ON MINIMAL n-UNIVERSAL GRAPHS

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A *graph* G_n consists of n distinct vertices $x_1, x_2, \ldots, x_n$ some pairs of which are joined by an edge. We stipulate that at most one edge joins any two vertices and that no edge joins a vertex to itself. If x_i and x_j are joined by an edge, we denote this by writing $x_i \circ x_j$.

Consider a second graph H_N, where $n \leqq N$. Following Rado [1], we say that a one-to-one mapping f of the vertices of G_n into the vertices of H_N defines an *embedding* if $x_i \circ x_j$ implies $f(x_i) \circ f(x_j)$, and conversely, for all $i, j = 1, 2, \ldots, n$. If there exists an embedding of G_n into H_N, we denote this by writing $G_n \prec H_N$. The particular graph H_N is said to be *n-universal* if $G_n \prec H_N$ for every graph G_n with n vertices.

For each positive integer n, let $\lambda(n)$ denote the least integer N for which there exists an n-universal graph H_N. (It is clear that $\lambda(n)$ is finite, since the graph consisting of disjoint copies of all the graphs with n vertices is n-universal.) The object in this note is to establish the following inequalities:

$$2^{\frac{1}{2}(n-1)} \leqq \lambda(n) \leqq \begin{cases} n \,.\, 2^{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd,} \\ \dfrac{3}{2\sqrt{2}}\, n \,.\, 2^{\frac{1}{2}(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

The first inequality is obtained by the following simple argument. There are at least $2^{\binom{n}{2}}/n!$ different graphs G_n, since the labellings assigned to the vertices are immaterial to the problem. Hence, if H_N is n-universal,

$$2^{\binom{n}{2}}/n! \leqq \binom{N}{n} \leqq N^n/n!,$$

since different graphs G_n must be mapped onto different subgraphs with n vertices of H_N. The lower bound for $\lambda(n)$ now follows immediately. Slight improvements may be obtained by using better estimates for the number of different graphs G_n.

To obtain an upper bound for $\lambda(n)$ we proceed as follows. Let T_n be any oriented complete graph with n vertices $y_1, y_2, \ldots, y_n$. Write $y_i \rightarrow y_j$ if the edge joining y_i and y_j is oriented from y_i to y_j ($i \neq j$). Let $Y_i = \{y_j : y_j \rightarrow y_i\}$. Construct a graph H whose vertices $z_{i,A}$ are in one-to-one correspondence with the ordered pairs (i, A), where $A \subset Y_i$. If $A \subset Y_i$, $B \subset Y_j$ and $y_i \rightarrow y_j$, then let $z_{i,A} \circ z_{j,B}$ in H if and only if $y_i \in B$.

We now show that H is n-universal. If G_n has vertices $x_1, x_2, \ldots, x_n$, we may set $f(x_i) = z_{i,A(i)}$, where $A(i) = \{y_j : y_j \rightarrow y_i \text{ and } x_j \circ x_i\}$. Then f is an embedding of G_n into H, since, if $y_i \rightarrow y_j$, we have

$$f(x_i) \circ f(x_j) \;\Leftrightarrow\; z_{i,A(i)} \circ z_{j,A(j)} \;\Leftrightarrow\; y_i \in A(j) \;\Leftrightarrow\; x_i \circ x_j.$$

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Therefore

$$\lambda(n) \leqq (\text{number of vertices of } H) = 2^{|Y_1|} + 2^{|Y_2|} + \dots + 2^{|Y_n|}.$$

To minimise this sum, let T_n be the oriented complete graph in which $y_i \to y_j$ if and only if $0 < j-i \leqq [\frac{1}{2}n]$, where the subtraction is modulo n or $n+1$ according as n is odd or even. For this choice of T_n it is not difficult to see that

$$|Y_1| = \dots = |Y_n| = \tfrac{1}{2}(n-1), \quad \text{if } n \text{ is odd},$$

$$|Y_1| = \dots = |Y_{\frac{1}{2}n}| = \tfrac{1}{2}n, \quad |Y_{\frac{1}{2}n+1}| = \dots = |Y_n|, \quad \text{if } n \text{ is even}.$$

Hence

$$\lambda(n) \leqq n \,.\, 2^{\frac{1}{2}(n-1)}, \quad \text{if } n \text{ is odd},$$

$$\lambda(n) \leqq \tfrac{1}{2}n \,.\, 2^{\frac{1}{2}n} + \tfrac{1}{2}n \,.\, 2^{\frac{1}{2}(n-2)} = \frac{3}{2\sqrt{2}}\, n \,.\, 2^{\frac{1}{2}(n-1)}, \quad \text{if } n \text{ is even}.$$

This completes the proof of the above inequalities.

I am indebted to the referee for suggestions leading to a substantial improvement in the upper bound for $\lambda(n)$.

REFERENCE

1. R. Rado, Universal graphs, *Acta Arith.* (to appear).

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