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NONUNIFORM EXPONENTIAL BEHAVIOUR AND TOPOLOGICAL EQUIVALENCE

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Abstract. We show that any evolution family with a strong nonuniform exponential dichotomy can always be transformed by a topological equivalence to a canonical form that contracts and/or expands the same in all directions. We emphasize that strong nonuniform exponential dichotomies are ubiquitous in the context of ergodic theory. The main novelty of our work is that we are able to control the asymptotic behaviour of the topological conjugacies at the origin and at infinity.

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1. Introduction. We consider an evolution family $U(t, s)$ with a strong nonuniform exponential dichotomy on a Banach space. This means that besides having contraction in the stable direction for positive time and in the unstable direction for negative time, the contraction is at most exponential. We show that a strong nonuniform exponential dichotomy can always be transformed by a topological equivalence to a canonical form that contracts and/or expands the same in all directions. This means that there exist homeomorphisms h_t such that

$$U(t, s) \circ h_s = h_t \circ V(t, s)$$

for every t and s, where $V(t, s)$ is the evolution family determined by an autonomous equation

$$x' = -x, \quad y' = y.$$

This result was first established in [**4**], where it was also shown that there exist always homeomorphisms h_t that are locally Hölder continuous. Here, we use instead an approach inspired in [**12**], which considers the particular case of uniform exponential dichotomies, that allows us to control the asymptotic behaviour of the maps h_t at zero and at infinity (the case of finite-dimensional spaces was considered earlier in [**9**]). We note that this type of control was not obtained in [**4**]. The proof consists of constructing explicitly the homeomorphisms h_t. We refer to [**1**, **10**, **14**] for earlier related work.

The notion of a (uniform) exponential dichotomy, introduced by Perron in [**11**], plays a central role in the stability theory of differential equations. In particular, there exist large classes of linear differential equations with exponential dichotomies. We refer the reader to the books [**6**, **7**, **13**] for details and references. On the other hand, the notion of an exponential dichotomy is too stringent for the dynamics and it is of interest to look for more general types of hyperbolic behaviour that can be much more typical. This is precisely what happens with the notion of a nonuniform exponential dichotomy. We refer the reader to [**5**] for a systematic study of some of its consequences, in particular concerning the existence and smoothness of invariant manifolds, the Grobman–Hartman theorem, and the existence of centre manifolds, among other topics. From the point of view of ergodic theory, the notion of a nonuniform exponential dichotomy is the typical situation for a dynamics with nonzero Lyapunov exponents. The nonuniform part of the dichotomy can be made arbitrarily small for almost all trajectories, although not necessarily zero. This is a simple consequence of Oseledets' multiplicative ergodic theorem in [**8**] (see [**2**] for a detailed discussion). On the other hand, by work in [**3**], for certain classes of measure-preserving transformations, the nonuniform part of the dichotomy cannot be made zero on a set of full topological entropy and full Hausdorff dimension.

We emphasize that strong nonuniform exponential dichotomies are also ubiquitous in the context of ergodic theory. More precisely, for a flow preserving a finite measure and having only nonzero Lyapunov exponents, almost all linear variational equations have a strong nonuniform exponential dichotomy (see [**2**] for details). We note that there are plenty examples of measure-preserving flows. For example, any Hamiltonian system restricted to a compact energy hypersurface is a measure-preserving flow. Moreover, any geodesic flow on a compact manifold with negative curvature is a measure-preserving flow and has nonzero Lyapunov exponents (in fact it is an Anosov flow). We refer the reader to [**2**] for the description of many other examples.

2. Nonuniform exponential behaviour. Let $\mathcal{L}(E)$ be the set of all bounded linear operators acting on a Banach space E. A family of linear operators

$$\mathcal{U} = \left\{U(t, s) : t, s \in \mathbb{R},\ t \geq s\right\} \subset \mathcal{L}(E)$$

is called an *evolution family* if:

(1) $U(t, t) = \mathrm{Id}$ and

$$U(t, s)U(s, r) = U(t, r) \quad \text{for} \quad t \geq s \geq r; \tag{1}$$

(2) for each $s \in \mathbb{R}$ and $x \in E$, the map $t \mapsto U(t, s)x$ is continuous.

Moreover, $\mathcal{U}$ is said to be a *reversible* evolution family if $U(t, s)$ is invertible for all $t \geq s$. In this case, we write $U(s, t) = U(t, s)^{-1}$ for $t \geq s$ and (1) holds for all $t, s, r \in \mathbb{R}$.

Let $\mathcal{F}$ be the set of all increasing continuous functions $\rho\colon \mathbb{R} \to \mathbb{R}$ with $\rho(0) = 0$. Given $\rho \in \mathcal{F}$, we write

$$\Phi(t, s) = \rho(t) - \rho(s).$$

A reversible evolution family $\mathcal{U}$ is said to admit a *strong ρ-nonuniform exponential dichotomy* if there exist complementary projections $P(t) + Q(t) = \mathrm{Id}$ for $t \in \mathbb{R}$ and constants $K, \lambda, \alpha > 0$ with $\lambda \le \alpha$ and $\varepsilon \ge 0$ such that

$$P(t)U(t, s) = U(t, s)P(s) \tag{2}$$

for $t, s \in \mathbb{R}$,

$$\|U(t, s)P(s)\| \le Ke^{-\lambda\Phi(t,s)+\varepsilon|\rho(s)|}, \quad \|U(s, t)Q(t)\| \le Ke^{-\lambda\Phi(t,s)+\varepsilon|\rho(t)|} \tag{3}$$

for $t, s \in \mathbb{R}$ with $t \ge s$ and

$$\|U(t, s)P(s)\| \le Ke^{-\alpha\Phi(t,s)+\varepsilon|\rho(s)|}, \quad \|U(s, t)Q(t)\| \le Ke^{-\alpha\Phi(t,s)+\varepsilon|\rho(t)|} \tag{4}$$

for $t, s \in \mathbb{R}$ with $t \le s$.

Moreover, $\mathcal{U}$ is said to admit a *strong ρ-exponential dichotomy* with respect to a family of norms $\|\cdot\|_t^*$ if there exist complementary projections $P(t) + Q(t) = \mathrm{Id}$ for $t \in \mathbb{R}$ and constants $K, \lambda, \alpha > 0$ with $\lambda \le \alpha$ such that (2) holds for $t, s \in \mathbb{R}$ as well as

$$\|U(t, s)P(s)x\|_t^* \le Ke^{-\lambda\Phi(t,s)}\|x\|_s^*, \quad \|U(s, t)Q(t)x\|_s^* \le Ke^{-\lambda\Phi(t,s)}\|x\|_t^*$$

for $x \in E$ and $t, s \in \mathbb{R}$ with $t \ge s$ and

$$\|U(t, s)P(s)x\|_t^* \le Ke^{-\alpha\Phi(t,s)}\|x\|_s^*, \quad \|U(s, t)Q(t)x\|_s^* \le Ke^{-\alpha\Phi(t,s)}\|x\|_t^*$$

for $x \in E$ and $t, s \in \mathbb{R}$ with $t \le s$.

The following result gives a characterization of the notion of a strong ρ-nonuniform exponential dichotomy.

THEOREM 1. *For a reversible evolution family $\mathcal{U}$, the following properties are equivalent:*

(1) *$\mathcal{U}$ admits a strong ρ-nonuniform exponential dichotomy;*
(2) *$\mathcal{U}$ admits a strong ρ-exponential dichotomy with respect to a family of norms $\|\cdot\|_t^*$ satisfying*

$$\|x\| \le \|x\|_t^* \le Ce^{\varepsilon|\rho(t)|}\|x\| \tag{5}$$

for some constant $C > 0$.

Proof. Assume first that $\mathcal{U}$ admits a strong ρ-nonuniform exponential dichotomy. For $x \in E$ and $t \in \mathbb{R}$, we define

$$\|x\|_t^* = \|x_1\|_t^* + \|x_2\|_t^*,$$

where

$$\|x_1\|_t^* = \sup_{\sigma \ge t}\left(\|U(\sigma, t)P(t)x\|e^{\lambda\Phi(\sigma,t)}\right) + \sup_{\sigma < t}\left(\|U(\sigma, t)P(t)x\|e^{\alpha\Phi(\sigma,t)}\right)$$

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and

$$\|x_2\|_t^* = \sup_{\sigma<t}\left(\|U(\sigma,t)Q(t)x\|e^{\lambda\Phi(t,\sigma)}\right) + \sup_{\sigma\geq t}\left(\|U(\sigma,t)Q(t)x\|e^{\alpha\Phi(t,\sigma)}\right).$$

First, we show that (5) holds. Since

$$\|x_1\|_t^* \geq \|P(t)x\|, \quad \|x_2\|_t^* \geq \|Q(t)x\|,$$

we have

$$\begin{aligned}\|x\| &= \|P(t)x + Q(t)x\| \\ &\leq \|P(t)x\| + \|Q(t)x\| \\ &\leq \|x_1\|_t^* + \|x_2\|_t^* = \|x\|_t^*.\end{aligned}$$

Moreover, using (3) and (4) we obtain

$$\begin{aligned}\|x_1\|_t^* &\leq \sup_{\sigma\geq t}\left(\|U(\sigma,t)P(t)x\|e^{\lambda\Phi(\sigma,t)}\right) + \sup_{\sigma<t}\left(\|U(\sigma,t)P(t)x\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\leq 2Ke^{\varepsilon|\rho(t)|}\end{aligned}$$

and

$$\begin{aligned}\|x_2\|_t^* &\leq \sup_{\sigma<t}\left(\|U(\sigma,t)Q(t)x\|e^{\lambda\Phi(t,\sigma)}\right) + \sup_{\sigma\geq t}\left(\|U(\sigma,t)Q(t)x\|e^{\alpha\Phi(t,\sigma)}\right) \\ &\leq 2Ke^{\varepsilon|\rho(t)|},\end{aligned}$$

which shows that (5) holds with $C = 2K$.

For $y = P(s)x$ and $t \geq s$, we have

$$\begin{aligned}\|U(t,s)y\|_t^* &= \sup_{\sigma\geq t}\left(\|U(\sigma,t)U(t,s)y\|e^{\lambda\Phi(\sigma,t)}\right) \\ &\quad + \sup_{\sigma<t}\left(\|U(\sigma,t)U(t,s)y\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\leq \sup_{\sigma\geq t}\left(\|U(\sigma,s)y\|e^{\lambda\Phi(\sigma,t)}\right) + \sup_{s\leq\sigma<t}\left(\|U(\sigma,s)y\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\quad + \sup_{\sigma<s}\left(\|U(\sigma,s)y\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\leq 2\sup_{\sigma\geq s}\left(\|U(\sigma,s)y\|e^{\lambda\Phi(\sigma,t)}\right) + \sup_{\sigma<s}\left(\|U(\sigma,s)y\|e^{\alpha\Phi(\sigma,t)}\right),\end{aligned}$$

where in the last inequality we have used that $\lambda \leq \alpha$. Hence,

$$\begin{aligned}\|U(t,s)y\|_t^* &\leq 2e^{-\lambda\Phi(t,s)}\sup_{\sigma\geq s}\left(\|U(\sigma,s)y\|e^{\lambda\Phi(\sigma,s)}\right) \\ &\quad + e^{-\alpha\Phi(t,s)}\sup_{\sigma<s}\left(\|U(\sigma,s)y\|e^{\alpha\Phi(\sigma,s)}\right) \\ &\leq 2e^{-\lambda\Phi(t,s)}\|x\|_s^*\end{aligned}$$

again sing $\lambda \leq \alpha$. Analogously, for $z = Q(t)x$ and $t \geq s$, we have

$$\begin{aligned}\|U(s,t)z\|_s^* &= \sup_{\sigma \leq s} \left(\|U(\sigma,s)U(s,t)y\|e^{\lambda\Phi(\sigma,s)}\right) \\ &\quad + \sup_{\sigma > s} \left(\|U(\sigma,s)U(s,t)z\|e^{\alpha\Phi(\sigma,s)}\right) \\ &\leq \sup_{\sigma \leq s} \left(\|U(\sigma,t)z\|e^{\lambda\Phi(\sigma,t)}\right) + \sup_{s<\sigma\leq t} \left(\|U(\sigma,t)z\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\quad + \sup_{\sigma > t} \left(\|U(\sigma,t)z\|e^{\alpha\Phi(\sigma,s)}\right) \\ &\leq 2\sup_{\sigma \leq t} \left(\|U(\sigma,t)z\|e^{\lambda\Phi(\sigma,s)}\right) + \sup_{\sigma > t} \left(\|U(\sigma,t)z\|e^{\alpha\Phi(\sigma,s)}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|U(s,t)z\|_s^* &\leq 2e^{-\lambda\Phi(t,s)} \sup_{\sigma \leq t} \left(\|U(\sigma,t)z\|e^{\lambda\Phi(\sigma,t)}\right) \\ &\quad + e^{-\alpha\Phi(t,s)} \sup_{\sigma > t} \left(\|U(\sigma,t)z\|e^{\alpha\Phi(\sigma,t)}\right) \\ &\leq 2e^{-\lambda\Phi(t,s)}\|x\|_t^*.\end{aligned}$$

One can show in a similar manner that

$$\|U(t,s)P(s)x\|_t^* \leq e^{-\alpha\Phi(t,s)}\|x\|_s^*$$

and

$$\|U(s,t)Q(t)x\|_s^* \leq e^{-\alpha\Phi(t,s)}\|x\|_t^*$$

for $t \leq s$. Therefore, $\mathcal{U}$ admits a strong ρ-exponential dichotomy with respect to the family of norms $\|\cdot\|_t^*$.

Conversely, assume that $\mathcal{U}$ admits a strong ρ-exponential dichotomy with respect to a family of norms $\|\cdot\|_t^*$ satisfying (5). Then,

$$\begin{aligned}\|U(t,s)P(s)x\| &\leq \|U(t,s)P(s)x\|_t^* \\ &\leq Ke^{-\lambda\Phi(t,s)}\|x\|_s^* \\ &\leq KCe^{-\lambda\Phi(t,s)+\varepsilon|\rho(s)|}\|x\|\end{aligned}$$

and

$$\begin{aligned}\|U(s,t)Q(t)x\| &\leq \|U(t,s)Q(s)x\|_s^* \\ &\leq Ke^{-\lambda\Phi(t,s)}\|x\|_t^* \\ &\leq KCe^{-\lambda\Phi(t,s)+\varepsilon|\rho(t)|}\|x\|\end{aligned}$$

for $x \in E$ and $t \geq s$. Similarly,

$$\|U(t,s)P(s)x\| \leq KCe^{-\alpha\Phi(t,s)+\varepsilon|\rho(s)|}\|x\|$$

and

$$\|U(s,t)Q(t)x\| \leq KCe^{-\alpha\Phi(t,s)+\varepsilon|\rho(t)|}\|x\|$$

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for $x \in E$ and $t \le s$. This shows that $\mathcal{U}$ admits a strong ρ-nonuniform exponential dichotomy. $\square$

3. Reduction to a canonical form. In this section, we show that an evolution family admitting a strong ρ-nonuniform exponential dichotomy can always be transformed to a canonical form that expands and/or contracts the same in all directions.

We first introduce the notion of a topological equivalence. Two reversible evolution families $\mathcal{U} = \{U(t,s)\}$ and $\mathcal{V} = \{V(t,s)\}$ are *ρ-nonuniformly topologically equivalent* if there exists a continuous map $h\colon \mathbb{R} \times E \to E$, an increasing onto map $L\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a constant $c \ge 0$ such that:

(1) $h_t = h(t,\cdot)\colon E \to E$ is a homeomorphism for $t \in \mathbb{R}$;
(2) $U(t,s) \circ h_s = h_t \circ V(t,s)$ for $t, s \in \mathbb{R}$;
(3) for $t \in \mathbb{R}$ and $x \in E$,

$$\|h_t(x)\| \le L(e^{c|\rho(t)|}\|x\|) \quad \text{and} \quad \|h_t^{-1}(x)\| \le L(e^{c|\rho(t)|}\|x\|).$$

THEOREM 2. *Given a reversible evolution family $\mathcal{U}$ and a C^1 onto function $\rho \in \mathcal{F}$, if $\mathcal{U}$ admits a strong ρ-nonuniform exponential dichotomy, then it is ρ-nonuniformly topologically equivalent to the evolution family*

$$V(t,s) = e^{\Phi(s,t)}P(0) + e^{\Phi(t,s)}Q(0).$$

Proof. Write $\omega = \rho'$ (note that ω is a continuous function). For each $t \in \mathbb{R}$ and $x \in E$, let

$$\|x\|_t = \int_t^\infty \|U_P(\xi,t)x\|_\xi^* \,\omega(\xi)\, d\xi + \int_{-\infty}^t \|U_Q(\xi,t)x\|_\xi^* \,\omega(\xi)\, d\xi,$$

where

$$U_P(t,s) = U(t,s)P(s) \quad \text{and} \quad U_Q(t,s) = U(t,s)Q(s).$$

Since $\omega > 0$, the map $x \mapsto \|x\|_t$ is a norm on E for each t.

LEMMA 1. *For $t \in \mathbb{R}$ and $x \in E$, we have*

$$\frac{1}{K\alpha}\|x\|_t^* \le \|x\|_t \le \frac{4K}{\lambda}\|x\|_t^*. \tag{6}$$

Proof of the lemma. Clearly,

$$\|x\|_t = \|P(t)x\|_t + \|Q(t)x\|_t,$$

where

$$\|P(t)x\|_t = \int_t^\infty \|U_P(\xi,t)x\|_\xi^* \omega(\xi)\, d\xi$$

and

$$\|Q(t)x\|_t = \int_{-\infty}^t \|U_Q(\xi,t)x\|_\xi^* \omega(\xi)\, d\xi.$$

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By Theorem 1, we have

$$\|P(t)x\|_t \leq 2K\|x\|_t^* \int_t^\infty e^{-\lambda\Phi(\xi,t)}\omega(\xi)\,d\xi = \frac{2K}{\lambda}\|x\|_t^*$$

and similarly,

$$\|Q(t)x\|_t \leq \frac{2K}{\lambda}\|x\|_t^*. \tag{7}$$

This yields the second inequality in (6). Moreover, for $\xi \geq t$ we have

$$\begin{aligned}\|P(t)x\|_t^*\omega(\xi) &\leq \|U_P(t,\xi)\|_t^*\|U_P(\xi,t)x\|_\xi^*\omega(\xi)\\ &\leq Ke^{\alpha\Phi(\xi,t)}\|U_P(\xi,t)x\|_\xi^*\omega(\xi).\end{aligned}$$

Therefore,

$$\|P(t)x\|_t \geq \frac{\|P(t)x\|_t^*}{K}\int_t^\infty e^{-\alpha\Phi(\xi,t)}\omega(\xi)\,d\xi = \frac{\|P(t)x\|_t^*}{\alpha K}$$

and analogously,

$$\|Q(t)x\|_t \geq \frac{\|Q(t)x\|_t^*}{\alpha K}. \tag{8}$$

The first inequality in (6) follows now readily from (7) and (8). □

Write $U(t) = U(t,0)$, $P = P(0)$ and $Q = Q(0)$.

LEMMA 2. *The following properties hold:*

(1) $t \mapsto \|U(t)Px\|_t$ *is strictly decreasing and* $t \mapsto \|U(t)Qx\|_t$ *is strictly increasing;*
(2) *for each* $x \in (PE \cup QE) \setminus \{0\}$, *there exists a unique* $t \in \mathbb{R}$ *such that* $\|U(t)x\|_t = 1$.

Proof of the lemma. Since

$$\|U(t)Px\|_t = \int_t^\infty \|U_P(\xi,0)x\|_\xi^*\omega(\xi)\,d\xi, \tag{9}$$

we have

$$\frac{d}{dt}\|U(t)Px\|_t = -\|U_P(t,0)x\|_t^*\omega(t) < 0.$$

Similarly, since

$$\|U(t)Qx\|_t = \int_{-\infty}^t \|U_Q(\xi,0)x\|_\xi^*\omega(\xi)\,d\xi,$$

we have

$$\frac{d}{dt}\|U(t)Qx\|_t = \|U_Q(t,0)x\|_t^*\omega(t) > 0.$$

This yields the first property.

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Now, we establish the second property. By (9), we have $\|U(t)Px\|_t \to 0$ when $t \to \infty$. On the other hand, for $t \le \xi \le 0$ we have

$$\begin{aligned}\|Px\|_0^*\omega(\xi) &\le \|U_P(0,\xi)\|_0^*\|U_P(\xi,0)x\|_\xi^*\omega(\xi)\\ &\le Ke^{\lambda\rho(\xi)}\|U_P(\xi,0)x\|_\xi^*\omega(\xi)\end{aligned}$$

and thus,

$$\begin{aligned}\|U(t)Px\|_t &\ge \int_t^0 \|U_P(\xi,0)x\|_\xi^*\omega(\xi)\,d\xi\\ &\ge \frac{\|Px\|_0^*}{K}\int_t^0 e^{-\lambda\rho(\xi)}\omega(\xi)\,d\xi = \frac{\|Px\|_0^*}{K\lambda}e^{-\lambda\rho(t)}.\end{aligned}$$

Therefore, $\|U(t)Px\|_t = \infty$ when $t \to -\infty$, provided that $Px \ne 0$. One can show in a similar manner that

$$\lim_{t\to\infty}\|U(t)Qx\|_t = \infty \quad \text{and} \quad \lim_{t\to-\infty}\|U(t)Qx\|_t = 0,$$

provided that $Qx \ne 0$. The desired property follows now readily from the first property. $\square$

We proceed with the proof of the theorem. Define maps

$$h_t^P\colon PE \to P(t)E \quad \text{and} \quad h_t^Q\colon QE \to Q(t)E$$

by

$$h_t^P(x) = \begin{cases} U(t)x/\|U(\rho^{-1}(\rho(t)+\log\|x\|_t^*))x\|_{\rho^{-1}(\rho(t)+\log\|x\|_t^*)}, & x \ne 0,\\ 0, & x = 0\end{cases} \tag{10}$$

and

$$h_t^Q(x) = \begin{cases} U(t)x/\|U(\rho^{-1}(\rho(t)-\log\|x\|_t^*))x\|_{\rho^{-1}(\rho(t)-\log\|x\|_t^*)}, & x \ne 0,\\ 0, & x = 0.\end{cases} \tag{11}$$

Finally, we define $h_t\colon E \to E$ by

$$h_t(x) = h_t^P(Px) + h_t^Q(Qx).$$

In order to show that these maps yield a topological equivalence, we divide the proof into steps.

Step 1. Invariance. We first show that

$$h_t\big(e^{\Phi(s,t)}Px + e^{\Phi(t,s)}Qx\big) = U(t,s)h_s(x). \tag{12}$$

For $x \in PE$, we have

$$\begin{aligned}h_t^P(e^{-\rho(t)}x) &= \frac{e^{-\rho(t)}U(t)x}{e^{-\rho(t)}\|U(\rho^{-1}(s))x\|_{\rho^{-1}(s)}}\\ &= \frac{U(t)x}{\|U(\rho^{-1}(\log\|x\|_t^*))x\|_{\rho^{-1}(\log\|x\|_t^*)}} = U(t)h_0^P(x),\end{aligned}$$

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where

$$s = \rho(t) + \log\|e^{-\rho(t)}x\|_t^* = \log\|x\|_t^*,$$

and analogously,

$$h_t^Q(e^{\rho(t)}x) = U(t)h_0^Q(x)$$

for $x \in QE$. Adding the former identities, we obtain $h_t \circ V(t) = U(t) \circ h_0$, where $V(t) = V(t, 0)$. This readily implies that

$$h_t \circ V(t, s) = U(t, s) \circ h_s$$

and so identity (12) holds.

Step 2. Injectivity of the maps h_t. Assume that $h_t^P(x) = h_t^P(y)$. Then,

$$\frac{x}{\|U(\tau_1)x\|_{\tau_1}} = \frac{y}{\|U(\tau_2)y\|_{\tau_2}} = \xi \in PE, \tag{13}$$

where

$$\tau_1 = \rho^{-1}(\rho(t) + \log\|x\|_t^*)$$

and

$$\tau_2 = \rho^{-1}(\rho(t) + \log\|y\|_t^*).$$

Therefore,

$$\|U(\tau_1)\xi\|_{\tau_1} = \|U(\tau_2)\xi\|_{\tau_2} = 1,$$

which implies that $\tau_1 = \tau_2$. Hence, $\|x\| = \|y\|$ and it follows from (13) that

$$c := \|U(\tau_1)x\|_{\tau_1} = \|U(\tau_2)y\|_{\tau_2}.$$

Therefore, $x = y = c\xi$. The injectivity of the maps h_t^Q can be proved in a similar manner. This readily implies that the maps h_t are one-to-one.

Step 3. Surjectivity of the maps h_t. Take $y \in P(t)E$. If $h_t^P(x) = y$, then

$$\frac{x}{\|U(\tau)x\|_\tau} = PU(t)^{-1}y,$$

where

$$\tau = \rho^{-1}(\rho(t) + \log\|x\|_t^*). \tag{14}$$

Therefore, $\|U(\tau)PU(t)^{-1}y\|_\tau = 1$ (the existence and uniqueness of τ is guaranteed by Lemma 2). By (14), we obtain

$$e^{\Phi(\tau,t)} = \|x\|_t^* = \|U(\tau)x\|_\tau \|PU(t)^{-1}y\|_t^*,$$

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that is,

$$\|U(\tau)x\|_\tau = \frac{\|x\|_t^*}{\|PU(t)^{-1}y\|_t^*} = \frac{e^{\Phi(\tau,t)}}{\|PU(t)^{-1}y\|_t^*}.$$

Moreover,

$$x = \|U(\tau)x\|_\tau PU(t)^{-1}y = e^{\Phi(\tau,t)}\frac{PU(t)^{-1}y}{\|PU(t)^{-1}y\|_t^*}.$$

Similarly, if $h_t^Q(x) = y$, then

$$x = e^{\Phi(t,\eta)}\frac{QU(t)^{-1}y}{\|QU(t)^{-1}y\|_t^*}$$

for a unique η (whose existence and uniqueness is guaranteed by Lemma 2).

This shows that h_t^P and h_t^Q are invertible, with inverses given by

$$(h_t^P)^{-1}(y) = e^{\Phi(\tau,t)}\frac{PU(t)^{-1}y}{\|PU(t)^{-1}y\|_t^*} \tag{15}$$

and

$$(h_t^Q)^{-1}(y) = e^{\Phi(t,\eta)}\frac{QU(t)^{-1}y}{\|QU(t)^{-1}y\|_t^*}. \tag{16}$$

Then, h_t is also invertible and its inverse $h_t^{-1}\colon E \to E$ is given by

$$h_t^{-1}(y) = (h_t^P)^{-1}(P(t)y) + (h_t^Q)^{-1}(Q(t)y).$$

Step 4. Existence of the map L. Take $x \in PE$. By (6) and (10), for

$$\tau = \rho^{-1}(\rho(t) + \log\|x\|_t^*)$$

we have

$$\|h_t^P(x)\|_t^* = \frac{\|U(t)x\|_t^*}{\|U(\tau)x\|_\tau} \le K\alpha\frac{\|U(t)x\|_t^*}{\|U(\tau)x\|_\tau^*}.$$

If $\|x\|_t^* \le 1$, then since $e^{\Phi(\tau,t)} = \|x\|_t^*$, we have $\tau \le t$ and using (5),

$$\begin{aligned}\|h_t^P(x)\| &\le \|h_t^P(x)\|_t^* \\ &\le \alpha K\frac{\|U_P(t,\tau)\|_t^*\|U(\tau)x\|_\tau^*}{\|U(\tau)x\|_\tau^*} \\ &\le \alpha K^2 e^{-\lambda\Phi(t,\tau)} = \alpha K^2(\|x\|_t^*)^\lambda \\ &\le \alpha K^2 C^\lambda e^{\varepsilon\lambda|\rho(t)|}\|x\|^\lambda.\end{aligned} \tag{17}$$

If $\|x\|_t^* > 1$, then $\tau > t$ and using (5),

$$\begin{aligned}\|h_t^P(x)\| \le \|h_t^P(x)\|_t^* \le K\alpha\|U_P(t,\tau)\|_t^* \le \alpha K^2 e^{\alpha\Phi(\tau,t)} \\ = \alpha K^2(\|x\|_t^*)^\alpha \le \alpha K^2 C^\alpha e^{\varepsilon\alpha|\rho(t)|}\|x\|^\alpha.\end{aligned} \tag{18}$$

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Similarly, using (11), we obtain

$$\|h_t^Q(x)\|_t^* = \frac{\|U(t)x\|_t^*}{\|U(\eta)x\|_\eta} \le \alpha K \frac{\|U(t)x\|_t^*}{\|U(\eta)x\|_\eta^*}$$

for $x \in QE$, where

$$\eta = \rho^{-1}(\rho(t) - \log\|x\|_t^*)$$

or, equivalently, $\|x\|_t^* = e^{\Phi(t,\eta)}$. If $\|x\|_t^* \le 1$, then $t \le \eta$ and using (5),

$$\begin{aligned} \|h_t^Q(x)\| &\le \|h_t^Q(x)\|_t^* \le \alpha K^2(\|x\|_t^*)^\lambda \\ &\le \alpha K^2 C^\lambda e^{\varepsilon\lambda|\rho(t)|}\|x\|^\lambda. \end{aligned} \tag{19}$$

On the other hand, if $\|x\|_t^* > 1$, then $t > \eta$ and using (5),

$$\begin{aligned} \|h_t^Q(x)\| &\le \|h_t^Q(x)\|_t^* \le \alpha K^2(\|x\|_t^*)^\alpha \\ &\le \alpha K^2 C^\alpha e^{\varepsilon\alpha|\rho(t)|}\|x\|^\alpha. \end{aligned} \tag{20}$$

Now we observe that by (15),

$$\|(h_t^P)^{-1}(y)\|_t^* = e^{\Phi(\tau,t)}, \tag{21}$$

where τ is determined by the identity

$$\begin{aligned} 1 &= \|U(\tau)PU(t)^{-1}y\|_\tau = \|U_P(\tau, t)y\|_\tau \\ &= \int_\tau^\infty \|U_P(\xi, 0)U(t)^{-1}y\|_\xi^*\omega(\xi)\,d\xi \\ &= \int_\tau^\infty \|U_P(\xi, t)y\|_\xi^*\omega(\xi)\,d\xi. \end{aligned}$$

If $t \ge \tau$, then

$$\begin{aligned} \int_\tau^t \|U_P(\xi, t)y\|_\xi^*\omega(\xi)\,d\xi &= \int_\tau^\infty \|U_P(\xi, t)y\|_\xi^*\omega(\xi)\,d\xi - \int_t^\infty \|U_P(\xi, t)y\|_\xi^*\omega(\xi)\,d\xi \\ &= 1 - \|y\|_t \end{aligned}$$

and thus $\|y\|_t > 1$. By (6) and Lemma 2, we obtain

$$\begin{aligned} \int_\tau^t \|U_P(\xi, t)y\|_\xi^*\omega(\xi)\,d\xi &\le K\alpha \int_\tau^t \|U_P(\xi, t)y\|_\xi\omega(\xi)\,d\xi \\ &= K\alpha \int_{\rho(\tau)}^{\rho(t)} \|U_P(\rho^{-1}(\eta), t)y\|_{\rho^{-1}(\eta)}\,d\eta \\ &\le K\alpha(\rho(t) - \rho(\tau)) \sup_{\rho(t)\le\eta\le\rho(\tau)} \|U_P(\rho^{-1}(\eta), t)y\|_{\rho^{-1}(\eta)} \\ &= K\alpha(\rho(t) - \rho(\tau))\|U_P(\tau, t)y\|_\tau \\ &= K\alpha\Phi(t, \tau). \end{aligned}$$

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Hence, using (5) and (6), yields that

$$1-\frac{K}{\lambda}Ce^{\varepsilon|\rho(t)|}\|y\|\leq 1-\frac{K}{\lambda}\|y\|_t^*\leq 1-\|y\|_t=K\alpha\Phi(t,\tau)$$

and thus,

$$\Phi(\tau,t)\leq\frac{K\alpha}{1-\frac{KC}{\lambda}e^{\varepsilon|\rho(t)|}\|y\|}\leq-\frac{\alpha\lambda e^{-\varepsilon|\rho(t)|}}{C\|y\|}.$$

By (21), we conclude that

$$\|(h_t^P)^{-1}(y)\|\leq\|(h_t^P)^{-1}(y)\|_t^*\leq\exp\left[-\frac{\alpha\lambda e^{-\varepsilon|\rho(t)|}}{C\|y\|}\right]. \tag{22}$$

On the other hand, if $t<\tau$, then

$$\begin{aligned}\int_t^\tau\|U_P(\xi,t)y\|_\xi^*\omega(\xi)\,d\xi&=\int_t^\infty\|U_P(\xi,t)y\|_\xi^*\omega(\xi)\,d\xi\\&\quad-\int_\tau^\infty\|U_P(\xi,t)y\|_\xi^*\omega(\xi)\,d\xi\\&=\|y\|_t-1\end{aligned}$$

and thus $\|y\|_t>1$. By (6) and Lemma 2, we obtain

$$\begin{aligned}\int_t^\tau\|U_P(\xi,t)y\|_\xi^*\omega(\xi)\,d\xi&\geq\frac{\lambda}{4K}\int_t^\tau\|U_P(\xi,t)y\|_\xi\omega(\xi)\,d\xi\\&=\frac{\lambda}{4K}\int_{\rho(t)}^{\rho(\tau)}\|U_P(\rho^{-1}(\eta),t)y\|_{\rho^{-1}(\eta)}\,d\eta\\&\geq\frac{\lambda}{4K}(\rho(\tau)-\rho(t))\inf_{\rho(t)\leq\eta\leq\rho(\tau)}\|U_P(\rho^{-1}(\eta),t)y\|_{\rho^{-1}(\eta)}\\&=\frac{\lambda}{4K}(\rho(\tau)-\rho(t))\|U_P(\tau,t)y\|_\tau\\&=\frac{\lambda}{4K}\Phi(\tau,t).\end{aligned}$$

Hence,

$$\begin{aligned}\Phi(\tau,t)&\leq\frac{4K}{\lambda}(\|y\|_t-1)\leq\frac{4K}{\lambda}\left(\frac{4K}{\lambda}\|y\|_t^*-1\right)\\&\leq\frac{4K}{\lambda}\left(\frac{4KC}{\lambda}e^{\varepsilon|\rho(t)|}\|y\|-1\right),\end{aligned}$$

which by (21) yields that

$$\begin{aligned}\|(h_t^P)^{-1}(y)\|&\leq\|(h_t^P)^{-1}(y)\|_t^*\\&\leq\exp\left[\frac{4K}{\lambda}\left(\frac{4K}{\lambda}CKe^{\varepsilon|\rho(t)|}\|y\|-1\right)\right].\end{aligned}\tag{23}$$

Similarly, using (16), we obtain

$$\|(h_t^Q)^{-1}(y)\|_t^*=e^{\Phi(t,\eta)}$$

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for $y \in QE$, where η is determined by the identity

$$\begin{aligned} 1 &= \int_{-\infty}^{\eta} \| U_Q(\xi, 0) U(t)^{-1} y\|_{\xi}^{*} \omega(\xi)\, d\xi \\ &= \int_{-\infty}^{\eta} \| U_Q(\xi, t) y\|_{\xi}^{*} \omega(\xi)\, d\xi. \end{aligned}$$

One can show in analogous manner that if $t < \eta$, then

$$\|(h_t^Q)^{-1}(y)\| \le \exp\left[-\frac{\alpha\lambda e^{-\varepsilon|\rho(t)|}}{C\|y\|}\right] \tag{24}$$

and if $t \ge \eta$, then

$$\|(h_t^Q)^{-1}(y)\| \le \exp\left[\frac{4K}{\lambda}\left(\frac{4K}{\lambda} CKe^{\varepsilon|\rho(t)|}\|y\| - 1\right)\right]. \tag{25}$$

It follows readily from (17)–(20), (22)–(25) that there exists a map L with the desired properties. This completes the proof of the theorem. □

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REFERENCES

1. J. Appell, V. Lakshmikantham, N. Van Minh and P. Zabreiko, A general model of evolutionary processes. Exponential dichotomy. I, II, *Nonlinear Anal.* **21** (1993), 207–218, 219–225.

2. L. Barreira and Ya. Pesin, *Nonuniform hyperbolicity*, Encyclopedia of Mathematics and its Applications, vol. 115 (Cambridge University Press, Cambridge, 2007).

3. L. Barreira and J. Schmeling, Sets of “non-typical” points have full topological entropy and full Hausdorff dimension, *Isr. J. Math.* **116** (2000), 29–70.

4. L. Barreira and C. Valls, Conjugacies for linear and nonlinear perturbations of non-uniform behavior, *J. Funct. Anal.* **253** (2007), 324–358.

5. L. Barreira and C. Valls, *Stability of nonautonomous differential equations*, Lect. Notes. in Math., vol. 1926 (Springer, Berlin, 2008).

6. J. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs, vol. 25 (American Mathematical Society, Providence, 1988).

7. D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes in Math., vol. 840 (Springer, Berlin-New York, 1981).

8. V. Oseledets, A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.* **19** (1968), 197–221.

9. K. Palmer, A characterization of exponential dichotomy in terms of topological equivalence, *J. Math. Anal. Appl.* **69** (1979), 8–16.

10. K. Palmer, The structurally stable linear systems on the half-line are those with exponential dichotomies, *J. Differ. Equ.* **33** (1979), 16–25.

11. O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, *Math. Z.* **32** (1930), 703–728.

12. L. H. Popescu, A topological classification of linear differential equations on Banach spaces, *J. Differ. Equ.* **203** (2004), 28–37.

13. G. Sell and Y. You, *Dynamics of evolutionary equations*, Applied Mathematical Sciences, vol. 143 (Springer, New York, 2002).

14. N. Van Minh, On a topological classification of nonautonomous differential equations, in *Qualitative theory of differential equations (Szeged, 1988)*, Colloq. Math. Soc. János Bolyai, vol. 53 (North-Holland, 1990), 421–426.