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HODGE THEORY ON COMPACT TWO-DIMENSIONAL SPACETIMES AND THE UNIQUENESS OF g_{ij} WITH A SPECIFIED R_{ij}

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1. Introduction. The main question we wish to address in this paper is to what extent does the Ricci curvature of a spacetime determine the metric of that spacetime. Although it is relatively easy to see that the full Riemann curvature uniquely determines the metric for a generic choice of curvature tensors (see [**4**], [**10**], [**11**], [**14**] and [**15**], and the references contained therein), very little has been discovered about whether, if ever, Ric (or the stress energy tensor in Einstein's equations for that matter) determines g. Most exact solution techniques for Einstein's equations look only for solutions that have the same symmetries as Ric. It is not true in general that g must inherit the symmetries of Ric. It is not even clear that there is a Ric such that every g with this Ricci tensor is known.

Recently some progress has been made in showing that certain choices for Ric have a unique (to within a constant conformal factor) Riemannian g with this Ricci tensor (see [**1**], [**2**], [**5**], [**12**], [**21**] and the references therein). [**12**] provides a good overview of these results. The considerations which lead to uniqueness theorems involving the full Riemann curvature are local and essentially signature independent. However neither of these aspects remain valid when uniqueness theorems for Ric are considered. To illustrate this we outline the proof given in [**21**] that there is at most one Riemannian metric (to within a constant conformal factor) which has a given tensor as its Ricci tensor if the tensor vanishes only on a set of measure zero and if the metric is on a compact orientable two dimensional manifold. Let g_1 and g_2 be two positive definite metrics with the same Ricci tensor. Since M is orientable one can define two global orientation forms Ω_i ($i = 1, 2$) such that Ω_i has unit length in g_i. There is an $\alpha:M \to \mathbf{R}$ such that $\Omega_1 = \exp(\alpha)\Omega_2$. Now we observe that g_1 and g_2 must be conformally related off of a set of measure zero since

$$\text{Ric} = (1/2)R_i g_i$$

where R_i is the scalar curvature of g_i (use the fact that M is two dimensional). A calculation in local coordinates shows that this conformal factor must be $\exp(\alpha)$. This means $g_1 - \exp(\alpha)g_2$ is zero almost every-

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where and thus identically zero by continuity. Again a local coordinates calculation using that g_1 and g_2 have the same Ricci tensor shows that α is harmonic. Because M is connected the Hodge-deRham theorem gives that α is a constant as desired.

Every step of this proof is valid for spacetimes except the last line which uses the Hodge-deRham theorem. In fact there can be many harmonic functions other than the constants on a compact spacetime. For example let $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ be the torus with the flat spacetime metric $dx^2 - dy^2$. Then any function of the form $f(x + y) + g(x - y)$ is harmonic where f and g are any smooth functions from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{R}$.

In this paper we wish to show that this example is somewhat atypical. We show that harmonic functions in a spacetime are simply related to first integrals of two vector fields (4.3). Then using the deep result of Peixoto that the Morse-Smale dynamical systems are generic on compact two manifolds (see 2.5 and 2.6 together with the reference given there), we are able to show that for a generic class of time orientable spacetimes the only harmonic functions are the constants (4.9 and 4.10). Thus for a suitable generic class of Ricci tensors on a compact orientable spacetime there is only one metric to within a constant conformal factor that has that tensor as its Ricci tensor (4.9 and 4.11). We give these results only for spacetimes on the torus since the only compact orientable surface that admits a spacetime structure is a torus. (This uses that $\mathfrak{X}(M) = 0$ (see [**20**] p. 207) and the classification of compact orientable two manifolds (see [**8**] p. 204 for example)).

The next natural question that can be asked is how much of the Hodge-deRham theory holds for these spacetimes. In 5.4 we show that almost all of it holds. The only part in doubt is whether every element of $H^1(M, \mathbf{R})$ has a harmonic representative (if it does have one, it is unique). As existence theorems are frequently more difficult to deal with, this question may not have a simple resolution.

The final remaining question is that of how much these results can be generalized to higher dimensional spacetimes. Perhaps the results most likely to generalize to higher dimensional spacetimes are the uniqueness results concerning harmonic forms representing a given cohomology class. It seems unlikely that similar techniques can be used to show anything for a generic class of spacetimes since the basic results concerning the genericity of dynamical systems does not hold for manifolds of higher dimension. However it does seem that a uniqueness theorem may be possible for certain special classes of spacetimes.

2. Notation and basic results. In this section we establish our notation and give the basic results that we will need later. We assume that all manifolds and tensors are C^∞. If a topology is needed on a set of tensors we use the Whitney C^∞ topology.

2.1 *Notation*. (a) M will be a connected orientable two dimensional manifold.

(b) $g = g_{ij}dx^i dx^j$ will denote a spacetime metric on M (the Einstein summation convention will be used throughout the paper).

(c) $\text{Ric}(g) = R_{ij}dx^i dx^j$ will denote the Ricci tensor associated with g.

(d) $S = S_{ij}dx^i dx^j$ will denote a symmetric rank two tensor such that

(i) $\{m \in M|S(m) = 0\}$ has measure zero.

(ii) $\det(S) \leqq 0$.

(e) SYM $= \{S|S$ satisfies the conditions given in (d)$\}$.

(f) $G(S) = \{g|\text{Ric}(g) = S\}$.

(g) d denotes the exterior derivative.

(h) Let M have a spacetime metric g, and let M be oriented.

(i) $*$ represents the Hodge dual which takes j forms to $2 - j$ forms.

(ii) $\delta = -*d*$ takes j forms to $j - 1$ forms.

(iii) $\nabla^2 = d\delta + \delta d$ is the Laplacian.

(iv) A form Θ is harmonic if $\nabla^2(\Theta) = 0$. (Note that the property of being harmonic is independent of orientation).

(i) $H(g) = \{f:M \to \mathbf{R}|f$ is harmonic in the metric $g\}$.

2.2 *Assumption*. We will always assume that the Ricci tensor satisfies 2.1 (d). (Note that the Ricci tensor always satisfies 2.1 (d) (ii) because the metric does.) Thus we define

$$\text{RIC} = \{S|S \in \text{SYM and } G(S) \neq \emptyset\}.$$

Next we state some of the basic facts we need to reduce the problem of determining $G(S)$ to determining $H(g)$. We first indicate a useful system of local coordinates in which it is easy to perform calculations.

2.3 LEMMA. *There is a coordinate system in which*

$$g = \exp(\beta)\, dxdy.$$

(*These coordinates are called null coordinates*; *they are coordinates in which* $\partial/\partial x$ *and* $\partial/\partial y$ *are characteristic vector fields for the Laplacian*.)

Proof. The idea of the proof of this result is as follows. Locally define two independent null vector fields N_1 and N_2 ($g(N_i, N_i) = 0$ for $i = 1, 2$). Note that the local existence of null vector fields may easily be obtained by applying the implicit function theorem to the function f from the sphere bundle (S in 3.2) to $\mathbf{R}$ defined by $f(v) = g(v, v)$ (0 is a regular value of f). Next let a_i be defined by

$$[N_1, N_2] = a_1N_1 + a_2N_2.$$

Then one finds f_1 and f_2 which are locally defined real valued functions such that

$$[f_1N_1, f_2N_2] = 0$$

by letting

$$f_i(x, y) = \int_0^x a_i(s, y)ds$$

in any coordinate system such that $N_i = \partial/\partial x$. The Frobenius theorem (see [6] p. 123 for example) says there are coordinates x and y such that $f_1 N_1 = \partial/\partial x$ and $f_2 N_2 = \partial/\partial y$. These are the desired coordinates.

We next reduce the uniqueness question to the study of the harmonic functions on M.

2.4 LEMMA. *Using the notation of* 2.1 (*including the hypothesis given in* 2.1 (d)) *the following are true*:

(a) $\mathrm{Ric}(g) = (R/2)g$ *where* $R = \mathrm{trace}(\mathrm{Ric}(g))$.

(b) *If* g_1 *and* g_2 *are both in* $G(S)$ *then*

$$g_1 = \pm\exp(\alpha)\, g_2$$

for some $\alpha:M \to \mathbf{R}$. *Also* $H(g_1) = H(g_2)$.

(c) *Let* $S = \mathrm{Ric}(g)$. *The function*

$$i:H(g) \times \{1, -1\} \to G(S)$$

defined by

$$i(\alpha, \epsilon) = \epsilon \exp(\alpha)\, g$$

is a set isomorphism.

Proof. We give an outline of the proof. To see (a) simply calculate Ric in local coordinates; null coordinates are most convenient. The reason (a) is true is that on a two dimensional spacetime R_{ijkl} has only one independent component. The first part of (b) follows from (a) using the argument given in the introduction. The second part of (b) follows from the fact that in dimension 2 the property of a function being harmonic is invariant under conformal transformations. To see (c) use again a calculation in null coordinates to find that

$$\mathrm{Ric}(g) = \mathrm{Ric}(\epsilon \exp(\alpha)\, g)$$

if and only if

$$\partial^2(\alpha)/\partial x \partial y = 0$$

if and only if α is harmonic.

We finish this section by stating the basic results about dynamical systems that we use in this paper. These results are due to Peixoto. He defines a class of dynamical systems on compact two manifolds which are called Morse-Smale dynamical systems because Smale later generalized

Peixoto's definition to apply to flows on manifolds of arbitrary dimension. His definition is given in [**17**] p. 104 and we give it here. Smale's generalization is given in [**18**] p. 798 (2.2). [**18**] gives a good overview of this subject.

2.5 *Definition*. A vector field v on a compact 2 dimensional manifold is called a *Morse-Smale dynamical system* if all of the following are satisfied.

(a) v is zero at most a finite number of points and each zero is generic.

(b) The α and ω limit sets of every trajectory can only be points where v is zero or closed orbits.

(c) No trajectory connects saddle points.

(d) There are a finite number of closed orbits each of which is hyperbolic.

Although we do use this definition in a minor technical calculation (4.6), the basic thrust of this paper does not use the definition of a Morse-Smale dynamical system explicitly. The main information we use about Morse-Smale dynamical systems are two results of Peixoto ([**17**] p. 119) which we give below.

2.6 THEOREM. *The set of Morse-Smale dynamical systems on a compact two dimensional manifold M is open and dense in the set of all dynamical systems on M.*

2.7 THEOREM. *Let M be compact and two dimensional. If a continuous function from M to* $\mathbf{R}$ *is constant on all the orbits of a Morse-Smale dynamical system, then the function is constant.*

3. Global null vector fields. In Section 4 we will want to apply the results of Peixoto concerning constants of the motion of Morse-Smale dynamical systems to null vector fields since they are the characteristic vector fields of the Laplacian. Unfortunately even two dimensional spacetimes on a torus need not have any globally defined null vector fields. A simple example of this can be obtained from the following spacetime metric on $\mathbf{R}^2$:

$$g = [\cos(\pi x)\, dx - \sin(\pi x)\, dy][\sin(\pi x)\, dx + \cos(\pi x)\, dy].$$

This metric is left unchanged by the transformation

$$x' = x + n \quad \text{and} \quad y' = y + m$$

where n and m are any integers. Thus g induces a metric on the two dimensional torus T^2 which may be identified with $\mathbf{R}^2/\mathbf{Z}^2$. The natural null vector fields

$$\sin(\pi x)\, \partial/\partial x + \cos(\pi x)\, \partial/\partial y \quad \text{and}$$

$$\cos(\pi x)\, \partial/\partial x - \sin(\pi x)\, \partial/\partial y$$

which are defined on $\mathbf{R}^2$ do not give rise to vector fields on T^2 because the transformation given above with $n = 1$ and $m = 0$ takes them to their negatives.

The first lemma we prove below shows that this represents the worst that can happen in that there are always defined global null vector fields to 'within a sign'. The idea behind the proof is as follows. One can picture the set of null vectors on a two dimensional manifold as a smooth assignment of a cross (the null cone) to each point in the manifold. If this cross is parallel transported around any closed path, it must come back onto itself. This will permute the 4 branches of the cross. There are precisely 8 possible permutations that can occur. Only 4 of those permutations can occur as the result of a Lorentz transformation, and only two can occur from an orientation preserving Lorentz transformation. Since the spacetime is assumed to be orientable, we have that the permutation comes from an orientation preserving Lorentz transformation, and thus this permutation either leaves each branch fixed or takes each branch to its negative. Hence we arrive at 3.1. 3.1 is good enough for some purposes (see 4.3) since the sign ambiguity in the vector field does not stand in the way of defining the concept of a derivative being 0.

However we ultimately will need global null vector fields. 3.2 will be our main tool to achieve this. We produce a two fold covering of M (denoted $\tilde{M}$) on which there will always be global null vector fields. We will then try to prove results on M by translating the whole problem to $\tilde{M}$. 4.10 illustrates this. The idea behind the two fold cover is as follows. As observed above, the difficulty in defining a null vector field on M is that we have a sign ambiguity at each point. Thus we build the new space $\tilde{M}$ so that it contains that sign information in it. This means there will be two points in $\tilde{M}$ which correspond to every point in M. If $\tilde{M}$ is the trivial bundle, then it will consist of two separate components. Each component will correspond to a consistent choice of sign for the null vector field. If $\tilde{M}$ is connected, then there is a path which takes a point with one sign label to the same point with the opposite sign label. The projection of this path onto M will represent a path along which parallel transport will take a null vector to its negative. Thus no globally defined null vector field will be possible.

Since there are some results which can not be shown by translation to a covering space (see 4.9), we give 3.3 to relate the existence of global null vector fields to the physically natural concept of time orientability. The idea behind this result is that if a closed path takes a null vector to its negative, then it will take every vector to its negative and hence take every timelike vector to its negative. Thus it should not be possible to define a global timelike vector field. We now give precise statements and proofs of the ideas discussed above.

3.1 LEMMA. *There are two real line bundles E_1 and E_2 over M which are subbundles of the tangent bundle $T(M)$ such that every null vector is in $E_1 \cup E_2$.*

Proof. Define E_i as follows. The spacetime structure (M, g) induces a connection on the bundle of frames which is a principal $Gl(2, \mathbf{R})$ bundle. Let

$$O^+(1, 1) = \{A:\mathbf{R}^2 \to \mathbf{R}^2 | A \text{ is linear, } \det(A) = 1 \text{ and } AJA^t = J\},$$

where $J = \mathrm{diag}(1, -1)$, which denotes the group of orientation preserving Lorentz transformations. This connection can be reduced to a connection on a principal $O^+(1, 1)$ subbundle P since (M, g) is an orientable spacetime (see [**13**] p. 83). We have (see [**13**] p. 56)

$$T(M) = P \times_{O^+(1,1)} \mathbf{R}^2.$$

Let L_i be $\{ (a, (-1)^i a)^t | a \in \mathbf{R}\}$ where $i = 1$ or 2. L_i is invariant under the action of $O^+(1, 1)$. Define

$$E_i = P \times_{O^+(1,1)} L_i.$$

We have that every null vector is in E_i for some i.

3.2 THEOREM. *Let E be either of the line bundles defined in* 3.1. *Let h be any positive definite metric on M. Define*

$$\widetilde{M} = \{v \in E | h(v, v) = 1\}.$$

Then

(a) *$\widetilde{M}$ is a manifold and $\widetilde{\pi}$ (π restricted to $\widetilde{M}$) is a two fold covering projection from $\widetilde{M}$ to M.*

(b) *Give $\widetilde{M}$ the spacetime metric $\widetilde{g} = \widetilde{\pi}^*(g)$. $\widetilde{M}$ has two global independent null vector fields.*

(c) *M has two global independent null vector fields if and only if the first Stiefel-Whitney class of E,*

$$w_1(E) \in H^1(M, \mathbf{Z}_2),$$

is zero.

Proof. Let $S(M)$ be the submanifold of $T(M)$ defined by

$$S(M) = \{v \in T(M) | h(v, v) = 1\}.$$

Let ι be the inclusion mapping from $S(M)$ into $T(M)$. An easy calculation shows that ι is transverse to E. Thus

$$\widetilde{M} = S(M) \cap E = \iota^{-1}(E)$$

is a manifold (see [**3**] p. 28 for example). It is clear that $\widetilde{\pi}$ is a two to one covering projection. Next we show (b). We construct the first null vector field on $\widetilde{M}$ as follows. Let

$$\tilde{N}_1 : \tilde{M} \to T(\tilde{M})$$
$$\tilde{N}_1(v) = D(\tilde{\pi})^{-1}(v)$$

where

$$D(\tilde{\pi})^{-1} : T_{\pi(v)}(M) \to T_v(\tilde{M}).$$

Since v is null, we have that $\tilde{N}$ is also null. Thus we have defined one global non-zero null vector field. To construct the second null vector field let

$$\tilde{N}_2(v) = D(\tilde{\pi})^{-1}(v')$$

where v' is the unique null vector not in E with $h(v', v') = 1$ and (v, v') positively oriented. To complete the theorem we must only show (c). Now E (and hence $\tilde{M}$) is trivial if and only if $w_1(E)$ is zero (see [**18**] p. 281, for example). Also E is trivial if and only if it has a non zero section. Thus if there are two independent null vector fields we may use one of them to produce the desired non zero section. If we have a non zero section then we have one non zero null vector field, and an argument similar to the one given above may be used to produce the second null vector field. The proof is now complete.

3.3 COROLLARY. *(M, g) is time orientable (has a non zero timelike vector field) $\Leftrightarrow$ (M, g) has two independent null vector fields.*

Proof. We start with $\Rightarrow$. Let v be a non zero timelike vector field. Define

$$f : \tilde{M} \to \mathbf{R}$$

by

$$f(x) = g(x, v).$$

Since only spacelike vectors are perpendicular to v, $f(x)$ is never 0. But $f(-x) = -f(x)$ so that $f(\tilde{M})$ has at least two components. Thus $\tilde{M}$ is not connected. $\tilde{M}$ is a two fold cover of a connected space, so it must be a trivial bundle over M which gives the two independent null vector fields. We finish with $\Leftarrow$. If v_1 and v_2 are two independent null vector fields, we have that

$$g(v_1 \pm v_2, v_1 \pm v_2) = \pm 2g(v_1, v_2) \neq 0$$

so that either $v_1 + v_2$ or $v_1 - v_2$ is a global timelike vector field.

3.4 *Remark.* $w_1(L_1) = w_1(L_2)$.

Proof. We have that $T(M)$ is trivial for the following reason. If M is compact then M is T^2 which has trivial tangent bundle. If M is not compact then M has a non-zero vector field. Using a construction similar to the construction in 3.2, we may find an independent vector field and thus $T(M)$ is also trivial. Now the result follows directly from product theorem for Stiefel-Whitney classes (see [**16**] p. 37) which states

$$0 = w_1(T(M)) = w_1(E_1 \oplus E_2) = w_1(E_1) + w_1(E_2).$$

4. Harmonic functions. In this section we will show that the only harmonic functions on a compact spacetime are the constant functions for generic time orientable M. We start with a general definition and then give a basic short exact sequence which determines the harmonic zero forms for an arbitrary two dimensional spacetime.

4.1 *Definition.* Let E be a line subbundle of $T(M)$. Define

$$\mathscr{F}_E(M) = \{f:M \to \mathbf{R} | df(v) = 0 \text{ for all } v \in E\}.$$

Note that $\mathscr{F}_E(M)$ is a vector space as is $H(g)$ and $H^1(M, \mathbf{R})$. Before we state the theorem we give some useful observations which are easily verified in null coordinates.

4.2 OBSERVATION. *Let* $*_1:\Lambda^1(M) \to \Lambda^1(M)$. *Then*

(a) $(*_1)^2 = I$

(b) $*_1$ *has eigenvalues* 1 *and* -1. Θ *is an eigen-form of* $*_1$ *if and only if* Θ *is null.*

4.3 THEOREM. *The following sequence is exact:*

$$0 \to \mathbf{R} \xrightarrow{\alpha} \mathscr{F}_{E_1}(M) \oplus \mathscr{F}_{E_2}(M) \xrightarrow{\beta} H(g) \xrightarrow{\gamma} H^1(M, \mathbf{R})$$

where $\alpha(a) = (a, -a)$, $\beta(f_1, f_2) = f_1 + f_2$ *and* $\gamma(h) = [*dh]$.

Proof. First observe that $\beta(f_1, f_2)$ is harmonic because in any system of null coordinates suitably labeled we have that $f_{1,1} = 0$ and $f_{2,2} = 0$ so that both f_1 and f_2 satisfy Laplace's equation which is $h_{1,2} = 0$. Also $*dh$ is closed since h harmonic implies that $*d*dh = 0$. Thus all the maps in the sequence are well defined. Next we show that $\gamma\beta = 0$. Let Θ be any null one form. We have that $*\Theta = \pm\Theta$. If f is in either $\mathscr{F}_{E_1}(M)$ or $\mathscr{F}_{E_2}(M)$ then df is null so that

$$*df = \pm df = d(\pm f).$$

This shows that $\gamma\beta(f) = 0$. Now we show that if $\gamma(h) = 0$ then $h = \beta(f_1, f_2)$. Let $*dh = dh'$. Then let

$$f_1 = (h + h')/2 \quad \text{and} \quad f_2 = (h - h')/2$$

(here we assume that $*$ acts as the identity on the null forms that are zero on E_1). We must only show that

$$f_1 \in \mathscr{F}_{E_1}(M) \quad \text{and} \quad f_2 \in \mathscr{F}_{E_2}(M).$$

We have

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$$2df_1 = dh + dh' = (1 + *)dh$$

so that

$$*(2df_1) = (* + 1)dh = 2(df_1).$$

Thus df_1 is zero on E_1 as desired. A similar argument shows that $*(2df_2) = -2df_2$, and thus df_2 is zero on E_2. We have now shown the sequence is exact at $H(g)$. Since it is clear that $\beta\alpha = 0$ we must only show that $f_1 + f_2 = 0$ implies that f_1 is constant to finish the proof of the theorem. But df_1 is zero on any vector in E_1 and $df_1 = -df_2$ is zero on any vector in E_2 so that df_1 is zero on every vector giving f_1 is constant as desired.

Next we explore some of the consequences of this theorem. In order to do this we need to discover some more information about the map γ. Not only is it not usually onto, it is most frequently zero.

4.4 LEMMA. *Suppose* $\iota:S^1 \to M$ *such that*

$$D(\iota)(T(S^1)) \subseteq E_j \quad \textit{for some } j = 1, 2.$$

Then $\gamma(h)$ *is zero on every element of* $\iota_*(H_1(S^1))$.

Proof. We must only calculate $\int_\iota \gamma(h)$. Now, by the proof of 4.3 we have that $(1 \pm *)\gamma(h)$ is zero on $D(\iota)(T(S^1))$ so that

$$\int_\iota \gamma(h) = \mp\int_\iota *\gamma(h) = \mp\int_\iota d(h) = 0.$$

This completes the lemma.

The following corollary is what we will need to calculate γ on the torus. It will be applied to the manifold which is obtained from the torus after a closed orbit of a null vector field is removed. We will see in 4.6 below that this manifold will have its fundamental group isomorphic to $\mathbf{Z}$ as required by the corollary.

4.5 COROLLARY. *Let* $\pi_1(M) = \mathbf{Z}$. *Suppose that either* N_1 *or* N_2 *has a compact integral submanifold. Then* $\gamma = 0$.

Proof. Since

$$H_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)] = \mathbf{Z}$$

(see **[19]** p. 394) we have

$$H^1(M, \mathbf{R}) = \mathrm{Hom}(H_1(M), \mathbf{R}) = \mathbf{R}$$

(see **[19]** p. 243). Let Σ be the compact integral submanifold. By 4.4 we need only show that the class of Σ in $H_1(M)$ is not zero. Since $\pi_1(M) \cong H^1(M)$, we have that if Σ is zero in $H_1(M)$ it represents a contractable path, and it will lift to a closed orbit in the universal cover $\widetilde{M}$ of M. Since

$$H^1(\widetilde{M}, \mathbf{Z}_2) = 0$$

we have that there is a global non-zero vector field on $\widetilde{M}$ which has Σ as an orbit. But the Poincaré-Bendixson theorem (see [**9**] p. 252) would mean that this vector field would have a zero. This contradiction proves the result.

Next we give the characterization of γ which is relevant to the generic result of this paper. To prepare for this result we first give a lemma about first integrals of Morse-Smale dynamical systems.

4.6 LEMMA. *Let v be a non-zero vector field on T^2, the torus. Let Σ be a closed orbit of v. Let $M = T^2 - \Sigma$.*

(a) *M is connected and $\pi_1(M) = \mathbf{Z}$.*

(b) *If v is a Morse-Smale dynamical system and if*

$$f:M \to \mathbf{R}$$

with $v(f) = 0$ then f is a constant.

Proof. To see (a) we let T be a tubular neighborhood of Σ in T^2 and then use the Mayer-Vietoris sequence for M and T (see [**19**] p. 186). We note that $H_2(M, \mathbf{R}) = 0$ since M is a non compact manifold, and thus we have the following long exact sequence:

$$\begin{aligned} 0 &\to H_2(T^2, \mathbf{R}) \to H_1(M \cap T, \mathbf{R}) \to H_1(M, \mathbf{R}) \oplus H_1(T, \mathbf{R}) \\ &\to H_1(T^2, \mathbf{R}) \to H_0(M \cap T, \mathbf{R}) \to H_0(M, \mathbf{R}) \oplus H_0(T, \mathbf{R}) \\ &\to H_0(T^2, \mathbf{R}) \to 0. \end{aligned}$$

Now we use that T is homotopic to S^1, the circle, and $M \cap T$ is homotopic to the disjoint union of two copies of S^1. This second statement follows from Milnor ([**16**] p. 115) which says T is diffeomorphic to the normal bundle to Σ and the fact that the normal bundle is trivial since the tangent bundles to both T^2 and S^1 are trivial. This gives us the exact sequence

$$\begin{aligned} 0 &\to \mathbf{R} \to \mathbf{R}^2 \to H_1(M, \mathbf{R}) \oplus \mathbf{R} \to \mathbf{R}^2 \xrightarrow{\alpha} \mathbf{R}^2 \\ &\to H_0(M, \mathbf{R}) \oplus \mathbf{R} \to \mathbf{R} \to 0. \end{aligned}$$

In particular we have the short exact sequence

$$0 \to \mathrm{Im}(\alpha) \to \mathbf{R}^2 \to H_0(M, \mathbf{R}) \to 0.$$

Here we used the fact that the short exact sequence of vector spaces splits in order to simplify the right end of the long exact sequence. Thus $H_0(M, \mathbf{R}) = \mathbf{R}$ if α is not zero. If α is zero then the intersection number of Σ with every element of $H_1(T^2)$ is zero. This means Σ is homotopic to zero in T^2. Thus the lift of Σ to $\mathbf{R}^2$, the universal cover of T^2, will still be a closed path. The Poincaré-Bendixon theorem then implies that v has a zero contradicting our assumption. Thus we have that α is not 0, and M is connected. Next we show that

$$H_1(M, \mathbf{R}) = \mathbf{R}.$$

If we consider the long exact sequence above as an algebraic chain complex with 0 homology, we find that the alternating sum of the dimensions of the vector spaces in this sequence is zero. This tells us that

$$\dim(H_1(M, \mathbf{R})) = \dim(H_0(M, \mathbf{R})) = 1.$$

Now we conclude (a) by showing $\pi_1(M) = \mathbf{Z}$. Since M is a non compact two manifold, it has the homotopy type of a one dimensional C. W. complex. The fundamental group of a one dimensional C. W. complex is a free group on k generators (see **[7]** p. 242). Using the Hurewitz isomorphism and the universal coefficients theorem as above, one can then conclude that

$$H_1(M, \mathbf{R}) = \mathbf{R}^k.$$

Thus $k = 1$ as desired.

(b) is a slight generalization of the result of Peixoto (**[17]** p. 119 stated in 2.7). We show (b) by showing that $df = 0$ everywhere. Since v is a Morse-Smale system with no fixed points, each α and ω limit set must be a closed orbit. Let $m \in M$. There is a neighborhood U of m such that for all $m' \in U$ we have $\alpha(m) = \alpha(m')$ and $\omega(m) = \omega(m')$. If Σ has a non empty stable manifold then it can only be in the α limit set of any point not in Σ. In this case f is constant on U since $\omega(m) \neq \emptyset$ so that

$$f(m) = f(x) = f(m')$$

where x is any element of $\omega(m)$. This is so since f is constant on orbits and continuous. If Σ has a non empty unstable submanifold then the same argument also shows $f(m) = f(m')$ if one uses the α limit points instead of ω limit points. Thus, in any case f is constant on U, and $df = 0$ as desired. The lemma is now complete.

4.7 THEOREM. *Let $M = T^2$, the torus, and let (M, g) be time orientable $(w_1(E) = 0)$. Let v be a never* 0 *null vector field. If v is a Morse-Smale dynamical system then γ is* 0.

Proof. Assume that v spans N_1. We will show that $*dh = dh$ for all harmonic h. Let $m \in T^2$. Let $\Sigma = \alpha(m)$ be the alpha limit set of m, and let $\Sigma' = \omega(m)$ be the omega limit set of m. They are not empty since M is compact. They are not points since v has no zeros. Thus they are distinct closed orbits since v is a Morse-Smale system. Let $M = T^2 - \Sigma$. Using 4.6 (a), the fact that Σ' is a closed orbit in M and 4.5 gives us that $*dh = dh'$ for some h' defined on M. Thus

$$(1 - *)dh = d(h - h').$$

Since $(1 - *)dh(v) = 0$ we have $v(h - h') = 0$. Then 4.6 (b) tells us that $h - h'$ is constant on M. Thus $(1 - *)dh = 0$ on M. But M is a dense submanifold of T^2 and $(1 - *)dh$ is continuous so $(1 - *)dh$ is zero on all of T^2. This completes the theorem.

We are now ready to give the main results of the paper. First we define a Morse-Smale spacetime and show that such spacetimes occur frequently. Then we present our results concerning these spacetimes.

4.8 *Definition.* (a) (M, g) is called a *Morse-Smale spacetime* if both of the null vector fields on $\tilde{M}$ are Morse-Smale dynamical systems.

(b) (M, S) is called *Morse-Smale* if (M, g) is Morse-Smale for any $g \in G(S)$ (see 2.1).

4.9 THEOREM. (a) *The set of time orientable Morse-Smale spacetimes is generic in the set of all time orientable spacetimes.*

(b) *The set of* (M, S) *that are time orientable Morse-Smale is generic among the time orientable* (M, S) *for which* $S \in$ RIC (*see* 2.2).

Proof. Let

$$\mathscr{F}^{\pm}(M) = \{f:M \rightarrow \mathbf{R} | f(m) \neq 0 \text{ for all } m\}$$

and

$$\mathscr{B}(M) = \{ \{v_1, v_2\} | v_1 \text{ and } v_2 \text{ are vector fields which are independent at each point of } M\}.$$

There is a function $\mathfrak{g}$

$$\mathfrak{g}:\mathscr{F}^{\pm}(M) \times \mathscr{B}(M) \rightarrow \mathscr{G}$$

where

$$\mathscr{G} = \{g | g \text{ is a spacetime metric on } T^2 \text{ with } w_1(E) = 0\}.$$

$\mathfrak{g}$ is given by

$$\mathfrak{g}(f, \{v_1, v_2\}) = f(v_1^* \otimes v_2^* + v_2^* \otimes v_1^*)$$

where $v_i^*(v_j) = \delta_{ij}$. $\mathfrak{g}$ is one to one and $\mathfrak{g}$ is onto because (M, g) is time orientable. Thus $\mathfrak{g}$ is a homeomorphism. Let

$$\mathfrak{X}(M) = \{v | v \text{ is a vector field on } T^2\}.$$

Then $\mathscr{B}(M)$ is open in $\mathfrak{X}(M) \vee \mathfrak{X}(M)$ since

$$\mathscr{B}(M) = \{ \{v_1, v_2\} | \Omega(v_1, v_2) > 0 \text{ at all } m \in M \text{ or } \Omega(v_1, v_2) < 0 \text{ for all } m \in M\}$$

where Ω is any fixed orientation form on M. Thus since

$$\mathscr{M}(M) = \{ \{v_1, v_2\} | v_1 \text{ and } v_2 \text{ are both Morse-Smale dynamical systems}\}$$

is generic in $\mathfrak{X}(M) \vee \mathfrak{X}(M)$ (see 2.6) we have $\mathscr{M}(M) \cap \mathscr{B}(M)$ is generic in $\mathscr{B}(M)$. Here we let

$$V \vee W = \{ \{v, w\} | v \in V \text{ and } w \in W\}.$$

This means that $\mathscr{F}^{\pm}(M) \times (\mathscr{M}(M) \cap \mathscr{B}(M))$ is generic in $\mathscr{F}^{\pm}(M) \times \mathscr{B}(M)$. This completes (a). (b) follows in exactly the same manner if one replaces $\mathscr{F}^{\pm}(M)$ by

$$\{f:M \to \mathbf{R} | f^{-1}(0) \text{ has measure zero}\}.$$

This completes the proof of our theorem.

4.10 THEOREM. *Let (T^2, g) be a Morse-Smale spacetime. The only harmonic functions on M are the constants.*

Proof. Any harmonic function h on M gives rise to a harmonic function $h\widetilde{\pi}$ on $\widetilde{M}$ (see 3.2). Since $\widetilde{M}$ is a two fold covering of M, all the conditions of this theorem lift to $\widetilde{M}$, so it is sufficient to show the result for $\widetilde{M}$. On $\widetilde{M}$, $w_1(E) = 0$. Also, $\widetilde{M}$ is a torus since it is a compact orientable spacetime. Thus we may use 4.3 together with 4.7 to find the exact sequence

$$0 \to \mathbf{R} \to \mathscr{F}_{E_1}(\widetilde{M}) \oplus \mathscr{F}_{E_2}(\widetilde{M}) \to H(\widetilde{g}) \to 0.$$

Now the result of Peixoto (2.7) says that this sequence is

$$0 \to \mathbf{R} \to \mathbf{R}^2 \to H(\widetilde{g}) \to 0.$$

Thus $\dim(H(\widetilde{g})) = 1$. The result is now complete since $H(\widetilde{g})$ clearly contains the constants.

4.11 THEOREM. *Let (T^2, g) be a Morse-Smale spacetime. If g' is a spacetime metric on M with* $\mathrm{Ric}(g') = \mathrm{Ric}(g)$, *then there is a constant k with $kg = g'$.*

Proof. Use 2.4 and 4.9.

5. Harmonic forms. Once one has a characterization of the harmonic functions it is easy to obtain a characterization of all the harmonic forms on a two dimensional manifold. We will restrict ourselves here to metrics that are Morse-Smale on compact manifolds for convenience. The crucial observation is the lemma below.

5.1 LEMMA. *Let Θ be a harmonic form on an arbitrary spacetime. Then $d\Theta$, $*\Theta$ and $\delta\Theta$ are all harmonic.*

Proof. If Θ is harmonic then $d\delta\Theta = -\delta d\Theta$. Thus

$$(d\delta + \delta d)d\Theta = d\delta d\Theta = -dd\delta\Theta = 0,$$

and $d\Theta$ is harmonic. Also if Θ is harmonic then

$$(*d*d + \epsilon d*d*)\Theta = 0$$

where $\epsilon = \pm 1$. We use the fact that $** = \eta I$ where $\eta = \pm 1$ (η depends on the signature of g, $\dim(M)$ and the rank of Θ). We have

$$\begin{aligned} *(*d*d + \epsilon d*d*)*\Theta &= *(*d*d* + \epsilon\eta d*d)\Theta \\ &= \epsilon\eta(\epsilon d*d* + *d*d)\Theta = 0. \end{aligned}$$

Thus $*\Theta$ is harmonic. It immediately follows that $\delta\Theta$ is also harmonic.

5.2 COROLLARY. *Θ is a harmonic $\dim(M)$ form $\Leftrightarrow \Theta = h\Omega$ where h is a harmonic function where Ω is an orientation on M with unit length.*

Thus we need only consider harmonic one forms to complete the description of the harmonic forms on a two dimensional spacetime. We summarize our results for compact Morse-Smale spacetimes below.

5.3 *Definition.* Let

$$\mathrm{HARM}^j(M) = \{\Theta | \Theta \text{ is a harmonic } j \text{ form}\}.$$

For positive definite compact metrics the Hodge-deRham theorem states that

$$\dim(\mathrm{HARM}^j(M)) = \dim(H^j(M, \mathbf{R})).$$

The result below says that this is nearly true for compact Morse-Smale spacetimes.

5.4 THEOREM. *Let m be a compact two dimensional Morse-Smale space-time. Then*

(a) $\dim(\mathrm{HARM}^0(M)) = \dim(H^0(M, \mathbf{R})) = 1$

(b) $\dim(\mathrm{HARM}^1(M)) \leqq \dim(H^1(M, \mathbf{R})) = 2$

(c) $\dim(\mathrm{HARM}^2(M)) = \dim(H^2(M, \mathbf{R})) = 1.$

Proof. (a) and (c) follow directly from 4.10 and 5.2. We must only show (b). Let Θ be a harmonic one form. We will first show Θ is closed and coclosed. $d\Theta$ is a harmonic two form by 5.1, and thus $d\Theta = c\Omega$ where c is constant by 4.10 and 5.2. Now

$$c\int_M \Omega = \int_M d\Theta = \int_{\partial M} \Theta = 0$$

since ∂M is empty. We also have that $\int_M \Omega > 0$ so that this equation implies $c = 0$. Thus $d\Theta = 0$. Using the same argument for $*\Theta$ (which is also harmonic) we find $d*\Theta = 0$ so that $\delta\Theta = 0$. To complete (b) we must only show that if $\Theta = df$ then $\Theta = 0$. If $\Theta = df$ then the above says that $\delta df = 0$ or f is harmonic. Thus f is constant and Θ is zero as desired.

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