DIMENSION OF IDEALS IN POLYNOMIAL RINGS

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1. Introduction. A well-known theorem asserts that if K is a field, $\mathfrak{P}$ a prime ideal in the polynomial ring $S = K[X_1, \ldots X_n]$ and d the transcendence degree of $S/\mathfrak{P}$ over K

$$n = \text{rank } \mathfrak{P} + d.$$

In the first half of this paper we extend this result to the case of arbitrary commutative noetherian K, as well as giving a purely homological proof of the classical theorem. In the second half we use our first result to compute the analogue of the dimension of the product and intersection of two affine varieties when K is a Dedekind ring. This seems to be of some interest in view of **(4)**.

We shall adhere to the following notations throughout: K will always be a commutative noetherian ring with unit and $S = K[X_1, \ldots X_n]$ the ring of polynomials in n indeterminates over K. For a prime ideal $\mathfrak{P}$ in S with $\mathfrak{P} \cap K = \mathfrak{p}$, the *dimension* of $\mathfrak{P} = d(\mathfrak{P})$, is the transcendence degree of the field of quotients of $S/\mathfrak{P}$ over the field of quotients of $K/\mathfrak{p}$; $r(\mathfrak{P})$ is the rank of $\mathfrak{P}$. If R is a local ring, dim R is the Krull dimension = the rank of the maximal ideal = the minimal number of non-zero generators of an ideal containing some power of the maximal one. Finally, if M is a module over a ring R, $hd_R M$ is the projective dimension of M **(2**, p. 109**)**, $w.\ hd_R M$ is the weak dimension of M **(2**, VI, Ex.3**)** and gl. dim R is the global dimension of R **(2**, p. 111**)**.

2. Dimension.

LEMMA 1. *Let R be a local ring, $\mathfrak{a}$ and $\mathfrak{b}$ proper ideals in R such that $\mathfrak{b} \supset \mathfrak{a}^s$ for some integer s. If $\mathfrak{b}$ can be generated by t elements, then* dim $R \leqslant$ dim $R/\mathfrak{a} + t$.

Proof. Let $\mathfrak{m}$ be the maximal ideal in R and dim $R/\mathfrak{a} = r$. Then there exist r elements $x_1, \ldots, x_r$ in $\mathfrak{m}$ such that $\mathfrak{m}^n \subset (x_1, \ldots, x_r) + \mathfrak{a}$ for some integer $n > 0$. Then $\mathfrak{m}^{ns} \subset (x_1, \ldots, x_r) + \mathfrak{a}^s \subset (x_1, \ldots, x_r) + \mathfrak{b}$. Thus $(x_1, \ldots, x_r) + \mathfrak{b}$ is an $\mathfrak{m}$-primary ideal generated by $r + t$ elements and therefore dim $R \leqslant r + t$.

THEOREM 2. *Let P be a prime ideal in the polynomial ring S, $\mathfrak{p}$ the prime ideal $K \cap \mathfrak{P}$ in K. Then*

$$d(\mathfrak{P}) + r(\mathfrak{P}) = n + r(\mathfrak{p}).$$

Proof. Let Q be the field of quotients of $K/\mathfrak{p}$. Then the natural epimorphism $K_\mathfrak{p} \to Q$ induces an epimorphism $\phi : K_\mathfrak{p}[X] \to Q[X]$ where X denotes the

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n-tuple $(X_1, \ldots, X_n)$. Let $\overline{\mathfrak{P}}$ be the prime ideal $K_{\mathfrak{p}}[X]\mathfrak{P}$ in $K_{\mathfrak{p}}[X]$. Since $\overline{\mathfrak{P}}$ contains $\mathfrak{p}K_{\mathfrak{p}}[X] = Ker\,\phi$, the ideal $\phi\,(\overline{\mathfrak{P}})$ is a prime ideal in $Q[X]$. Now $K_{\mathfrak{p}}[X]/\overline{\mathfrak{P}}$ has the same field of quotients as $K[X]/\mathfrak{P}$. Thus the transcendence degree of

$$Q[X]/\phi(\overline{\mathfrak{P}}) \approx K_{\mathfrak{p}}[X]/\overline{\mathfrak{P}}$$

over Q is $d(\mathfrak{P})$. Since Q is a field, we have the well-known classical result (for which we give a homological proof in Proposition 3) that

$$n = d(\mathfrak{P}) + \text{rank}\,\phi\,(\overline{\mathfrak{P}}).$$

Therefore to complete the proof it suffices to show that rank $\phi(\overline{\mathfrak{P}}) =$ rank $\mathfrak{P}$ − rank $\mathfrak{p}$.

The epimorphism $\phi : K_{\mathfrak{p}}[X] \to Q[X]$ induces an epimorphism

$$K_{\mathfrak{p}}[X]_{\overline{\mathfrak{P}}} \to Q[X]_{\phi(\overline{\mathfrak{P}})} \text{ with kernel } \quad \mathfrak{p}K_{\mathfrak{p}}[X]_{\overline{\mathfrak{P}}}\,.$$

If we let

$$R = K_{\mathfrak{p}}[X]_{\overline{\mathfrak{P}}} \text{ we have } R/\mathfrak{p}R \approx Q[X]_{\phi(\overline{\mathfrak{P}})}\,.$$

Using the facts that passage to rings of quotients and polynomial rings do not change the ranks of prime ideals (**5**, p. 57, p. 67) we deduce the following equalities: dim $R/\mathfrak{p}R$ = rank $\phi(\overline{\mathfrak{P}})$, rank $\mathfrak{p}$ = rank $\mathfrak{p}R$ and dim R = rank $\overline{\mathfrak{P}}$. Therefore the equality we wish to prove becomes dim R = dim $R/\mathfrak{p}R$ + rank $\mathfrak{p}R$. Let $x_1, \ldots, x_r$ in $\mathfrak{p}K_{\mathfrak{p}}$ be a system of parameters in $K_{\mathfrak{p}}$. Then for some integer s, we have that $(\mathfrak{p}K_{\mathfrak{p}})^s \subset (x_1, \ldots, x_r)$ and therefore $(\mathfrak{p}R)^s \subset (x_1, \ldots, x_r)R$. Applying Lemma 1 we have that dim $R \leqslant$ dim $R/\mathfrak{p}R$ + rank $\mathfrak{p}R$. The reverse inequality follows trivially from the definition of dim R.

PROPOSITION 3. *Let K be a field, $\mathfrak{P}$ a prime ideal in S. Then*

$$n = d(\mathfrak{P}) + r(\mathfrak{P}).$$

Proof. Let R be the local ring $S_{\mathfrak{P}}$. Since $F = R/\mathfrak{P}R$ is isomorphic (as a K-algebra) to the field of quotients of $S/\mathfrak{P}$, the transcendence degree of F over K is $d(\mathfrak{P}) = d$. Let $t_1, \ldots, t_d$ in R have the property that their images in F form a transcendence base for F over K. We consider F to be a module over the polynomial ring $R[Y_1, \ldots, Y_d]$ by defining $Y_i f = t_i f$ for all $f \in F$. Then to prove Proposition 3 we shall compute $hd_{R[Y]}F$ in two different ways.

Since $K[Y] \subset R[Y]$, the set $N = K[Y] - \{0\}$ is a multiplicatively closed subset of $R[Y] - \{0\}$. Then $R[Y]_N = (R \otimes_K K[Y])_N = R \otimes_K K(Y)$ and [1] $F_N = F$. Thus, applying (**2**, VII, Ex. 10) and (**2**, VI, Ex. 3b) we have [2]

$$hd_{R[Y]}F = w.hd_{R[Y]}F = w.hd_{R[Y]_N}F_N = hd_{R \otimes K(Y)}F.$$

Since K is a field, we know that

[1]For the definition of F_N see (**2**, VII Ex. **9**).

[2]The unadorned $\otimes$ refers to a tensor product over K.

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(1) $$\operatorname{Ext}^q_{R \otimes R}(R, \operatorname{Hom}_{K(Y)}(F, C)) \approx \operatorname{Ext}^q_{R \otimes K(Y)}(F, C)$$

for all q and all $R \otimes K(Y)$-modules C **(2**, IX, 4.3**)**. Let T be the multiplicatively closed subset of $S \otimes S$ consisting of all $h \otimes k$ where $h,k \notin \mathfrak{P}$. Consider the exact sequence

$$0 \to \mathfrak{J} \to S \otimes S \xrightarrow{\phi} S \to 0$$

where $\phi(f \otimes g) \to fg$ and $\mathfrak{J} = \operatorname{Ker} \phi$. Since $\phi(T)$ does not contain 0, the set T does not meet $\mathfrak{J}$. Therefore we obtain the exact sequence

$$0 \to \mathfrak{J}_T \to (S \otimes S)_T \to S_T \to 0.$$

It is easily seen that $(S \otimes S)_T = R \otimes R$ and that $S_T = R$, that is, we have an exact sequence

$$0 \to \mathfrak{J}_T \to R \otimes R \to R \to 0.$$

Clearly each ideal $\mathfrak{J}^{(k)}$ generated in $S \otimes S$ by $X_i \otimes 1 - 1 \otimes X_i$ for $i = 1, \ldots k$, is a prime ideal and $\mathfrak{J}^{(n)} = \mathfrak{J}$. Thus, each of the ideals $\mathfrak{J}_T^{(k)}$ in $R \otimes R$ is generated by $X_i \otimes 1 - 1 \otimes X_i$ for $i = 1, \ldots k$ and is a prime ideal, and also $\mathfrak{J}_T^{(n)} = \mathfrak{J}_T$. Hence, the hypotheses of **(2**, VIII, 4.2**)** are satisfied, and we find[3] that $hd_{R \otimes R} R = n$, so that in view of (1)

$$hd_{R \otimes K(Y)} F \leqslant n.$$

Furthermore, by the discussion on p. 153 of **(2)**, it also follows that

$$\operatorname{Ext}^n_{R \otimes R}(R, D) \approx D/\mathfrak{J}_T D.$$

Let $D = \operatorname{Hom}_{K(Y)}(F, F)$. Then $(r \otimes r'g)(f) = r(g(r'f))$ for all $r \otimes r' \in R \otimes R$, $g \in \operatorname{Hom}_{K(Y)}(F, F)$ and $f \in F$. Now every element in $\mathfrak{J}_T$ is a sum of elements of the form $r \otimes 1(r' \otimes 1 - 1 \otimes r')$ **(2**, p. 168**)**. So every element in $\mathfrak{J}_T \operatorname{Hom}_{K(Y)}(F, F)$ is a sum of elements of the form $LL'g - LgL'$, where L and L' stand for the linear transformation r and r' induce on F by multiplication. Since the Y's act on F as a transcendence basis of F over K, and since F is a finitely generated extension of K, it follows that $[F : K(Y)] < \infty$. Thus, every element of D has a well-defined trace. Then since $LL' = L'L$ the trace of $LL'g - LgL' = L'(Lg) - (Lg)L'$ is zero. Therefore every element of $\mathfrak{J}_T D$ has trace zero and so $D \neq \mathfrak{J}_T D$. Consequently

$$\operatorname{Ext}^n_{R \otimes R}(R, \operatorname{Hom}_{K(Y)}(F, F)) \neq 0$$

and so by (1) $hd_{R[Y]} F = n$.

We now show that $hd_{R[Y]} F = d + r(\mathfrak{P})$ which will complete the proof.

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[3]Since

$$hd_{R \otimes R} R \geqslant \text{gl. dim } R$$

(2, IX, 7.6**)**, we have that gl. dim. $R < \infty$. Applying **(1**, Theorem 1.10**)**, we obtain a homological proof of the well-known fact that R is a regular local ring.

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Let $t_1, \ldots, t_d$ be the elements in R such that $Y_i f = t_i f$ for all $f \in F$. If we let $Z_i = Y_i - t_i$ $(i = 1, \ldots, d)$ we have that the Z_i are algebraically independent over R, $R[Z_1, \ldots, Z_d] = R[Y]$ and $Z_i F = 0$ for all i. Thus, the operation of $R[Z]$ on F is defined by the natural epimorphism $R[Z] \to R/\mathfrak{P}R = F$ which sends each Z_i to zero. Then $\mathfrak{P}'$, the kernel of this epimorphism, is the prime ideal generated by $(\mathfrak{P}R, Z_1, \ldots, Z_d)$. Since R is a regular local ring with maximal ideal $\mathfrak{P}R$, we know that R is an integral domain, $\mathfrak{P}R = (u_1, \ldots, u_r)$ with $r = r(\mathfrak{P}) = \dim R$, and each $(u_1, \ldots, u_j)$ for $j = 1, \ldots, r$ is a prime ideal of R. It follows, therefore, that $\mathfrak{P}'$ has the $r(\mathfrak{P}) + d$ generators $u_1, \ldots, u_r, Z_1, \ldots, Z_d$, and that $(u_1, \ldots, u_j)$ for $j = 1, \ldots, r$ and $(u_1, \ldots, u_r, Z_1, \ldots, Z_i)$ for $i = 1, \ldots, d$ are prime ideals in $R[Y]$. We can, therefore, apply **(2**, VIII, 4.2**)** to find the required value for $hd_{R[Y]}F$.[4]

3. Dedekind rings. We assume throughout this section (except in Lemma 6) that K is a Dedekind ring, that is, a commutative noetherian integrally closed integral domain in which every non-zero prime ideal is maximal.

THEOREM 4. *Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be prime ideals in S such that $(\mathfrak{P}_1, \mathfrak{P}_2) \neq S$, and let $\mathfrak{J}$ be the ideal generated by $\mathfrak{P}_1 \otimes 1$ and $1 \otimes \mathfrak{P}_2$ in $S \otimes S$. Then if $\mathfrak{U}$ is a minimal prime of $\mathfrak{J}$ we have*

$$d(\mathfrak{U}) = d(\mathfrak{P}_1) + d(\mathfrak{P}_2).$$

We need two lemmas before beginning the proof.

LEMMA 5. *If $\mathfrak{P}_i \cap K = 0$, $i = 1, 2$, then $\mathfrak{U} \cap K = 0$.*

Proof. We have the exact sequence

$$0 \to \mathfrak{J} \to S \otimes S \to (S/\mathfrak{P}_1) \otimes S/\mathfrak{P}_2) \to 0.$$

Since the $\mathfrak{P}_i$'s are prime ideals such that $\mathfrak{P}_i \cap K = 0$, the $S/\mathfrak{P}_i$'s are integral domains containing isomorphic copies of K. Thus, the $S/\mathfrak{P}_i$'s are torsion-free K-modules which also makes $(S/\mathfrak{P}_1) \otimes (S/\mathfrak{P}_2)$ a torsion-free K-module **(2**, VII, 4.5**)**. Hence, at any rate $K \cap \mathfrak{J} = 0$. Now $\mathfrak{U}/\mathfrak{J}$ is a minimal prime belonging to zero in $(S/\mathfrak{P}_1) \otimes (S/\mathfrak{P}_2)$ and so consists entirely of zero-divisors. Thus, if $\mathfrak{U} \cap K \neq 0$, then $(S/\mathfrak{P}_1) \otimes (S/\mathfrak{P}_2)$ has K-torsion, which is a contradiction.

Before stating the next lemma we recall that a ring has been called regular in **(1)** if its local ring at each non-zero prime is regular. With this definition of regularity we have

LEMMA 6. *Let K be a regular ring and $\mathfrak{P}_0$ a prime ideal in S with $\mathfrak{P}_0 \cap K = \mathfrak{p}_0$. If $f \notin \mathfrak{P}_0$ and $\mathfrak{P}_0'$ is a minimal prime of $(\mathfrak{P}_0, f)$ with $\mathfrak{P}_0' \cap K = \mathfrak{p}_0'$ we have*

$$d(\mathfrak{P}_0') = d(\mathfrak{P}_0) - 1 + r(\mathfrak{p}_0') - r(\mathfrak{p}_0).$$

[4]In **(3**, §5, Remark 1**)** it is shown by spectral sequence arguments that $hd_{R[Z]}F = d + hd_R F$. This yields an alternative proof that $hd_{R[Y]}F = d + r(\mathfrak{P})$.

Proof. We begin by showing that S is regular. For any prime ideal $\mathfrak{P}$ in S with $\mathfrak{p} = \mathfrak{P} \cap K$, we have $S_{\mathfrak{p}} = K_{\mathfrak{p}}[X]_{\overline{\mathfrak{P}}}$, where $\overline{\mathfrak{P}} = \mathfrak{P}K_{\mathfrak{p}}[X]$. By **(1**, Theorem 1.10**)**, **(3**, Theorem 6**)** and **(1**, Theorem 4.5**)**, respectively, we have

$$\text{gl. dim } K_{\mathfrak{p}} < \infty \to \text{gl. dim } K_{\mathfrak{p}}[X] < \infty$$
$$\to K_{\mathfrak{p}}[X] \text{ regular} \to S_{\mathfrak{p}} \text{ regular local ring.}$$

Now let

$$\overline{\mathfrak{P}}_0' = \mathfrak{P}_0' S_{\mathfrak{p}_0'} \text{ and } \overline{\mathfrak{P}}_0 = \mathfrak{P}_0 S_{\mathfrak{p}_0'}.$$

Then in

$$S_{\mathfrak{P}_0'}/\overline{\mathfrak{P}}_0$$

the ideal $\overline{\mathfrak{P}}_0'/\overline{\mathfrak{P}}_0$ is a minimal prime ideal of a non-zero principal ideal, and so by the Krull Hauptidealsatz $r(\overline{\mathfrak{P}}_0'/\overline{\mathfrak{P}}_0) \leqslant 1$. But

$$S_{\mathfrak{P}_0'}/\overline{\mathfrak{P}}_0$$

is an integral domain which shows that $r(\overline{\mathfrak{P}}_0'/\overline{\mathfrak{P}}_0) = 1$, that is, there are no prime ideals between $\overline{\mathfrak{P}}_0'$ and $\overline{\mathfrak{P}}_0$. By **(1**, Proposition 2.8**)** we then find $r(\mathfrak{P}_0') = r(\overline{\mathfrak{P}}_0') = r(\overline{\mathfrak{P}}_0) + 1 = r(\mathfrak{P}_0) + 1$. An application of Theorem 2 then yields the result.

Proof of Theorem 4. Let $\mathfrak{P}_i \cap K = \mathfrak{p}_i$. Since the non-zero prime ideals of K are maximal and $(\mathfrak{P}_1, \mathfrak{P}_2) \neq S$ only three cases arise: (a) $\mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p} \neq 0$, (b) $\mathfrak{p}_1 = \mathfrak{p}_2 = 0$, (c) $\mathfrak{p}_1 \neq 0, \mathfrak{p}_2 = 0$.

(a) Let $\mathfrak{p}^* = \mathfrak{p}S$. Then if $\mathfrak{P}$ is any prime ideal of S with $\mathfrak{P} \cap K = \mathfrak{p}$, we have

$$d(\mathfrak{P}) = d(\mathfrak{P}/\mathfrak{p}^*) \tag{2}$$

since $S/\mathfrak{P} \approx S/\mathfrak{p}^*/\mathfrak{P}/\mathfrak{p}^*$ and $S/\mathfrak{p}^* \approx K/\mathfrak{p}[X_1, \ldots, X_n]$.

Clearly the ideal $\tilde{\mathfrak{J}} = ((\mathfrak{P}_1/\mathfrak{p}^*) \otimes 1, 1 \otimes (\mathfrak{P}_2/\mathfrak{p}^*))$ in $(S/\mathfrak{p}^*) \otimes_K (S/\mathfrak{p}^*) = (S/\mathfrak{p}^*) \otimes_{K/\mathfrak{p}} (S/\mathfrak{p}^*)$ is $\mathfrak{J}/\mathfrak{p}(S \otimes S)$, and so $\mathfrak{U}$ maps onto a minimal prime ideal $\tilde{\mathfrak{U}}$ of $\tilde{\mathfrak{J}}$. But $K/\mathfrak{p}$ is a field and so by the field case of Theorem 4 cf. for example, **(6**; 1 §4, 1**)**

$$d(\tilde{\mathfrak{U}}) = d(\mathfrak{P}_1/\mathfrak{p}^*) + d(\mathfrak{P}_2/\mathfrak{p}^*).$$

Now $\mathfrak{U} \cap K = \mathfrak{P}_i \cap K = \mathfrak{p}$ and so (2) finishes the proof.

(b) Let Q be the field of quotients of K. Then $S_{K^*} = Q[X_1, \ldots, X_n]$ and $(S \otimes S)_{K^*} = Q[X_1, \ldots, X_n] \otimes_Q Q[X_1, \ldots, X_n]$. Moreover, by Lemma 5 $\mathfrak{P}_i \cap K = \mathfrak{U} \cap K = 0$, so that the ideals $\overline{\mathfrak{P}}_i = \mathfrak{P}_i S_{K^*}$, $\overline{\mathfrak{J}} = \mathfrak{J}(S \otimes S)_{K^*}$ and $\overline{\mathfrak{U}} = \mathfrak{U}(S \otimes S)_{K^*}$ are proper ideals with $r(\overline{\mathfrak{P}}_i) = r(\mathfrak{P}_i)$ and $r(\overline{\mathfrak{U}}) = r(\mathfrak{U})$. Since $\overline{\mathfrak{J}}$ is generated by $\overline{\mathfrak{P}}_1 \otimes 1$ and $1 \otimes \overline{\mathfrak{P}}_2$ and $\overline{\mathfrak{U}}$ is still a minimal prime ideal of $\overline{\mathfrak{J}}$ the field case of Theorem 4 again applies to give

$$d(\overline{\mathfrak{U}}) = d(\overline{\mathfrak{P}}_1) + d(\overline{\mathfrak{P}}_2).$$

By Theorem 2

$$d(\mathfrak{U}) = 2n - r(\mathfrak{U}) = 2n - r(\overline{\mathfrak{U}}) = d(\overline{\mathfrak{U}})$$
$$d(\mathfrak{P}_i) = n - r(\mathfrak{P}_i) = n - r(\overline{\mathfrak{P}}_i) = d(\overline{\mathfrak{P}}_i)$$

proving Theorem 4 in this case.

(c) Here $\mathfrak{J} \supset \mathfrak{p}_1(S \otimes S)$ and so the prime ideal $\mathfrak{U} \cap (1 \otimes S) \supset (\mathfrak{P}_2, \mathfrak{p}_1)$. Hence $\mathfrak{U} \cap (1 \otimes S)$ contains some minimal prime $\mathfrak{P}_2'$ of $(\mathfrak{P}_2, \mathfrak{p}_1)$. If $\mathfrak{J}'$ is the ideal $(\mathfrak{P}_1 \otimes 1, 1 \otimes \mathfrak{P}_2')$ in $S \otimes S$ we clearly have

$$\mathfrak{U} \supset \mathfrak{J}' \supset \mathfrak{J},$$

which shows that $\mathfrak{U}$ is also a minimal prime ideal of $\mathfrak{J}'$. But then by (b)

$$d(\mathfrak{U}) = d(\mathfrak{P}_1) + d(\mathfrak{P}_2').$$

To compute $d(\mathfrak{P}_2')$ we pass to

$$K_{\mathfrak{p}_1}[X]$$

and as usual we let $\overline{\mathfrak{P}}_2'$, $\overline{\mathfrak{P}}_2$, $\bar{\mathfrak{p}}_1$ denote the extensions of $\mathfrak{P}_2'$, $\mathfrak{P}_2$, $\mathfrak{p}_1$ to this ring. Since $\mathfrak{p}_1$ is a maximal ideal, $\mathfrak{P}_2' \cap K = \mathfrak{p}_1$ also, and so $d(\mathfrak{P}_2') = d(\overline{\mathfrak{P}}_2')$. Now

$$\bar{\mathfrak{p}}_1 = \mathfrak{p}_1 K_{\mathfrak{p}_1}[X] = (\mathfrak{p}_1 K_{\mathfrak{p}_1})K_{\mathfrak{p}_1}[X].$$

But K is a Dedekind ring which means that

$$K_{\mathfrak{p}_1}$$

is integrally closed and has dimension one. Thus it is a regular local ring (**5**, Chap. 4, Theorem 8) of dimension one[5]. Therefore (**5**, Chap. 4, Proposition 7)

$$\mathfrak{p}_1 K_{\mathfrak{p}_1} = \pi K_{\mathfrak{p}_1},$$

for some

$$\pi \text{ in } K_{\mathfrak{p}_1},$$

from which it follows that

$$\bar{\mathfrak{p}}_1 = \pi K_{\mathfrak{p}_1}[X].$$

Thus $\overline{\mathfrak{P}}_2'$ is a minimal prime of $(\overline{\mathfrak{P}}_2, \pi)$ and since

$$\overline{\mathfrak{P}}_2 \cap K_{\mathfrak{p}_1} = 0$$

we know that $\pi \notin \overline{\mathfrak{P}}_2$. Lemma 6 with

$$K = K_{\mathfrak{p}_1}, \mathfrak{P}_0 = \overline{\mathfrak{P}}_2, \mathfrak{P}_0' = \overline{\mathfrak{P}}_2', \mathfrak{p}_0 = 0, \mathfrak{p}_0' = \bar{\mathfrak{p}}_1$$

[5]Since

$$\text{gl. dim } K_{\mathfrak{p}_1} \leqslant \text{gl. dim } K = 1$$

the results of (**1**) could also have been used here.

then becomes available and shows

$$d(\mathfrak{P}_2') = d(\overline{\mathfrak{P}_2'}) = d(\overline{\mathfrak{P}}_2) = d(\mathfrak{P}_2)$$

which completes the proof of Theorem 4.

THEOREM 5. *Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be two prime ideals in S with $(\mathfrak{P}_1, \mathfrak{P}_2) \neq S$. Then if $\mathfrak{B}$ is any minimal prime of $(\mathfrak{P}_1, \mathfrak{P}_2)$ we have*

$$d(\mathfrak{B}) \geqslant d(\mathfrak{P}_1) + d(\mathfrak{P}_2) - n.$$

Proof. Following the usual "diagonal" method we consider the natural map $\phi : S \otimes S \rightarrow S$. The pre-image of $(\mathfrak{P}_1, \mathfrak{P}_2)$ is $(\mathfrak{J}, \mathfrak{K})$ where $\mathfrak{J} = (\mathfrak{P}_1 \otimes 1, 1 \otimes \mathfrak{P}_2)$ and $\mathfrak{K} = (X_i \otimes 1 - 1 \otimes X_i)$. Clearly $\mathfrak{W}$, the pre-image of $\mathfrak{B}$, is a minimal prime ideal of $(\mathfrak{J}, \mathfrak{K})$.

Since $\mathfrak{K} \cap K = 0$ the mapping ϕ restricted to K is an isomorphism so that $\mathfrak{W} \cap K = \mathfrak{B} \cap K$. Furthermore $(S \otimes S)/\mathfrak{W} = S/\mathfrak{B}$ and we see once again that $d(\mathfrak{W}) = d(\mathfrak{B})$. But $\mathfrak{W} \supset \mathfrak{J}$, therefore, for some minimal prime ideal $\mathfrak{U}$ of $\mathfrak{J}$ we have $\mathfrak{W} \supset (\mathfrak{U}, \mathfrak{K}) \supset (\mathfrak{J}, \mathfrak{K})$. Hence $\mathfrak{W}$ is a minimal prime ideal of

$$(\mathfrak{U}, \mathfrak{K}) = (\mathfrak{U}, X_1 \otimes 1 - 1 \otimes X_1, \ldots, X_n \otimes 1 - 1 \otimes X_n).$$

Thus repeated application of Lemma 6 yields

$$d(\mathfrak{W}) \geqslant d(\mathfrak{U}) - n,$$

and invoking Theorem 4 we obtain

$$d(\mathfrak{B}) = d(\mathfrak{W}) \geqslant d(\mathfrak{P}_1) + d(\mathfrak{P}_2) - n.$$

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