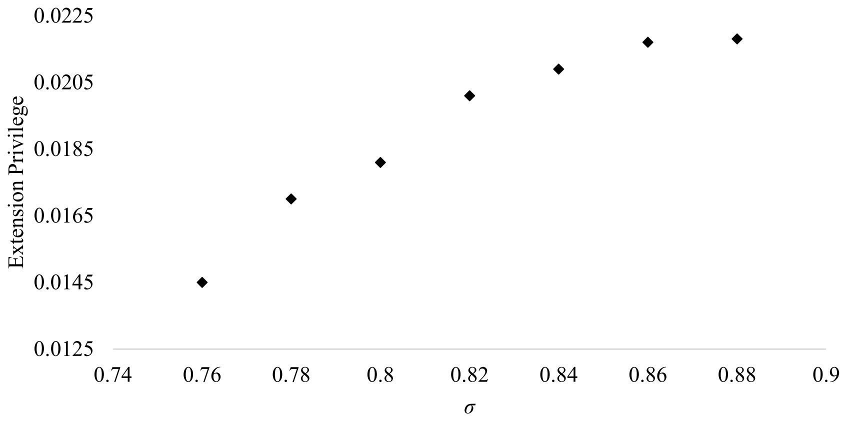

FIGURE 4. Extension privilege at time T_1 with different σ.

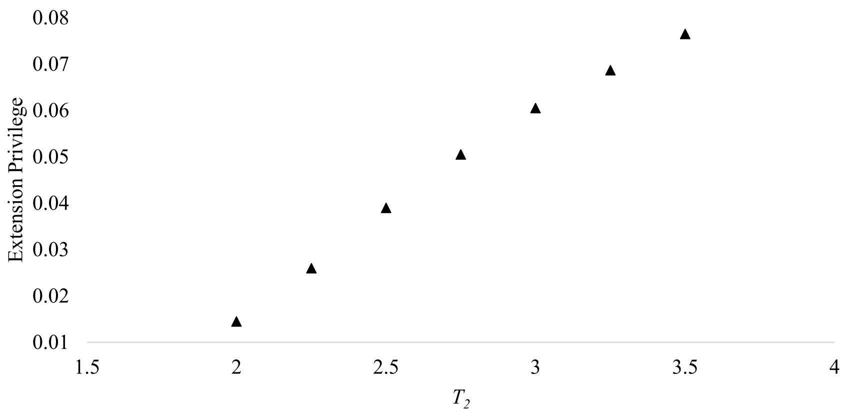

FIGURE 5. Extension privilege at time T_1 with different T_2.

and T_2 increase, respectively. We observe that the extension privilege increases as the values of v_0, σ and T_2 increase.

4. Conclusion

This paper considers the pricing of holder-extendable call options under the Heston dynamics using a FFT, and it compares this with the MCS and FDM benchmarks. The FFT pricing formula is expressible as a finite sum of expectations of the indicator functions, where the partition uses the two critical values introduced by Longstaff [23]. The evaluation of the expectations involves one-dimensional and two-dimensional Fourier transforms via the corresponding univariate and bivariate characteristic functions, respectively. Under the Heston model, there exist closed-form solutions of the characteristic functions; hence, in comparison with MCS and an explicit FDM, the application of the FFT yields significant computational savings, typically, of at least an order of magnitude. We also observe that, overall, the Heston

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model performs better than the Black–Scholes model in pricing holder-extendable call options.

In addition to stochastic volatility, this study can be further developed by incorporating jumps with stochastic volatility together with stochastic interest rates, in the spirit of Santa-Clara and Yan [30], and by implementing other optimization strategies. Finally, the use of extendable options in problems involving real options may lead to fruitful investigations.

Acknowledgements

We would like to thank the anonymous referees for their valuable comments and suggestions. We also gratefully acknowledge the contribution of the late Dr. Nick Constantinou to this paper. The author S. N. I. Ibrahim acknowledges the support from Universiti Putra Malaysia through Geran Putra IPM (9437300) and Geran Putra (9570100). The author A. Díaz-Hernández thanks Universidad Anahuac Mexico-Norte and the University of Essex for their financial support during a month-long visit for research.

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