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MODULAR CORRESPONDENCES ON $X(11)$

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In this paper, we show how to give a geometric interpretation of the modular correspondence T_3 on the modular curve $X(11)$ of level 11 using projective geometry. We use Klein's theorem that $X(11)$ is isomorphic to the nodal curve of the Hessian of the cubic threefold Λ defined by $V^2W+W^2X+X^2Y+Y^2Z+Z^2V=0$ in $\mathbf{P}^4(\mathbf{C})$ and geometry which we learned from a paper of W. L. Edge. We show that the correspondence T_3 is essentially the correspondence which associates to a point p of the curve $X(11)$ the four points where the singular locus of the polar quadric of p with respect to Λ meets $X(11)$. Our control of the geometry is good enough to enable us to compute the eigenvalues of T_3 acting on the cohomology of $X(11)$. This is the first example of an explicit geometric description of a modular correspondence without valence. The results of this article will be used in subsequent articles to associate two new abelian varieties to a cubic threefold, to desingularize the Hessian of a cubic threefold and to study self-conjugate polygons formed by the quadrisecants of the nodal curve of the Hessian.

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0. Introduction

In this paper we show how to describe certain modular correspondences on $X(11)$ explicitly in terms of the geometry of a projective embedding of $X(11)$ discovered by Felix Klein [**2, 6, 9, 10, 11**]. This is done in Theorem 1 and Theorem 2 of Section 2. Both correspondences considered here are without valence and are therefore beyond the scope of methods developed by Klein (c.f. [**11**, Ch. VI]) for describing correspondences geometrically as "Schnittsystem-Correspondenzen." At any rate, Klein did not consider the problem of obtaining explicit equations of modular correspondences on $X(11)$. In subsequent papers, we will use the results developed here to associate new abelian varieties to cubic threefolds, to desingularize the Hessian of a cubic threefold and to study self-conjugate polygons formed by the quadrisecants of the nodal curve of the Hessian.

In Section 1 we introduce the correspondence Ξ which will turn out (Theorem 1 of Section 2) to be essentially (i.e. up to composition with an automorphism of $X(11)$) the Hecke correspondence T_3 for $\Gamma(11)$. In Section 2 we determine the eigenvalues of Ξ acting on the cohomology of the modular curve (Proposition 1). This amounts to a determination of the eigenvalues of a Hecke operator using only projective geometry. In Theorem 2 of Section 2, we show that the modular correspondence

$$\Gamma(11)\eta^2\begin{pmatrix}9 & 0\\ 0 & 1\end{pmatrix}\Gamma(11),$$

where η is a certain element of $SL_2(\mathbf{Z})$, is the correspondence which associates to each point p of the modular curve the 12 points other than p where the four quadrisecants through p meet the modular curve. Our arguments depend on criteria for recognizing modular correspondences developed in [**1**], which also contains examples of the theory on $X(5)$ and $X(7)$.

I received valuable help from my late friend and teacher Prof. Michio Kuga in conversations about this work. In particular, when I pointed out to him the correspondence Ξ and asked whether it might be a modular correspondence, he suggested to me that it might be T_3.

1. Elementary properties of the correspondence Ξ

Felix Klein showed [**2, 6, 9, 10, 11**] that the modular curve $X(11)$ is isomorphic to the curve $\mathscr{C}$ in $\mathbf{P}^4$ consisting of all points $[V, W, X, Y, Z]$ such that the matrix

$$\begin{bmatrix} W & V & 0 & 0 & Z\\ V & X & W & 0 & 0\\ 0 & W & Y & X & 0\\ 0 & 0 & X & Z & Y\\ Z & 0 & 0 & Y & V\end{bmatrix} \tag{1.1}$$

has rank 3. Furthermore the group of 660 automorphisms of $\mathscr{C}$ acts as a group of collineations of $\mathbf{P}^4$. This matrix is, up to a scalar factor, the matrix of second partial derivatives of the cubic form $V^2W+W^2X+X^2Y+Y^2Z+Z^2V$. More recently, W. L. Edge [**6**] has studied Klein's curve $\mathscr{C}$ and the geometry surrounding it and the group of 660 collineations. In particular he draws attention to a scroll $\mathscr{S}$ of quadrisecants of $\mathscr{C}$ which arises in the following way.

Let p be a point of $\mathbf{P}^4$ and denote by Q_p the polar quadric of p with respect to the cubic threefold Λ defined by

$$V^2W+W^2X+X^2Y+Y^2Z+Z^2V=0.$$

Then Q_p is defined by the quadratic form associated to the matrix (1) with $p=[V, W, X, Y, Z]$. The locus of all p such that Q_p is a cone is the *Hessian* of Λ, which we denote by H and which is defined by the determinant of (1.1). If p lies on H, the singular locus of Q_p is denoted $\lambda(p)$. Then $\lambda(p)$ is a line if p lies on $\mathscr{C}$ and is a point otherwise. If p and q are two points of H, then p lies on $\lambda(q)$ if and only if q lies on $\lambda(p)$. So the correspondence $p\mapsto\lambda(p)$ is a birational involution of the Hessian. If p lies on $\mathscr{C}$ then the line $\lambda(p)$ meets $\mathscr{C}$ in four points. We denote by Ξ the correspondence on $\mathscr{C}$

which associates to each point p of $\mathscr{C}$ the four points where $\lambda(p)$ meets $\mathscr{C}$. Since p lies on $\lambda(q)$ if and only if q lies on $\lambda(p)$, we see that Ξ is a symmetric correspondence of bidegree (4, 4).

Remark. The mapping λ induces a rank 2 vector bundle on $\mathscr{C}$ which is invariant under the group of automorphisms of $\mathscr{C}$. It would be interesting to study this bundle from the standpoint of the theory of moduli of vector bundles on the modular curve. A similar phenomenon occurs in connection with a scroll of trisecants on the modular curve $X(7)$ of level 7 (cf. [**1**]). In view of the connection of both scrolls to modular correspondences, it would be interesting to try to generalize the scrolls, the bundles and the correspondences to modular curves of higher level.

Let G denote the group of 660 collineations. For future reference let us note that the collineations

$$\gamma=\begin{bmatrix} \zeta & 0 & 0 & 0 & 0 \\ 0 & \zeta^9 & 0 & 0 & 0 \\ 0 & 0 & \zeta^4 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^5 \end{bmatrix}, \tag{1.2}$$

where $\zeta=e^{2\pi i/11}$, and

$$\sigma=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.3}$$

belong to G and preserve the cubic form $V^2W+W^2X+X^2Y+Y^2Z+Z^2V$.

Lemma 1. *Ξ commutes with all of the collineations in G.*

Proof. As Klein showed, the cubic threefold and its Hessian are invariant under G. Since G acts as collineations and since the operation of taking polars is convariant, we have for all x *in* $\mathbf{P}^4$ and all g in G that $g\cdot Q_x=Q_y$ where $y=g\cdot x$. Therefore for all x in H, we have $\lambda(g\cdot x)=g\cdot\lambda(x)$. In case x lies on $\mathscr{C}$, we have therefore that

$$\Xi(g\cdot x)=\lambda(g\cdot x)=g(\lambda(x)\cap\mathscr{C})=g\circ\Xi(x).$$

Corollary 1. *Every irreducible component of Ξ commutes with G and is cuspidal.*

Proof. This follows at once from the Corollary of Lemma 11 of Section 5 of [**1**].

Corollary 2. *Ξ has no components of the form $\{x\}\times\mathscr{C}$ or $\mathscr{C}\times\{y\}$.*

Proof. Such a component, and therefore the point x, would be invariant under G.

Lemma 2. *The correspondence Ξ has no fixed points.*

Proof. Let $p=[V,W,X,Y,Z]$ be a point of $\mathscr{C}$. If p lies in $\lambda(p)$ then

$$\begin{bmatrix} W & V & 0 & 0 & Z \\ V & X & W & 0 & 0 \\ 0 & W & Y & X & 0 \\ 0 & 0 & X & Z & Y \\ Z & 0 & 0 & Y & V \end{bmatrix}\cdot\begin{bmatrix} V \\ W \\ X \\ Y \\ Z \end{bmatrix}=0.$$

But the entries of the above product are just the partial derivatives of $V^2W+W^2X+X^2Y+Y^2Z+Z^2V$. It is easy to check, however, that the cubic threefold Λ is non-singular. So p cannot lie in $\lambda(p)$ and in particular cannot be a fixed point of Ξ.

Lemma 3. *The correspondence Ξ^3 has no fixed points among the cusps of $X(11)$.*

Proof. Since Ξ commutes with the elements of G and since the cusps of $X(11)$ form a single G-orbit, it is enough to prove this for one cusp. Let $p=[1,0,0,0,0]$. One checks that p lies on $\mathscr{C}$. Since p is fixed by the collineation γ of order 11 in G defined in equation (1.2), it follows that p is a cusp of $X(11)$. One then checks that $\lambda(p)$ is spanned by $[0,0,1,0,0]$ and $[0,0,0,1,0]$. Since these two points are the only fixed points of γ on $\lambda(p)$ and since $\Xi(p)$ is a divisor of degree 4 invariant under the cyclic group of order 11 generated by γ, it follows that $\Xi(p)$ is supported on these two points. Actually, we can use the collineation σ defined in (1.3) to describe these two points as $\sigma^2(p)$ and $\sigma^3(p)$. Since $\sigma\cdot\gamma^9=\gamma\cdot\sigma$ and since Ξ commutes with the elements of G, it follows that Ξ^2 is supported on p, $\sigma(p)$ and $\sigma^4(p)$. Since the supports of $\Xi(p)$ and of $\Xi^2(p)$ are disjoint, we conclude from the symmetry of Ξ that Ξ^3 does not fix p.

2. Eigenvalues of Ξ and characterization of Ξ as a modular correspondence

Proposition 1. *The eigenvalues of Ξ acting on the space of holomorphic differentials of $\mathscr{C}$ are -1 with multiplicity 16 and 2 with multiplicity 10.*

Proof. According to Hecke [7], the representation of G on the space Ω^1 of holomorphic differentials decomposes into irreducible components of degrees 5, 10 and 11 respectively. (Actually, this observation was made earlier by A. Hurwitz [8]). Since Ξ commutes with the elements of G, Ξ must act as a scalar in each component. Let α, β and γ denote the eigenvalues of Ξ acting on the 5, 10 and 11 dimensional components of Ω^1 respectively. Since Ξ is symmetric and since the representation of G and the action of Ξ on the cohomology group $H^1(\mathscr{C},\mathbf{R})$ are isomorphic to the actions of G and of Ξ on the real vector space underlying Ω^1, it follows that α, β and γ are real. Hence the characteristic polynomial of $H^1(\Xi)$ is

$$h(t)=(t-\alpha)^{10}\cdot(t-\beta)^{20}\cdot(t-\gamma)^{22}$$

where $H^1(\Xi)$ denotes the endomorphism of $H^1(\mathscr{C},\mathbf{R})$ incuded by Ξ. Since $H^1(\Xi)$ leaves $H^1(\mathscr{C},\mathbf{Z})$ invariant, $h(t)$ has rational integral coefficients. Let us decompose $H^1(\mathscr{C},\mathbf{Q})$ as a G-module into isotypic components. There will be three such components, say, A, B and C. We may suppose that A belongs to the 5 dimensional component of Ω^1, B to the 10 dimensional component and C to the 11 dimensional component. The endomorphism ring of the G-module A is isomorphic to the quadratic field $\mathbf{Q}(\sqrt{-11})$ while the endomorphism rings of B and C are isomorphic to $M_2(\mathbf{Q})$. It follows that α, β and γ all satisfy monic quadratic polynomials with rational integral coefficients and that α belongs to $\mathbf{Q}(\sqrt{-11})$. Since α is real, α is in fact a rational integer. It follows that

$$(t-\beta)^{20}\cdot(t-\gamma)^{22}$$

must also have rational integral coefficients. If β is not rational then γ must be conjugate to β over $\mathbf{Q}$ and

$$(t-\beta)^{20}\cdot(t-\gamma)^{20}$$

must have rational coefficients. But then $(t-\gamma)^2$ must also have rational coefficients, whence γ is rational. Therefore, β and γ must be rational integers. By the Lefschetz fixed point formula and by Lemma 2 we have

$$0=4-tr\,H^1(\Xi)+4=8-10\alpha-20\beta-22\gamma$$

and therefore

$$5\alpha+10\beta+11\gamma=4. \tag{2.1}$$

In particular, γ is congruent to -1 modulo 5 and α and γ have the same parity. Let Υ denote the correspondence $\Xi^2-4\Delta$ where Δ denotes the identity correspondence. Let N denote the number of fixed points of Υ, counted with their multiplicities. By the Lefschetz fixed point formula we have

$$\begin{aligned} N&=12-tr\,H^1(\Upsilon)+12\\ &=24-10\cdot(\alpha^2-4)-20\cdot(\beta^2-4)-22\cdot(\gamma^2-4)\end{aligned}$$

so that

$$5\alpha^2+10\beta^2+11\gamma^2=116-\tfrac{1}{2}N. \tag{2.2}$$

Therefore $\gamma^2 \leqq 9$ and since $\gamma \equiv -1 \pmod 5$, we conclude that $\gamma=-1$. Therefore $116-\frac{1}{2}N \geqq 11$ and $N \leqq 210$. Since the divisor of fixed points of Υ is invariant under G, it is a sum of G-orbits. The orbits of G on $\mathscr{C}$ have orders 60, 220, 330 and 660 respectively. It follows that N is a multiple of 60, say, $N=60w$, where $0 \leqq w \leqq 3$. Suppose $w=3$. Then $\frac{1}{2}N=90$ and $5\alpha^2+10\beta^2+11\gamma^2=26$, which implies $\alpha^2=\beta^2=\gamma^2=1$. Since $\gamma \equiv -1 \pmod 5$, we have $\gamma=-1$ and $\alpha=\beta=1$ by equation (2.1). But by Lemma 4, Ξ^3 does not contain Δ and the number of fixed points of Ξ^3 is therefore given by the Lefschetz number $128-10\alpha^3-20\beta^3-22\gamma^3=120$. But then every cusp of $X(11)$ would be a fixed point of Ξ^3, contradicting Lemma 3. So w cannot equal 3. Since $\gamma=-1$, we have from (2.1) and (2.2) that

$$\begin{aligned} \alpha+2\beta&=3 \\ \alpha^2+2\beta^2&=21-6w. \end{aligned} \tag{2.3}$$

If $w=0$, then $\alpha^2+2\beta^2=21$, which contradicts the fact that α and β are integers. If $w=1$ then $\alpha^2+\beta^2=15$, which is also impossible. It follows that $w=2$ and

$$\alpha^2+2\beta^2=9. \tag{2.4}$$

From equations (2.3) and (2.4) we have that either $\alpha=-1$ and $\beta=2$ or $\alpha=3$ and $\beta=0$. If $\alpha=3$ and $\beta=0$ then the number of fixed points of Ξ^3 is

$$128-10\cdot 3^3-20\cdot 0^3-22\cdot -1^3$$

which is <0. That can only happen if Ξ^3 contains the identity correspondence Δ, which is impossible by Lemma 3. Therefore, $\alpha=-1$ and $\beta=2$ and the proposition is proved.

Remark. Felix Klein was able to realize certain modular correspondences as "Schnittsystem-Correspondenzen," all of which are correspondences with valence. It follows from Proposition 1 that Ξ is a correspondence without valence and therefore cannot be obtained by the methods of Klein. To the best of my knowledge, this is the first example which has been obtained of a geometric interpretation of a modular correspondence without valence .

Corollary. *Denote by Θ the correspondence defined by*

$$\Theta=\Xi\circ\Xi-4\Delta.$$

Then every fixed point of Θ is a cusp of $X(11)$.

Proof. The correspondence Θ has bidegree $(12,12)$. Using Proposition 1, we see that the eigenvalues of $H^1(\Theta)$ are -3 with multiplicity 16 and 0 with multiplicity 10. According to the Lefschetz fixed point formula, the number of fixed points of Θ is

$$12 - tr\, H^1(\Theta) + 12 = 24 - 2\cdot(16\cdot(-3) + 10\cdot 0) = 120.$$

Since Θ commutes with the elements of G, the set of fixed points must be a union of orbits of G. The G-orbits on $X(11)$ have orders 60,220,330 and 660, with the cusps of $X(11)$ forming the unique orbit of order 60. Therefore the 120 fixed points of Θ must consist of the cusps of $X(11)$ taken twice, which proves the corollary.

Theorem 1. *The correspondence Ξ is a modular correspondence. In fact, Ξ is associated with the double coset*

$$\Gamma(11)\eta\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}\Gamma(11)$$

where η is any element of $\Gamma = SL_2(\mathbf{Z})$ which is congruent to $\left(\begin{smallmatrix}2 & 0\\ 0 & 6\end{smallmatrix}\right)$ modulo 11.

Proof. By the corollary of Lemma 11 of Section 5 of [**1**], every component of the correspondence Ξ is cuspidal and commutes with the elements of G. Let Θ be the correspondence considered in the corollary of Proposition 1. Let Ξ_0 be any irreducible component of Ξ and let (a,b) be the bidegree of Ξ_0. If $a=1$, then Ξ_0 is the graph of a function f from $\mathscr{C}$ to itself. By Corollary 2 of Lemma 1, the function f cannot be a constant. Since $\mathscr{C}$ has genus>1, the function f must be an automorphism of $\mathscr{C}$. Since it must commute with the elements of G, it must be the identity automorphism. But by Lemma 2, Ξ has no fixed no fixed points and *a fortiori* cannot contain the identity correspondence. Therefore, $a\neq 1$. If $a=3$ then Ξ is the sum of Ξ_0 and another correspondence of bidegree $(1,4-b)$, which is impossible as we have just shown. Therefore, if Ξ is reducible, we must have $a=2$ and by symmetry of Ξ we must also have $b=2$. Let Θ_0 denote the correspondence $\iota(\Xi_0)\circ\Xi_0 - 2\Delta$. Then Θ_0 is contained in Θ. Since the fixed points of Θ are, by the corollary of Proposition 1, all cusps, the same is therefore true for Θ_0. It now follows from Corollary 1 of Lemma 12 of [**1**] and Theorem 1 of [**1**] that Ξ_0 is a modular correspondence. But this contradicts Lemma 8 of [**1**] since there does not exist a positive integer D such that $\psi(D)=2$. Therefore the correspondence Ξ is irreducible. It now follows the Corollary 2 of Lemma 12 of [**1**] that Ξ is a modular correspondence and the theorem follows at once from Theorem 2 of [**1**].

Corollary. *The tangent to a point p of $\mathscr{C}$ is a quadrisecant of $\mathscr{C}$ if and only if p is a cusp of $X(11)$.*

Proof. The tangent to $\mathscr{C}$ at p is a quadrisecant if and only if p is a branch point of $\iota(\Xi)$. We know from Theorem 1 and from Theorem 1 of [1] that Ξ is almost unramified. Therefore the only branch points of $\iota(\Xi)$ are cusps. Conversely, the point $p_1 =$

$[1,0,0,0,0]$ lies on $\mathscr{C}$ and is fixed by the automorphism defined by equation 1.2. That automorphism has order 11 and therefore p is a cusp of $X(11)$. Direct computation of the tangent line to $\mathscr{C}$ at p_1 shows that it coincides with the quadrisecant corresponding to the point $[0,0,0,1,0]$ of $\mathscr{C}$. Since the cusps form a G-orbit, it follows that the tangent to $\mathscr{C}$ at any cusp is a quadrisecant of $\mathscr{C}$.

Remark. This corollary was proved by Edge (c.f. [**6**], Section 8, p. 654]) by a different method. His determination of the number of tangent quadrisecants to be 120 is recovered in effect by the corollary of Proposition 1.

Theorem 2. *Let Θ be the correspondence on $\mathscr{C}$ which associates to each point p of $\mathscr{C}$ the* 12 *points other than p in which the four quadrisecants through p meet $\mathscr{C}$. Then Θ is the modular correspondence on $\mathscr{C}$ associated with the double coset*

$$\Gamma(11)\eta^2\begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}\Gamma(11)$$

where η is as in Theorem 1.

Proof. The correspondence Θ is, by definition, $\Xi\circ\Xi-4\Delta$. Since Ξ commutes with the elements of G and $\Gamma=SL_2(\mathbf{Z})$ normalizes $\Gamma(11)$, we have

$$\Gamma(11)\eta\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\Gamma(11)\cdot\Gamma(11)\eta\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\Gamma(11)$$

$$=\Gamma(11)\eta^2\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\Gamma(11)\cdot\Gamma(11)\begin{pmatrix}3 & 0\\0 & 1\end{pmatrix}\Gamma(11)$$

$$\supseteq\Gamma(11)\eta^2\begin{pmatrix}9 & 0\\0 & 1\end{pmatrix}\Gamma(11).$$

By Lemma 8 of Section 4 of [**1**], $[\Gamma(11)\eta^2(\begin{smallmatrix}9 & 0\\0 & 1\end{smallmatrix})\Gamma(11)]$ is an irreducible correspondence of bidegree $(12,12)$ on $X(11)$ and is therefore contained in Θ. Since Θ also has bidegree $(12,12)$, the two coincide and the theorem is proved.

Remark. It would be interesting to generalize the description of Ξ to modular curves of higher level. The problem of generalizing the description of the modular curve in terms of the cubic invariant has been discussed in the author's paper [**4**]. However, even without knowing the relation of the modular curves to cubic invariants, one can consider the modular curve as defined by quartic equations (c.f. [**5, 10, 11**]) and ask whether certain modular correspondences are associated to scrolls of linear spaces.

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