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On a Certain Residual Spectrum of Sp_8

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Abstract. Let $G = \mathrm{Sp}_{2n}$ be the symplectic group defined over a number field F. Let $\mathbb{A}$ be the ring of adeles. A fundamental problem in the theory of automorphic forms is to decompose the right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2\big(G(F) \setminus G(\mathbb{A})\big)$. Main contributions have been made by Langlands. He described, using his theory of Eisenstein series, an orthogonal decomposition of this space of the form: $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big) = \bigoplus_{(M,\pi)} L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)}$, where (M,π) is a Levi subgroup with a cuspidal automorphic representation π taken modulo conjugacy. (Here we normalize π so that the action of the maximal split torus in the center of G at the archimedean places is trivial.) and $L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big)_{(M,\pi)}$ is a space of residues of Eisenstein series associated to (M,π). In this paper, we will completely determine the space $L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big)_{(M,\pi)}$, when $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_2$. This is the first result on the residual spectrum for non-maximal, non-Borel parabolic subgroups, other than GL_n.

1 Introduction

Let $G = \mathrm{Sp}_{2n}$ be the symplectic group defined over a number field F. Let $\mathbb{A}$ be the ring of adeles. A fundamental problem in the theory of automorphic forms is to decompose the right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2\big(G(F) \setminus G(\mathbb{A})\big)$.

The space $L^2\big(G(F) \setminus G(\mathbb{A})\big)$ has both a discrete spectrum and a continuous spectrum:

$$L^2\big(G(F) \setminus G(\mathbb{A})\big) = L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big) \oplus L^2_{\mathrm{cont}}\big(G(F) \setminus G(\mathbb{A})\big).$$

Since the continuous spectrum is well understood, we are mainly interested in the discrete spectrum. Main contributions have been made by Langlands [25]. He described, using his theory of Eisenstein series, an orthogonal decomposition of this space of the form:

$$L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big) = \bigoplus_{(M,\pi)} L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big)_{(M,\pi)},$$

where (M,π) is a Levi subgroup with a cuspidal automorphic representation π taken modulo conjugacy (Here we normalize π so that the action of the maximal split torus in the center of G at the archimedean places is trivial.) and $L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big)_{(M,\pi)}$ is a space of iterated residues of Eisenstein series associated to (M,π).

Here we note that the subspace

$$\bigoplus_{(G,\pi)} L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A})\big)_{(G,\pi)},$$

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is the space of cuspidal representations $L^2_{\mathrm{cusp}}\big(G(F)\setminus G(\mathbb{A})\big)$. Its orthogonal complement in $L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big)$ is called the *residual spectrum* and we denote it by $L^2_{\mathrm{res}}\big(G(F)\setminus G(\mathbb{A})\big)$. Therefore we have an orthogonal decomposition

$$L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big) = L^2_{\mathrm{cusp}}\big(G(F)\setminus G(\mathbb{A})\big) \oplus L^2_{\mathrm{res}}\big(G(F)\setminus G(\mathbb{A})\big).$$

For the problems in calculating the residual spectrum, we refer to the introduction by Kim [18].

In this paper, we will completely determine the space

$$L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_M,$$

when $G = \mathrm{Sp}_8$, $M \simeq \mathrm{GL}_2\times\mathrm{GL}_2$. This is the first result on the residual spectrum for non-maximal, non-Borel parabolic subgroups, other than GL_n.

The result is similar to the residual spectrum of Sp_4, coming from the Borel subgroup [17]. However, we need to use the root system of the non-maximal torus and the R-group attached to general parabolic subgroups. Also, the point β_3 in Figure 1 contributes to the residual spectrum, unlike the result in [17]. This agrees with the conjecture made in [16]. The conjecture in [16] is for odd orthogonal groups. However, it is easy to formulate a similar conjecture for symplectic groups. See Remark 9.6.

In order to describe our result, let $\pi = \pi_1\otimes\pi_1$ be a cuspidal representation of $M(\mathbb{A})$. Let $I(\gamma,\pi) = \mathrm{Ind}_P^G\,|\det|^{\frac{3}{2}}\,\pi_1\otimes|\det|^{\frac{1}{2}}\,\pi_1$ be the induced representation. Let $J(\gamma,\pi_v)$ be the unique quotient of $I(\gamma,\pi_v)$ for each v. (If π_v is tempered, it is the usual Langlands' quotient). It is the image of the intertwining operator $R(\sigma\tau\sigma\tau,\gamma,\pi_v)$. (See Section 9 for detail.) Let $J(\gamma,\pi) = \bigotimes_v J(\gamma,\pi_v)$.

Let $I(\beta_3,\pi) = \mathrm{Ind}_P^G\,|\det|^{\frac{1}{2}}\,\pi_1\otimes|\det|^{\frac{1}{2}}\,\pi_2$ be the induced representation. Let $J(\beta_3,\pi_v)$ be the unique quotient of $I(\beta_3,\pi_v)$ for each v. It is the image of the intertwining operator $R(\tau\sigma\tau,\beta_3,\pi_v)$. Let $J(\beta_3,\pi) = \bigotimes_v J(\beta_3,\pi_v)$.

Let $I(\beta_4,\pi) = \mathrm{Ind}_P^G\,|\det|\,\pi_1\otimes\pi_1$. By inducing in stages,

$$I(\beta_4,\pi_v) = \mathrm{Ind}_{\mathrm{GL}_2\times\mathrm{Sp}_4}^{\mathrm{Sp}_8}\,|\det|\otimes(\pi_{1v}\otimes\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4}\,\pi_{1v}).$$

Write $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4}\,\pi_{1v} = \pi_{+,+,v}\oplus\pi_{+,-,v}\oplus\pi_{-,+,v}\oplus\pi_{-,-,v}$ as in Section 5, where $\pi_{+,+,v}$ is generic with respect to ψ_v. Here we fix an additive character $\psi = \otimes\psi_v$ of $\mathbb{A}/F$.

Let $\epsilon(\pi_{+,+,v}) = \epsilon(\pi_{-,-,v}) = 1$, $\epsilon(\pi_{+,-,v}) = \epsilon(\pi_{-,+,v}) = -1$, and let $J_{\cdot,\cdot,v}$ be the Langlands' quotient of $\mathrm{Ind}_{\mathrm{GL}_2\times\mathrm{Sp}_4}^{\mathrm{Sp}_8}\,|\det|\otimes(\pi_{1v}\otimes\pi_{\cdot,\cdot,v})$. Let

$$J_v = \{J_{+,+,v}, J_{+,-,v}, J_{-,+,v}, J_{-,-,v}\}$$

and if $\rho\in J_v$, $\epsilon(\rho)$ be the corresponding sign and define $J(\pi)$ to be the collection $J(\pi) = \{\Pi = \otimes\Pi_v \mid \Pi_v\in J_v \text{ for all } v,\ \Pi_v = J_{+,+,v} \text{ for almost all } v,\ \prod_v\epsilon(\Pi_v) = 1\}$. Then

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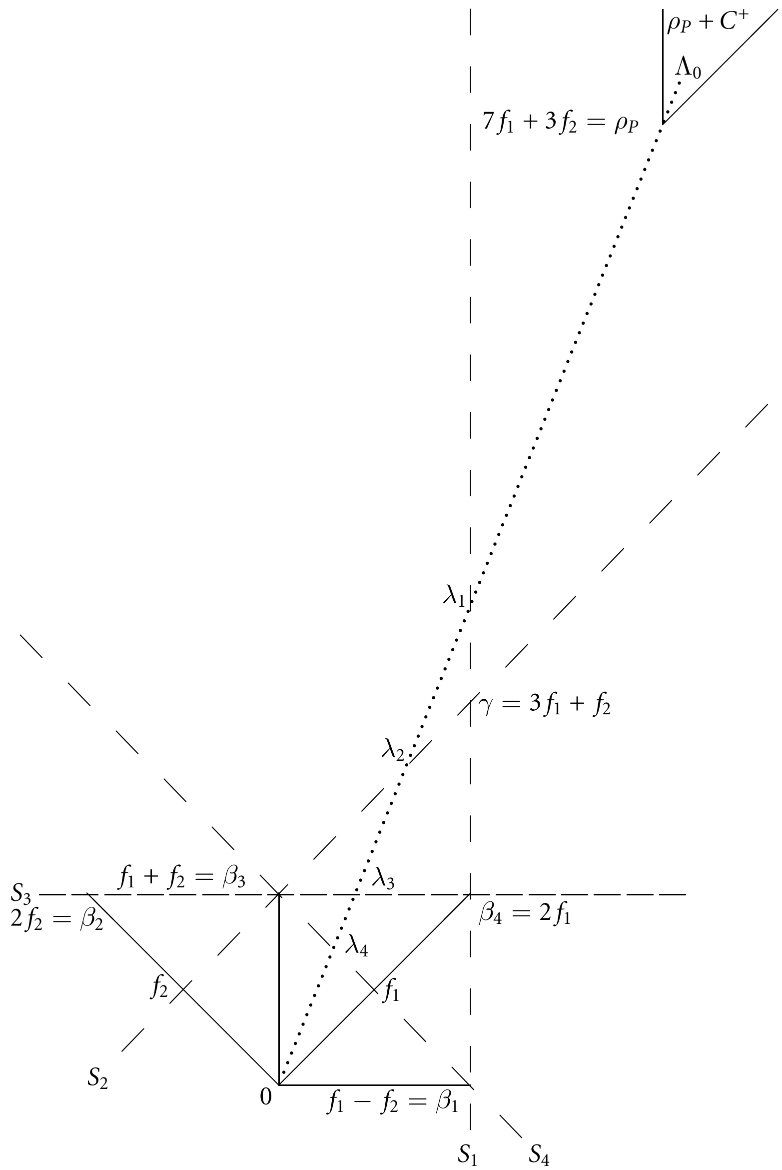

Figure 1: The real plane with the singular hyperplanes S_i as dashed lines and the contour that we are following as a dotted line segment.

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Theorem 1.1

$$L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_M = \Big(\bigoplus_{\pi} J(\gamma,\pi)\Big) \oplus \Big(\bigoplus_{\pi} J(\beta_3,\pi)\Big) \oplus J(\pi),$$

where

- *In the first sum,* $\pi = \pi_1 \otimes \pi_1$, π_1 *runs through cuspidal representations of* GL_2 *with the trivial central character such that* $L(\frac{1}{2},\pi_1) \neq 0$.
- *In the second sum,* $\pi = \pi_1 \otimes \pi_2$, $\pi_1 \not\simeq \pi_2$, $\omega_{\pi_1} = \omega_0$, $\omega_{\pi_2} = \omega_0$, $L(\frac{1}{2},\pi_1) \neq 0$, $L(\frac{1}{2},\pi_2) \neq 0$.
- *In the third summand,* $\pi = \pi_1 \otimes \pi_1$, π_1 *runs through self-contragredient monomial cuspidal representations of* GL_2.

Here the condition $\prod_v \epsilon(\Pi_v) = 1$ comes from subtle analysis of the normalized intertwining operator.

In a future work, we would like to study the residual spectrum coming from the Levi subgroup $\mathrm{GL}_1 \times \mathrm{Sp}_4 \subset \mathrm{Sp}_6$ and $\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{Sp}_4 \subset \mathrm{Sp}_8$. In the last two cases, the non-generic cuspidal representations of Sp_4 will generate singular hyperplanes at $\frac{3}{2}$ and 2, unlike generic cuspidal representations (*cf.* see [22]).

2 Symplectic Groups and Their Parabolic Subgroups

This section is essentially from Goldberg [9]. Let $G = \mathrm{Sp}_{2n}$. Let J_n be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

Let $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Then $\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid {}^t g J'_{2n} g = J'_{2n}\}$. Let A_0 be the maximal split torus consisting of diagonal matrices in G. Then

$$A_0 = \left\{ \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_n & & & \\ & & & & \lambda_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \lambda_2^{-1} \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} \,\middle|\, \lambda_i \in F^* \right\}.$$

Some denote a block diagonal matrix, $\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix}$, by $\mathrm{diag}\{X_1,\dots,X_k\}$. For a block of scalar matrix, $\begin{pmatrix} \lambda_1 I_{k_1} & & \\ & \ddots & \\ & & \lambda_j I_{k_j} \end{pmatrix}$, some write $\mathrm{diag}\{\lambda_1,\dots,\lambda_j\}$ if the dimensions k_i are clearly understood.

Let $\Phi(G,A_0)$ be the roots of G with respect to A_0. We prefer the Borel subgroup to be the subgroup of upper triangular matrices in G, so we choose the ordering on

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the roots accordingly. Let Δ be the simple roots in $\Phi(G, A_0)$ given by $\Delta = \{\alpha_j\}_{j=1}^{n}$, with $\alpha_j = e_j - e_{j+1}$ for $1 \leq j \leq n-1$, and $\alpha_n = 2e_n$. We let $\langle\ ,\ \rangle$ be the standard Euclidean inner product on $\Phi(G, A_0)$. Here Φ is a root system of type C_n.

Let $W(G/A_0)$ be the Weyl group of G with respect to A_0. Then $W(G/A_0) \simeq S_n \ltimes \mathbb{Z}_2^n$, where S_n acts by permutations on the λ_i, $i = 1, \dots, n$. We will use standard cycle notation for the elements of S_n. Thus $(i\,j)$ interchanges λ_i and λ_j. If c_i is the nontrivial element in the i-th copy of $\mathbb{Z}_2$, then c_i takes λ_i to λ_i^{-1}. The element c_i is called a sign change because its action on $\Phi(G, A_0)$ takes e_i to $-e_i$.

The parabolic subgroups of Sp_{2n} are of the form:

$$\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{Sp}_{2l}$$

where $n_1 + n_2 + \cdots + n_k + l = n$, $l \geq 1$ and Sp_2 is understood to be SL_2.

3 Roots and Weyl Group

Consider Sp_8. Let $\Delta = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, 2e_4\}$, $\theta = \{e_1 - e_2, e_3 - e_4\}$. Then $P = P_\theta = MN$ and $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_2$. Let A be the maximal torus in M. Then

$$A = \left\{ t(a_1, a_2) = \left(\begin{array}{cccc|cccc} a_1 &&&&&&& \\ & a_1 &&&&&& \\ && a_2 &&&&& \\ &&& a_2 &&&& \\ \hline &&&& a_2^{-1} &&& \\ &&&&& a_2^{-1} && \\ &&&&&& a_1^{-1} & \\ &&&&&&& a_1^{-1} \end{array} \right) \right\}.$$

Let $f_1 \colon t(a_1, a_2) \mapsto a_1$ and $f_2 \colon t(a_1, a_2) \mapsto a_2$. Let $\Phi(G, A)$ be the set of roots with respect to A. The positive roots are:

$$\beta_1 = f_1 - f_2, \quad \beta_2 = 2f_2,$$
$$\beta_3 = f_1 + f_2 \quad \text{and} \quad \beta_4 = 2f_1,$$

where $f_1 - f_2 = (e_1 - e_3)|_A$, $f_1 + f_2 = (e_1 + e_4)|_A$, $2f_1 = (e_1 + e_2)|_A$ and $2f_2 = (e_3 + e_4)|_A$. Let $\sigma = (1\,3)(2\,4)$, $\tau = c_3c_4$. Then $\sigma(\pi_1 \otimes \pi_2) = \pi_2 \otimes \pi_1$ and $\tau(\pi_1 \otimes \pi_2) = \pi_1 \otimes \tilde{\pi}_2$, where $\tilde{\pi}_2$ is the contragredient of π_2. Let $W(M)$ be the Weyl group of M. Then $W(M) = \{1, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma, \tau\sigma\tau, \sigma\tau\sigma\tau\}$. As usual, let $X(A)$ be the group of all rational characters of A defined over F, and let $\mathfrak{a}^* = X(A) \otimes \mathbb{R}$, $\mathfrak{a}^*_{\mathbb{C}} = X(A) \otimes \mathbb{C}$. The positive Weyl chamber in $\mathfrak{a}^*$ is

$$C^+ = \{\Lambda \in \mathfrak{a}^* \mid \langle \Lambda, \beta^\vee \rangle > 0, \text{ for all } \beta \text{ positive roots}\}.$$

We can see easily that $C^+ = \{af_1 + b(f_1 + f_2) \mid a, b > 0\}$. Let ρ_P be the half-sum of positive roots. Then $\rho_P = 7f_1 + 3f_2$.

Remark 3.1 See Figure 1. See Table 1 in order to see how the Weyl group elements act on the positive roots. This root system is C_2.

w	β_1	β_2	β_3	β_4	$w^{-1}(\pi_1 \otimes \pi_2)w$
1	β_1	β_2	β_3	β_4	$\pi_1 \otimes \pi_2$
σ	$-\beta_1$	β_4	β_3	β_2	$\pi_2 \otimes \pi_1$
τ	β_3	$-\beta_2$	β_1	β_4	$\pi_1 \otimes \tilde{\pi}_2$
$\sigma\tau$	β_3	$-\beta_4$	$-\beta_1$	β_2	$\tilde{\pi}_2 \otimes \pi_1$
$\tau\sigma$	$-\beta_3$	β_4	β_1	$-\beta_2$	$\pi_2 \otimes \tilde{\pi}_1$
$\sigma\tau\sigma$	$-\beta_3$	β_2	$-\beta_1$	$-\beta_4$	$\tilde{\pi}_1 \otimes \pi_2$
$\tau\sigma\tau$	β_1	$-\beta_4$	$-\beta_3$	$-\beta_2$	$\tilde{\pi}_2 \otimes \tilde{\pi}_1$
$\sigma\tau\sigma\tau$	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$\tilde{\pi}_1 \otimes \tilde{\pi}_2$

Table 1: Weyl group, together with their actions on the positive roots and on $\pi = \pi_1 \otimes \pi_2$.

4 Eisenstein Series and Pseudo-Eisenstein Series

This section essentially follows Kim [15]. Let $G = \mathrm{Sp}_8$, $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_2$. $P = MN$ is the parabolic subgroup. Let $\pi = \pi_1 \otimes \pi_2$ be a cuspidal representation of $M(\mathbb{A})$. For each $\Lambda = 2s_1 f_1 + 2s_2 f_2 \in \mathfrak{a}_{\mathbb{C}}^*$, we can define the induced representation $I(\Lambda, \pi) = \mathrm{Ind}_P^G |\det|^{s_1} \pi_1 \otimes |\det|^{s_2} \pi_2$ (See [33]), and we form the Eisenstein series:

$$E(g, \phi, \Lambda) = \sum_{\delta \in P(F)\backslash G(F)} \phi(\delta g),$$

where $\phi \in I(\Lambda, \pi)$. It converges absolutely for $\Re\Lambda \in \rho_P + C^+$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g, \phi, \Lambda)$ along P is given by

$$E_0(g, \phi, \Lambda) = \int_{N(F)\backslash N(\mathbb{A})} E(ng, \phi, \Lambda)\, dn = \sum_{w \in W(M)} M(w, \Lambda, \pi)\phi(g),$$

where $W(M)$ is the Weyl group of M and

$$M(w, \Lambda, \pi)\phi(g) = \int_{N_w(\mathbb{A})} \phi(w^{-1} ng)\, dn,$$

where $N_w = N \cap w\bar{N}w^{-1}$, $\bar{N}$ is the unipotent radical opposed to N. Then $M(w, \Lambda, \pi)$ defines a linear map from $I(\Lambda, \pi)$ to $I(w\Lambda, w\pi)$ and satisfies the functional equation of the form

$$M(w_1 w_2, \Lambda, \pi) = M(w_1, w_2\Lambda, w_2\pi)M(w_2, \Lambda, \pi).$$

Let S be a finite set of places of F, including all the archimedean places such that for every $v \notin S$, π_v and ψ_v are unramified and if $\phi = \otimes\phi_v$, for $v \notin S$, ϕ_v is the unique K_v-fixed function normalized by $\phi_v(e_v) = 1$. We have

$$M(w, \Lambda, \pi) = \bigotimes_v M(w, \Lambda, \pi_v).$$

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Then by applying Gindikin-Karpelevic method [24], we can see that for $v \notin S$,

$$M(w,\Lambda,\pi_v)\phi_v = \prod_{\beta>0, w\beta<0} \frac{L(\frac{1}{2}\langle\Lambda,\beta^\vee\rangle,\pi_v,\beta^\vee)}{L(\frac{1}{2}\langle\Lambda,\beta^\vee\rangle+1,\pi_v,\beta^\vee)}\tilde{\phi}_v,$$

where $\tilde{\phi}_v$ is the K_v-fixed function in the space of $I(w\Lambda, w\pi)$ satisfying $\tilde{\phi}_v(e_v) = 1$, and

$$L(s,\pi_v,\beta^\vee) = \begin{cases} L(s,\pi_{1v}\times\tilde{\pi}_{2v}), & \text{if } \beta=\beta_1=f_1-f_2,\\ L(s,\pi_{2v})L(2s,\omega_{\pi_{2v}}), & \text{if } \beta=\beta_2=2f_2,\\ L(s,\pi_{1v}\times\pi_{2v}), & \text{if } \beta=\beta_3=f_1+f_2,\\ L(s,\pi_{1v})L(2s,\omega_{\pi_{1v}}) & \text{if } \beta=\beta_4=2f_1.\end{cases}$$

Note that $L(s,\pi,\beta_1^\vee)$ has a pole at $s=0$, or $s=1$ iff $\pi_2 \simeq \tilde{\pi}_1$. $L(s,\pi,\beta_2^\vee)$ has a pole at $s=\frac{1}{2}$ iff $\omega_{\pi_2}=\omega_0$ and $L(\frac{1}{2},\pi_2)\neq 0$ [6, 8, 29, 30] where ω_0 is the trivial character. Let

$$S_i = \begin{cases} \{\Lambda\epsilon\mathfrak{a}_{\mathbb{C}}^* \mid \langle\Lambda,\beta_i^\vee\rangle = 2\}, & \text{if } i=1 \text{ or } i=3,\\ \{\Lambda\epsilon\mathfrak{a}_{\mathbb{C}}^* \mid \langle\Lambda,\beta_i^\vee\rangle = 1\}, & \text{if } i=2 \text{ or } i=4.\end{cases}$$

Thus, we get Figure 1.

For any v, let

$$r_v(w) = \prod_{\beta>0,w\beta<0} \frac{L(\frac{1}{2}\langle\Lambda,\beta^\vee\rangle,\pi_v,\beta^\vee)}{L(\frac{1}{2}\langle\Lambda,\beta^\vee\rangle+1,\pi_v,\beta^\vee)\epsilon(\frac{1}{2}\langle\Lambda,\beta^\vee\rangle,\pi_v,\beta^\vee,\psi_v)}.$$

We normalize the intertwining operator $M(w,\Lambda,\pi_v)$ for all v by

$$M(w,\Lambda,\pi_v) = r_v(w)R(w,\Lambda,\pi_v).$$

Let $R(w,\Lambda,\pi) = \bigotimes_v R(w,\Lambda,\pi_v)$ and $R(w,\Lambda,\pi)$ satisfies the functional equation

$$R(w_1w_2,\Lambda,\pi) = R(w_1,w_2\Lambda,w_2\pi)R(w_2,\Lambda,\pi).$$

Lemma 4.1 *Anent the holomorphy of rank-one local intertwining operators we have that:*

(i) $R(s,\pi_{1v}\otimes\pi_{2v},\omega_0)$ *is the intertwining operator for* $\mathrm{GL}_2\times\mathrm{GL}_2\subset\mathrm{GL}_4$. *It is holomorphic for* $\Re(s)\geq 0$.

(ii) $R(s,\pi_{1v},\omega_0)$ *is the intertwining operator for* $\mathrm{GL}_2\subset\mathrm{Sp}_4$. *It is holomorphic for* $\Re(s)\geq 0$.

Proof See [29] for the first assertion. See [17] for the second assertion. ■

For any $w \in W(M)$, $wMw^{-1}=M$ and so $(M,w\pi)$ is conjugate to (M,π). Let $I(\pi)$ be the set of entire functions ϕ of Paley-Wiener type such that $\phi(\Lambda)\in I(\Lambda,\pi)$ for each Λ. Let

$$\theta_\phi(g) = \Big(\frac{1}{2\pi i}\Big)^2 \int_{\Re\Lambda=\Lambda_0} E\big(g,\phi(\Lambda),\Lambda\big)\,d\Lambda,$$

where $\Lambda_0 \in \rho_P + C^+$. It is called the *pseudo-Eisenstein series.* Then we have

Lemma 4.2 **(Langlands [25])** *$L^2\big(G(F)\setminus G(\mathbb{A})\big)_{(M,\pi)}$, is the space spanned by θ_ϕ for all $\phi \in I(w\pi)$ as $w\pi$ runs through all distinct conjugates of π.*

Let $L^2_{\text{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_{(M,\pi)}$ be the discrete part of $L^2\big(G(F)\setminus G(\mathbb{A})\big)_{(M,\pi)}$. It is the set of iterated residues of $E\big(g,\phi(\Lambda),\Lambda\big)$ of order 2.

In order to decompose $L^2_{\text{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_{(M,\pi)}$, we use the inner product formula of two pseudo-Eisenstein series: Let π and π' be conjugate representations and $\phi \in I(\pi)$, $\phi' \in I(\pi')$. Then

$$\begin{aligned}\langle\theta_\phi,\theta_{\phi'}\rangle &= \frac{1}{(2\pi i)^2}\int_{\Re\Lambda=\Lambda_0}\sum_{w\in W(\pi,\pi')}\big(M(w^{-1},-w\bar\Lambda,w\pi)\phi'(-w\bar\Lambda),\phi(\Lambda)\big)\,d\Lambda\\ &= \frac{1}{(2\pi i)^2}\int_{\Re\Lambda=\Lambda_0}\sum_{w\in W(\pi,\pi')}\big(M(w,\Lambda,\pi)\phi(\Lambda),\phi'(-w\bar\Lambda)\big)\,d\Lambda\end{aligned}$$

where $W(\pi,\pi') = \{w\in W(M)\mid w\pi=\pi'\}$.

Let $\{d\pi \mid d\in D\}$ be the set of distinct conjugates of π. In order to deal with the distinct conjugates of π simultaneously, we consider, for $\phi\in I(\pi)$,

$$A(\phi,\phi',\Lambda) = \sum_{d\in D}\sum_{w\in W(\pi,d\pi)}\big(M(w,\Lambda,\pi)\phi(\Lambda),\phi'_d(-w\bar\Lambda)\big),$$

where $\phi'_d \in I(d\pi)$. Since $W(M) = \bigcup_{d\in D} W(\pi,d\pi)$, for simplicity, we write it as

$$A(\phi,\phi',\Lambda) = \sum_{w\in W(M)}\big(M(w,\Lambda,\pi)\phi(\Lambda),\phi'(-w\bar\Lambda)\big).$$

We also have the adjoint formula for the intertwining operators

$$\begin{aligned}\big(M(w,\Lambda,\pi)\phi(\Lambda),\phi'(-w\bar\Lambda)\big) &= \big(\phi(\Lambda),M(w^{-1},-w\bar\Lambda,w\pi)\phi'(-w\bar\Lambda)\big)\\ \big(R(w,\Lambda,\pi)\phi(\Lambda),\phi'(-w\bar\Lambda)\big) &= \big(\phi(\Lambda),R(w^{-1},-w\bar\Lambda,w\pi)\phi'(-w\bar\Lambda)\big).\end{aligned}$$

We use this adjoint formula and calculate the residue of $A(\phi,\phi',\Lambda)$ to obtain the residual spectrum $L^2_{\text{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_{(M,\pi)}$. Let

$$A^i(\phi,\phi',\Lambda) = \mathrm{Res}_{S_i} A(\phi,\phi',\Lambda).$$

In order to get the discrete spectrum, we have to deform the contour $\Re\Lambda = \Lambda_0$ to $\Re\Lambda = 0$. Since the poles of the functions $M(w,\Lambda,\pi)$ all lie on S_i which is defined by real equations we can represent the process of deforming the contour with a dotted line segment and each singular hyperplane S_i as a dashed line in Figure 1.

We need to calculate the following iterated residues (see [27, 17]):

$$\begin{gathered}
\mathrm{Res}_{\beta_1} \mathrm{Res}_{S_1} A(\phi, \phi', \Lambda), \\
\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_1} A(\phi, \phi', \Lambda), \\
\mathrm{Res}_{\gamma} \mathrm{Res}_{S_1} A(\phi, \phi', \Lambda), \\
\mathrm{Res}_{f_2} \mathrm{Res}_{S_2} A(\phi, \phi', \Lambda), \\
\mathrm{Res}_{\beta_3} \mathrm{Res}_{S_2} A(\phi, \phi', \Lambda), \\
\mathrm{Res}_{\beta_3} \mathrm{Res}_{S_3} A(\phi, \phi', \Lambda) \quad \text{and} \\
\mathrm{Res}_{f_1} \mathrm{Res}_{S_4} A(\phi, \phi', \Lambda).
\end{gathered}$$

Notation 4.3 Let us write

$$\begin{gathered}
L(s, \omega_0) = \frac{c_2(F)}{s-1} + l_0 + l_1(s-1) + \cdots, \\
a_{-1} = \mathrm{Res}_{s=1} \frac{L(s, \pi_1 \times \tilde{\pi}_1)}{L(s+1, \pi_1 \times \tilde{\pi}_1)\epsilon(s, \pi_1 \times \tilde{\pi}_1)}, \\
b_{-1} = \mathrm{Res}_{s=1} \frac{L(s, \omega_0)}{L(s+1, \omega_0)\epsilon(s, \omega_0)}, \\
c_1(F) = \mathrm{Res}_{s=1} L(s, \pi_1 \times \tilde{\pi}_1), \\
c_2(F) = \mathrm{Res}_{s=1} L(s, \omega_0).
\end{gathered}$$

We set

$$M^i(w, \Lambda, \pi) = \begin{cases}
\frac{1}{a_{-1}} \mathrm{Res}_{S_1} M(w, \Lambda, \pi) & \text{if } i = 1, \\
\frac{L(\frac{3}{2}, \pi_2)\epsilon(\frac{1}{2}, \pi_2)}{b_{-1} L(\frac{1}{2}, \pi_2)} \mathrm{Res}_{S_2} M(w, \Lambda, \pi) & \text{if } i = 2, \\
\frac{1}{a_{-1}} \mathrm{Res}_{S_3} M(w, \Lambda, \pi) & \text{if } i = 3, \\
\frac{L(\frac{3}{2}, \pi_1)\epsilon(\frac{1}{2}, \pi_1)}{b_{-1} L(\frac{1}{2}, \pi_1)} \mathrm{Res}_{S_4} M(w, \Lambda, \pi) & \text{if } i = 4.
\end{cases}$$

5 Along S_1

$M(w, \Lambda, \pi)$ has a pole on S_1 only when $\pi_2 \simeq \pi_1$. From Table 1, we see that $M(w, \Lambda, \pi)$ has a pole when $w = \sigma, \tau\sigma, \sigma\tau\sigma, \sigma\tau\sigma\tau$. For $\Lambda = 2z\beta_3 + \beta_1 = (2z+1)f_1 + (2z-1)f_2$, $\langle \Lambda, \beta_2^\vee \rangle = 2z - 1$, $\langle \Lambda, \beta_3^\vee \rangle = 4z$ and $\langle \Lambda, \beta_4^\vee \rangle = 2z + 1$. Then

Lemma 5.1

$$M^1(\sigma,\Lambda,\pi)\phi = R(\sigma,\Lambda,\pi)\phi,$$

$$M^1(\tau\sigma,\Lambda,\pi)\phi = \frac{L(z+\frac{1}{2},\pi_1)L(2z+1,\omega_{\pi_1})R(\tau\sigma,\Lambda,\pi)\phi}{L(z+\frac{3}{2},\pi_1)L(2z+2,\omega_{\pi_1})\epsilon(z+\frac{1}{2},\pi_1)\epsilon(2z+1,\omega_{\pi_1})},$$

$$M^1(\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L(2z,\pi_1\times\pi_1)L(z+\frac{1}{2},\pi_1)L(2z+1,\omega_{\pi_1})R(\sigma\tau\sigma,\Lambda,\pi)\phi}{L(2z+1,\pi_1\times\pi_1)L(z+\frac{3}{2},\pi_1)L(2z+2,\omega_{\pi_1})\epsilon\bigstar_1},$$

$$M^1(\tau\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L\bigstar_1 R(\tau\sigma\tau\sigma,\Lambda,\pi)\phi}{L\bigstar_2\epsilon\bigstar_2},$$

where

$$\epsilon\bigstar_1 = \epsilon(2z,\pi_1\times\pi_1)\epsilon\Big(z+\frac{1}{2},\pi_1\Big)\epsilon(2z+1,\omega_{\pi_1}),$$

$$\epsilon\bigstar_2 = \epsilon\Big(z-\frac{1}{2},\pi_1\Big)\epsilon(2z-1,\omega_{\pi_1})\epsilon(2z,\pi_1\times\pi_1)\epsilon\Big(z+\frac{1}{2},\pi_1\Big)\epsilon(2z+1,\omega_{\pi_1}),$$

$$L\bigstar_1 = L\Big(z-\frac{1}{2},\pi_1\Big)L(2z-1,\omega_{\pi_1})L\big(2z,\mathrm{Sym}^2(\pi_1)\big),$$

$$L\bigstar_2 = L\big(2z+1,\mathrm{Sym}^2(\pi_1)\big)L\Big(z+\frac{3}{2},\pi_1\Big)L(2z+2,\omega_{\pi_1}).$$

Remark 5.2 Note that $L(s,\pi_1\times\pi_1) = L\big(s,\mathrm{Sym}^2(\pi_1)\big)L(s,\omega_{\pi_1})$ where $\mathrm{Sym}^2(\pi_1)$ is the symmetric square, which is an automorphic representation of GL_3 [7]. Hence there is a cancellation between $L(2z,\pi_1\times\pi_1)$ and $L(2z,\omega_{\pi_1})$. Likewise, there is a cancellation between $L(2z+1,\pi_1\times\pi_1)$ and $L(2z+1,\omega_{\pi_1})$.

Proposition 5.3 *If $\pi_1\simeq\tilde{\pi}_1$, $\omega_{\pi_1}\neq\omega_0$, then $A^1(\phi,\phi',\Lambda)$ has a pole at $\Lambda=\beta_4$, i.e. $z=\frac{1}{2}$, that is square integrable, but does not have a pole at $\Lambda=\beta_1$ or $\Lambda=\gamma$.*

Proof From Lemma 5.1, we can see by direct observation that there is not a pole at $\Lambda=\beta_1$, *i.e.* $z=0$, nor is there a pole at $\Lambda=\gamma$, *i.e.* $z=1$. So let us consider the pole at $\Lambda=\beta_4$, *i.e.* $z=\frac{1}{2}$. Then

$$\begin{aligned}\mathrm{Res}_{\beta_4} M^1(\sigma\tau\sigma,\Lambda,\pi)\phi &= \frac{\big(\frac{1}{2}c_1(F)\big)L(1,\pi_1)L(2,\omega_{\pi_1})R(\sigma\tau\sigma,\beta_4,\pi)\phi}{L(2,\pi_1\times\pi_1)L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}\\ &= \frac{c_1(F)L(1,\pi_1)L(2,\omega_{\pi_1})R(\sigma\tau\sigma,\beta_4,\pi)\phi}{2L(2,\pi_1\times\pi_1)L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})},\end{aligned}$$

$$\begin{aligned}\mathrm{Res}_{\beta_4} M^1(\tau\sigma\tau\sigma,\Lambda,\pi)\phi &= \frac{L(0,\pi_1)\big(\frac{1}{2}c_1(F)\big)L(2,\omega_{\pi_1})R(\tau\sigma\tau\sigma,\beta_4,\pi)\phi}{L(2,\pi_1\times\pi_1)L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(0,\pi_1)\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}\\ &= \frac{c_1(F)L(1,\pi_1)L(2,\omega_{\pi_1})R(\tau\sigma\tau\sigma,\beta_4,\pi)\phi}{2L(2,\pi_1\times\pi_1)L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}.\end{aligned}$$

Since $\sigma\tau\sigma\beta_4 = -\beta_4 = -2\beta_1 - 1\beta_2$, $\sigma\tau\sigma\tau\beta_4 = -\beta_4$, we have that

$$\operatorname{Res}_{\beta_4} \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda)$$

is square integrable. Here

$$\begin{aligned}
&\operatorname{Res}_{\beta_4} \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda) \\
&\quad = \frac{\big(c_1(F)\big)^2 L(1,\pi_1)L(2,\omega_{\pi_1})}{2\big(L(2,\pi_1\times\pi_1)\big)^2 L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}\langle\bigstar_1\rangle \\
&\qquad + \frac{\big(c_1(F)\big)^2 L(1,\pi_1)L(2,\omega_{\pi_1})}{2\big(L(2,\pi_1\times\pi_1)\big)^2 L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}\langle\bigstar_2\rangle \\
&\quad = \frac{\big(c_1(F)\big)^2 L(1,\pi_1)L(2,\omega_{\pi_1})}{2\big(L(2,\pi_1\times\pi_1)\big)^2 L(2,\pi_1)L(3,\omega_{\pi_1})\epsilon(1,\pi_1)\epsilon(2,\omega_{\pi_1})}\langle\bigstar_3\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\langle\bigstar_1\rangle &= \langle R(\sigma\tau\sigma,\beta_4,\pi)\phi(\beta_4),\phi'(\beta_4)\rangle, \\
\langle\bigstar_2\rangle &= \langle R(\tau\sigma\tau\sigma,\beta_4,\pi)\phi(\beta_4),\phi'(\beta_4)\rangle, \\
\langle\bigstar_3\rangle &= \langle R(\sigma\tau\sigma,\beta_4,\pi)\big(I + R(\tau,\beta_4,\pi)\big)\phi(\beta_4),\phi'(\beta_4)\rangle,
\end{aligned}$$

because $R(\tau\sigma\tau\sigma,\beta_4,\pi) = R(\sigma\tau\sigma,\tau\beta_4,\tau\pi)R(\tau,\beta_4,\pi) = R(\sigma\tau\sigma,\beta_4,\pi)R(\tau,\beta_4,\pi)$ since $\sigma\tau\sigma\tau = \tau\sigma\tau\sigma$. ∎

Remark 5.4 If $\pi_1 \simeq \tilde{\pi}_1$, $\omega_{\pi_1} \neq \omega_0$, then $\pi_1 \simeq \pi_1 \otimes \omega_{\pi_1}^{-1}$. Hence π_1 is a monomial cuspidal representation. Since $\omega_{\pi_1}^2 = 1$, ω_{π_1} determines a quadratic extension E/F. Then, there exists a grössencharacter χ of E such that $\pi_1 = \pi(\chi)$ (See [7, 23]).

Remark 5.5 As we deform the contour from β_4 to β_1, the normalized operator $R(\tau\sigma\tau\sigma,\Lambda,\pi_v)$ may have a pole, because the rank-one operator $R(\tau,\sigma\tau\sigma\Lambda,\sigma\tau\sigma\pi_v)$ is an operator on the negative Weyl chamber for $0 \leq z < \frac{1}{2}$.

However, we ignored the fact, since the pole can be easily removed:

Denote

$$\begin{aligned}
&\tilde{A}(\tau\sigma\tau\sigma,\Lambda,\pi_v) \\
&\quad = \frac{M(\tau\sigma\tau\sigma,\Lambda,\pi_v)}{L(z-\frac{1}{2},\pi_{1v})L(2z-1,\omega_{\pi_{1v}})L(2z,\pi_{1v}\times\pi_{1v})L(z+\frac{1}{2},\pi_{1v})L(2z+1,\omega_{\pi_{1v}})}.
\end{aligned}$$

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Then

$$M^1(\tau\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L(z-\frac{1}{2},\pi_1)L(2z-1,\omega_{\pi_1})L(2z+1,\omega_{\pi_1})}{L_S(2z+1,\pi_1\times\pi_1)L_S(z+\frac{3}{2},\pi_1)L_S(2z+2,\omega_{\pi_1})}$$

$$\times L_S\big(2z,\mathrm{Sym}^2(\pi_1)\big)\prod_{v\in S}L\Big(z+\frac{1}{2},\pi_{1v}\Big)L(2z,\pi_{1v}\times\pi_{1v})$$

$$\times \bigotimes_{v\notin S}\tilde{\phi}_v\otimes\bigotimes_{v\in S}\tilde{A}(\tau\sigma\tau\sigma,\Lambda,\pi_v).$$

By [5], $\tilde{A}(\tau\sigma\tau\sigma,\Lambda,\pi_v)$ is entire.

Hence for $0<z<\frac{1}{2}$, $M^1(\tau\sigma\tau\sigma,\Lambda,\pi)$ has no pole.

For $z=0$, we write $M^1(\tau\sigma\tau\sigma,\Lambda,\pi)$ as follows:

$$M^1(\tau\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L_S(z-\frac{1}{2},\pi_1)L_S(2z-1,\omega_{\pi_1})L_S\big(2z,\mathrm{Sym}^2(\pi_1)\big)}{L_S\big(2z+1,\mathrm{Sym}^2(\pi_1)\big)L_S(z+\frac{3}{2},\pi_1)L_S(2z+2,\omega_{\pi_1})}$$

$$\times \bigotimes_{v\notin S}\tilde{\phi}_v\otimes\bigotimes_{v\in S}M(\tau\sigma\tau\sigma,\Lambda,\pi_v).$$

Here $M(\tau\sigma\tau\sigma,\Lambda,\pi_v) = M(\tau,\sigma\tau\sigma\Lambda,\sigma\tau\sigma\pi_v)M(\sigma\tau\sigma,\Lambda,\pi_v)$. $M(\sigma\tau\sigma,\Lambda,\pi_v)$ is holomorphic at $z=0$. Also $M(\tau,\sigma\tau\sigma\Lambda,\sigma\tau\sigma\pi_v)$ is holomorphic at $z=0$, since $L(z-\frac{1}{2},\pi_{1v})L(2z-1,\omega_{\pi_{1v}})$ has no pole at $z=0$. Therefore, $M^1(\tau\sigma\tau\sigma,\Lambda,\pi)$ is holomorphic at $z=0$.

Similarly, we will ignore the problem of a pole of $R(\sigma\tau\sigma\tau,\Lambda,\pi_v)$ in Proposition 5.6 and Proposition 6.3.

Proposition 5.6 *If $\omega_{\pi_1}=\omega_0$, $L(\frac{1}{2},\pi_1)\neq 0$ then $A^1(\phi,\phi',\Lambda)$ has a possible pole at $\Lambda=\beta_1$, i.e. $z=0$, at $\Lambda=\beta_4$, i.e. $z=\frac{1}{2}$ and at $\Lambda=\gamma$, i.e. $z=1$. Furthermore,*

(i) $\mathrm{Res}_{\beta_1}\mathrm{Res}_{S_1}A(\phi,\phi',\Lambda)=0$,

(ii) $\mathrm{Res}_{\beta_4}\mathrm{Res}_{S_1}A(\phi,\phi',\Lambda)=0$,

(iii) $\mathrm{Res}_{\gamma}\mathrm{Res}_{S_1}A(\phi,\phi',\Lambda)$ *is square integrable.*

Proof (i)

$$\mathrm{Res}_{\beta_1}M^1(\tau\sigma,\Lambda,\pi)\phi = \frac{L(\frac{1}{2},\pi_1)\big(\frac{1}{2}c_2(F)\big)R(\tau\sigma,\beta_1,\pi)\phi}{L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)}$$

$$= \frac{c_2(F)L(\frac{1}{2},\pi_1)R(\tau\sigma,\beta_1,\pi)\phi}{2L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)},$$

$$\mathrm{Res}_{\beta_1}M^1(\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{-L(\frac{1}{2},\pi_1)\big(\frac{1}{2}c_2(F)\big)R(\sigma\tau\sigma,\beta_1,\pi)\phi}{L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)}$$

$$= \frac{-c_2(F)L(\frac{1}{2},\pi_1)R(\sigma\tau\sigma,\beta_1,\pi)\phi}{2L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)}.$$

Since $\pi_2 \simeq \pi_1$ and $\omega_{\pi_1} = \omega_0$, we have that $\tau\sigma\pi = \pi$. So $R(\sigma, \tau\sigma\beta_1, \tau\sigma\pi)$ is the identity. Hence $R(\sigma\tau\sigma, \Lambda, \pi) = R(\sigma, \tau\sigma\Lambda, \tau\sigma\pi)R(\tau\sigma, \Lambda, \pi)$ implies that $\operatorname{Res}_{\beta_1} \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda) = 0$.

(ii)

$$\begin{aligned}\operatorname{Res}_{\beta_4} M^1(\sigma\tau\sigma, \Lambda, \pi)\phi &= \frac{\left(\frac{1}{2}c_1(F)\right) L(1, \pi_1)L(2, \omega_0)R(\sigma\tau\sigma, \beta_4, \pi)\phi}{L(2, \pi_1 \times \pi_1)L(2, \pi_1)L(3, \omega_0)\epsilon(1, \pi_1)\epsilon(2, \omega_0)} \\ &= \frac{c_1(F)L(1, \pi_1)L(2, \omega_0)R(\sigma\tau\sigma, \beta_4, \pi)\phi}{2L(2, \pi_1 \times \pi_1)L(2, \pi_1)L(3, \omega_0)\epsilon(1, \pi_1)\epsilon(2, \omega_0)},\end{aligned}$$

$$\begin{aligned}\operatorname{Res}_{\beta_4} M^1(\tau\sigma\tau\sigma, \Lambda, \pi)\phi &= \frac{L(0, \pi_1)(-1)\left(\frac{1}{2}c_1(F)\right) L(2, \omega_0)R(\tau\sigma\tau\sigma, \beta_4, \pi)\phi}{L(2, \pi_1 \times \pi_1)L(2, \pi_1)L(3, \omega_0)\epsilon(0, \pi_1)\epsilon(1, \pi_1)\epsilon(2, \omega_0)} \\ &= \frac{-c_1(F)L(1, \pi_1)L(2, \omega_0)R(\tau\sigma\tau\sigma, \beta_4, \pi)\phi}{2L(2, \pi_1 \times \pi_1)L(2, \pi_1)L(3, \omega_0)\epsilon(1, \pi_1)\epsilon(2, \omega_0)}.\end{aligned}$$

Since $\pi_2 \simeq \pi_1$ and $\omega_{\pi_1} = \omega_0$, we have that $\sigma\tau\sigma\pi = \pi$. So $R(\tau, \sigma\tau\sigma\beta_4, \sigma\tau\sigma\pi)$ is the identity. Hence $R(\sigma\tau\sigma\tau, \Lambda, \pi) = R(\tau, \sigma\tau\sigma\Lambda, \sigma\tau\sigma\pi)R(\sigma\tau\sigma, \Lambda, \pi)$ implies that $\operatorname{Res}_{\beta_4} \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda) = 0$.

(iii)

$$\begin{aligned}&\operatorname{Res}_\gamma M^1(\tau\sigma\tau\sigma, \Lambda, \pi)\phi \\ &= \frac{L(\frac{1}{2}, \pi_1)\left(\frac{1}{2}c_2(F)\right) L(2, \pi_1 \times \pi_1)L(3, \omega_0)R(\tau\sigma\tau\sigma, \gamma, \pi)\phi}{L(2, \omega_0)L(3, \pi_1 \times \pi_1)L(\frac{5}{2}, \pi_1)L(4, \omega_0)\epsilon(\frac{1}{2}, \pi_1)\epsilon(2, \pi_1 \times \pi_1)\epsilon(\frac{3}{2}, \pi_1)\epsilon(3, \omega_0)} \\ &= \frac{c_2(F)L(\frac{1}{2}, \pi_1)L(2, \pi_1 \times \pi_1)L(3, \omega_0)R(\tau\sigma\tau\sigma, \gamma, \pi)\phi}{2L(2, \omega_0)L(3, \pi_1 \times \pi_1)L(\frac{5}{2}, \pi_1)L(4, \omega_0)\epsilon(\frac{1}{2}, \pi_1)\epsilon(2, \pi_1 \times \pi_1)\epsilon(\frac{3}{2}, \pi_1)\epsilon(3, \omega_0)}.\end{aligned}$$

So

$$\begin{aligned}&\operatorname{Res}_\gamma \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda) \\ &\quad= \frac{c_1(F)c_2(F)L(\frac{1}{2}, \pi_1)L(2, \pi_1 \times \pi_1)L(3, \omega_0)}{2L(2, \pi_1 \times \pi_1)L(2, \omega_0)L(3, \pi_1 \times \pi_1)L(\frac{5}{2}, \pi_1)L(4, \omega_0)\epsilon\bigstar_3}\langle\bigstar_4\rangle \\ &\quad= \frac{c_1(F)c_2(F)L(\frac{1}{2}, \pi_1)L(3, \omega_0)}{2L(2, \omega_0)L(3, \pi_1 \times \pi_1)L(\frac{5}{2}, \pi_1)L(4, \omega_0)\epsilon\bigstar_3}\langle\bigstar_4\rangle,\end{aligned}$$

where

$$\langle\bigstar_4\rangle = \langle R(\tau\sigma\tau\sigma, \gamma, \pi)\phi(\gamma), \phi'(\gamma)\rangle,$$

$$\epsilon\bigstar_3 = \epsilon\left(\frac{1}{2}, \pi_1\right)\epsilon(2, \pi_1 \times \pi_1)\epsilon\left(\frac{3}{2}, \pi_1\right)\epsilon(3, \omega_0).$$

Since $\sigma\tau\sigma\tau\gamma = -\gamma = -3\beta_1 - 2\beta_2$, we have that $\operatorname{Res}_\gamma \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda)$ is square integrable. ∎

6 Along S_2

$M(w,\Lambda,\pi)$ has a pole on S_2 only when $\omega_{\pi_2}=\omega_0$, $L(\frac{1}{2},\pi_2)\neq 0$. From Table 1, $M(w,\Lambda,\pi)$ has a pole when $w=\tau,\sigma\tau,\tau\sigma\tau,\sigma\tau\sigma\tau$. For $\Lambda=2zf_1+f_2$, $\langle\Lambda,\beta_1^\vee\rangle=2z-1$, $\langle\Lambda,\beta_3^\vee\rangle=2z+1$ and $\langle\Lambda,\beta_4^\vee\rangle=2z$. Note that if $\omega_{\pi_2}=\omega_0$, then $\pi_2\simeq\tilde{\pi}_2$. Then

Lemma 6.1

$$M^2(\tau,\Lambda,\pi)\phi=R(\tau,\Lambda,\pi)\phi,$$

$$M^2(\sigma\tau,\Lambda,\pi)\phi=\frac{L(z+\frac{1}{2},\pi_1\times\pi_2)R(\sigma\tau,\Lambda,\pi)\phi}{L(z+\frac{3}{2},\pi_1\times\pi_2)\epsilon(z+\frac{1}{2},\pi_1\times\pi_2)},$$

$$M^2(\tau\sigma\tau,\Lambda,\pi)\phi=\frac{L(z+\frac{1}{2},\pi_1\times\pi_2)L(z,\pi_1)L(2z,\omega_{\pi_1})R(\tau\sigma\tau,\Lambda,\pi)\phi}{L(z+\frac{3}{2},\pi_1\times\pi_2)L(z+1,\pi_1)L(2z+1,\omega_{\pi_1})\epsilon\bigstar_4},$$

$$M^2(\sigma\tau\sigma\tau,\Lambda,\pi)\phi=\frac{L\bigstar_3R(\sigma\tau\sigma\tau,\Lambda,\pi)\phi}{L\bigstar_4\epsilon\bigstar_5},$$

where

$$\epsilon\bigstar_4=\epsilon\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big)\epsilon(z,\pi_1)\epsilon(2z,\omega_{\pi_1}),$$

$$\epsilon\bigstar_5=\epsilon\Big(z-\frac{1}{2},\pi_1\times\pi_2\Big)\epsilon\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big)\epsilon(z,\pi_1)\epsilon(2z,\omega_{\pi_1}),$$

$$L\bigstar_3=L\Big(z-\frac{1}{2},\pi_1\times\pi_2\Big)L(z,\pi_1)L(2z,\omega_{\pi_1}),$$

$$L\bigstar_4=L\Big(z+\frac{3}{2},\pi_1\times\pi_2\Big)L(z+1,\pi_1)L(2z+1,\omega_{\pi_1}).$$

Proposition 6.2 *If $\omega_{\pi_1}=\omega_0$, $L(\frac{1}{2},\pi_1)\neq 0$ and $\pi_1\not\simeq\pi_2$, then $A^2(\phi,\phi',\Lambda)$ has a simple pole at $\Lambda=\beta_3$, i.e., $z=\frac{1}{2}$ and $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_2}A(\phi,\phi',\Lambda)$ is square integrable.*

Proof From Lemma 6.1, $M^2(\tau\sigma\tau,\Lambda,\pi)$ and $M^2(\sigma\tau\sigma\tau,\Lambda,\pi)$ have a pole at $\Lambda=\beta_3$. Then

$$\begin{aligned}\operatorname{Res}_{\beta_3}M^2(\tau\sigma\tau,\beta_3,\pi)\phi&=\frac{L(1,\pi_1\times\pi_2)L(\frac{1}{2},\pi_1)\big(\frac{1}{2}c_2(F)\big)R(\tau\sigma\tau,\beta_3,\pi)\phi}{L(2,\pi_1\times\pi_2)L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(1,\pi_1\times\pi_2)\epsilon(\frac{1}{2},\pi_1)}\\&=\frac{c_2(F)L(1,\pi_1\times\pi_2)L(\frac{1}{2},\pi_1)R(\tau\sigma\tau,\beta_3,\pi)\phi}{2L(2,\pi_1\times\pi_2)L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(1,\pi_1\times\pi_2)\epsilon(\frac{1}{2},\pi_1)},\end{aligned}$$

$$\begin{aligned}&\operatorname{Res}_{\beta_3}M^2(\sigma\tau\sigma\tau,\beta_3,\pi)\phi\\&\quad=\frac{L(0,\pi_1\times\pi_2)L(\frac{1}{2},\pi_1)\big(\frac{1}{2}c_2(F)\big)R(\sigma\tau\sigma\tau,\beta_3,\pi)\phi}{L(2,\pi_1\times\pi_2)L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon\bigstar_6}\\&\quad=\frac{c_2(F)L(1,\pi_1\times\pi_2)L(\frac{1}{2},\pi_1)R(\sigma\tau\sigma\tau,\beta_3,\pi)\phi}{2L(2,\pi_1\times\pi_2)L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(1,\pi_1\times\pi_2)\epsilon(\frac{1}{2},\pi_1)},\end{aligned}$$

where

$$\epsilon\bigstar_6 = \epsilon(0,\pi_1\times\pi_2)\epsilon(1,\pi_1\times\pi_2)\epsilon\Big(\frac{1}{2},\pi_1\Big).$$

Here we have used the fact that $L(0,\pi_1\times\pi_2) = \epsilon(0,\pi_1\times\pi_2)L(1,\pi_1\times\pi_2)$.

$$\begin{aligned}\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_2} A(\phi,\phi',\Lambda) = (c)\langle R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle\\ + (c)\langle R(\sigma\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle,\end{aligned}$$

where

$$c = \frac{c_2(F)L(1,\pi_1\times\pi_2)L(\frac{1}{2},\pi_1)}{2L(2,\pi_1\times\pi_2)L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(1,\pi_1\times\pi_2)\epsilon(\frac{1}{2},\pi_1)}.$$

We note that $\tau\sigma\tau(\pi)\simeq\pi_2\otimes\pi_1$, $\sigma\tau\sigma\tau(\pi)=\pi$. Let $\pi'=\pi_2\otimes\pi_1$. Hence, ϕ' in the first summand belongs to $I(\pi')$ but ϕ' in the second summand belongs to $I(\pi)$. Note our short-hand notation in the definition of $A(\phi,\phi',\Lambda)$. Here, $R(\sigma\tau\sigma\tau,\beta_3,\pi_v) = R(\tau\sigma\tau,\beta_3,\pi'_v)R(\sigma,\beta_3,\pi_v)$ and $R(\sigma,\beta_3,\pi_v)\colon I(\beta_3,\pi_v)\to I(\beta_3,\pi'_v)$ is an isomorphism. Hence the image of $R(\tau\sigma\tau,\beta_3,\pi_v)$ and the image of $R(\sigma\tau\sigma\tau,\beta_3,\pi_v)$ are equivalent. Since $\tau\sigma\tau\beta_3 = -\beta_3 = -1\beta_1 - 1\beta_2$, $\sigma\tau\sigma\tau\beta_3 = -\beta_3$, we have that $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_2} A(\phi,\phi',\Lambda)$ is square integrable. ■

Proposition 6.3 *If $\pi_1\simeq\pi_2$, then $A^2(\phi,\phi',\Lambda)$ has a double pole at $\Lambda=\beta_3$, i.e. $z=\frac{1}{2}$, but does not have a pole at $\Lambda=f_2$.*

Proof By direct observation of Lemma 6.1, there is not a pole at $\Lambda=f_2$, *i.e.* $z=0$. So let us consider the double pole at $\Lambda=\beta_3$, *i.e.* $z=\frac{1}{2}$. In order to calculate the residue, we use the following notations where $\pi_1\simeq\pi_2\simeq\tilde{\pi}_2$,

$$\frac{L(z+\frac{1}{2},\pi_1\times\pi_1)}{L(z+\frac{3}{2},\pi_1\times\pi_1)\epsilon(z+\frac{1}{2},\pi_1\times\pi_1)} = \frac{a_{-1}}{z-\frac{1}{2}} + a_0 + a_1\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$\frac{L(2z,\omega_0)}{L(2z+1,\omega_0)\epsilon(2z,\omega_0)} = \frac{\frac{1}{2}b_{-1}}{z-\frac{1}{2}} + b_0 + 2b_1\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$\frac{L(z,\pi_1)}{L(z+1,\pi_1)\epsilon(z,\pi_1)} = d_0 + 2d_1\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$R(\tau\sigma\tau,\Lambda,\pi) = R(\tau\sigma\tau,\beta_3,\pi) + N\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$R(\sigma,\Lambda,\pi) = I + P\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$\frac{L(z-\frac{1}{2},\pi_1\times\pi_1)}{L(z+\frac{1}{2},\pi_1\times\pi_1)\epsilon(z-\frac{1}{2},\pi_1\times\pi_1)} = -1 + h_1\Big(z-\frac{1}{2}\Big) + \cdots,$$

$$\phi(\Lambda) = \phi(\beta_3) + \Big(z-\frac{1}{2}\Big)D\phi(\beta_3) + \cdots,$$

$$\begin{aligned}R(\sigma\tau\sigma\tau,\Lambda,\pi) &= R(\sigma,\tau\sigma\tau\Lambda,\tau\sigma\tau\pi)R(\tau\sigma\tau,\Lambda,\pi)\\ &= R(\tau\sigma\tau,\beta_3,\pi) + \Big(z-\frac{1}{2}\Big)\Big(N + P\big(R(\tau\sigma\tau,\beta_3,\pi)\big)\Big) + \cdots.\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}_{\beta_3} M^2(\sigma\tau,\Lambda,\pi)\phi &= \frac{c_1(F)R(\sigma\tau,\beta_3,\pi)\phi(\beta_3)}{L(2,\pi_1\times\pi_1)}\\
&= a_{-1}R(\sigma\tau,\beta_3,\pi)\phi(\beta_3),\\
\operatorname{Res}_{\beta_3} M^2(\tau\sigma\tau,\Lambda,\pi)\phi &= \frac{1}{2}a_{-1}b_{-1}d_0R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3)\\
&\quad+\frac{1}{2}a_{-1}b_{-1}d_0N\phi(\beta_3)\\
&\quad+a_{-1}b_{-1}d_1R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad+a_{-1}b_0d_0R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad+\frac{1}{2}a_0b_{-1}d_0R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\\
\operatorname{Res}_{\beta_3} M^2(\sigma\tau\sigma\tau,\Lambda,\pi)\phi &= -\frac{1}{2}a_{-1}b_{-1}d_0R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3)\\
&\quad-\frac{1}{2}a_{-1}b_{-1}d_0N\phi(\beta_3)\\
&\quad-\frac{1}{2}a_{-1}b_{-1}d_0PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad+\frac{1}{2}a_{-1}b_{-1}d_0h_1R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad-a_{-1}b_{-1}d_1R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad-a_{-1}b_0d_0R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\
&\quad-\frac{1}{2}a_0b_{-1}d_0R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),
\end{aligned}$$

$$(*)\qquad \begin{aligned}\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_2} A(\phi,\phi',\Lambda) &= a_{-1}b_{-1}d_0\langle R(\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_1)\rangle\\
&\quad-\frac{1}{2}a_{-1}b_{-1}^2d_0^2\langle PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle\\
&\quad+\frac{1}{2}a_{-1}b_{-1}^2d_0^2h_1\langle R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle. \quad\blacksquare\end{aligned}$$

Remark 6.4 Since $\sigma\tau\beta_3 = -\beta_1 = -1\beta_1 + 0\beta_2$, $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_2} A(\phi,\phi',\Lambda)$ is not square integrable.

7 Along S_3

$M(w,\Lambda,\pi)$ has a pole on S_3 only when $\pi_2 \simeq \tilde\pi_1$. From Table 1, $M(w,\Lambda,\pi)$ has a pole when $w = \sigma\tau, \sigma\tau\sigma, \tau\sigma\tau, \sigma\tau\sigma\tau$. For $\Lambda = 2z\beta_1 + \beta_3 = (2z+1)f_1 + (-2z+1)f_2$, $\langle\Lambda,\beta_1^\vee\rangle = 4z$, $\langle\Lambda,\beta_2^\vee\rangle = -2z+1$ and $\langle\Lambda,\beta_4^\vee\rangle = 2z+1$. Then

Lemma 7.1

$$M^3(\sigma\tau,\Lambda,\pi)\phi = \frac{L(-z+\frac{1}{2},\tilde\pi_1)L(-2z+1,\omega_{\tilde\pi_1})R(\tau,\Lambda,\pi)\phi}{L(-z+\frac{3}{2},\tilde\pi_1)L(-2z+2,\omega_{\tilde\pi_1})\epsilon\bigstar_7},$$

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$$M^3(\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L(2z,\pi_1\times\pi_1)L(z+\frac{1}{2},\pi_1)L(2z+1,\omega_{\pi_1})R(\sigma\tau\sigma,\Lambda,\pi)\phi}{L(2z+1,\pi_1\times\pi_1)L(z+\frac{3}{2},\pi_1)L(2z+2,\omega_{\pi_1})\epsilon\bigstar_8},$$

$$M^3(\tau\sigma\tau,\Lambda,\pi)\phi = \frac{L\bigstar_5 R(\tau\sigma\tau,\Lambda,\pi)\phi}{L\bigstar_6\epsilon\bigstar_9},$$

$$M^3(\sigma\tau\sigma\tau,\Lambda,\pi)\phi = \frac{L\bigstar_7 R(\sigma\tau\sigma\tau,\Lambda,\pi)\phi}{L\bigstar_8\epsilon\bigstar_{10}},$$

where

$$\epsilon\bigstar_7 = \epsilon\Big(-z+\frac{1}{2},\tilde{\pi}_1\Big)\,\epsilon(-2z+1,\omega_{\tilde{\pi}_1}),$$

$$\epsilon\bigstar_8 = \epsilon(2z,\pi_1\times\pi_1)\epsilon\Big(z+\frac{1}{2},\pi_1\Big)\,\epsilon(2z+1,\omega_{\pi_1}),$$

$$\epsilon\bigstar_9 = \epsilon\Big(-z+\frac{1}{2},\tilde{\pi}_1\Big)\,\epsilon(-2z+1,\omega_{\tilde{\pi}_1})\epsilon\Big(z+\frac{1}{2},\pi_1\Big)\,\epsilon(2z+1,\omega_{\pi_1}),$$

$$\epsilon\bigstar_{10} = \epsilon(2z,\pi_1\times\pi_1)\epsilon\Big(-z+\frac{1}{2},\tilde{\pi}_1\Big)\,\epsilon(-2z+1,\omega_{\tilde{\pi}_1})\epsilon\Big(z+\frac{1}{2},\pi_1\Big)\,\epsilon(2z+1,\omega_{\pi_1}),$$

$$L\bigstar_5 = L\Big(-z+\frac{1}{2},\tilde{\pi}_1\Big)\,L(-2z+1,\omega_{\tilde{\pi}_1})L\Big(z+\frac{1}{2},\pi_1\Big)\,L(2z+1,\omega_{\pi_1}),$$

$$L\bigstar_6 = L\Big(-z+\frac{3}{2},\tilde{\pi}_1\Big)\,L(-2z+2,\omega_{\tilde{\pi}_1})L\Big(z+\frac{3}{2},\pi_1\Big)\,L(2z+2,\omega_{\pi_1}),$$

$$L\bigstar_7 = L(2z,\pi_1\times\pi_1)L\Big(-z+\frac{1}{2},\tilde{\pi}_1\Big)\,L(-2z+1,\omega_{\tilde{\pi}_1})L\Big(z+\frac{1}{2},\pi_1\Big)\,L(2z+1,\omega_{\pi_1}),$$

$$L\bigstar_8 = L(2z+1,\pi_1\times\pi_1)L\Big(-z+\frac{3}{2},\tilde{\pi}_1\Big)\;\times$$

$$L(-2z+2,\omega_{\tilde{\pi}_1})L\Big(z+\frac{3}{2},\pi_1\Big)\,L(2z+2,\omega_{\pi_1}).$$

Proposition 7.2 *If $\omega_{\pi_1}\neq\omega_0$, then we do not have a pole at $\Lambda=\beta_3$, i.e. $z=0$.*

Proof By direct observation of Lemma 7.1, there is not a pole at $\Lambda=\beta_3$, *i. e.* $z=0$. Note that if $\pi_1\simeq\tilde{\pi}_1$, there is a cancellation of poles between $L(2z,\pi_1\times\pi_1)$ and $L(2z+1,\pi_1\times\pi_1)$. ■

Proposition 7.3 *If $\omega_{\pi_1}=\omega_0$, then $A^3(\phi,\phi',\Lambda)$ has a double pole at $\Lambda=\beta_3$, i.e. $z=0$.*

Proof From Lemma 7.1, we can see that $M^3(w,\Lambda,\pi)$ has a double pole at $\Lambda=\beta_3$ *i.e.* $z=0$ when $\omega_{\pi_1}=\omega_0$. In order to calculate the residue, we use the following

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notations where $\pi_2 \simeq \tilde{\pi}_1 \simeq \pi_1$.

$$\frac{L(-2z+1,\omega_0)}{L(-2z+2,\omega_0)\epsilon(-2z+1,\omega_0)} = \frac{-\frac{1}{2}b_{-1}}{z} + b_0 - 2b_1 z + \cdots ,$$
$$\frac{L(-z+\frac{1}{2},\pi_1)}{L(-z+\frac{3}{2},\pi_1)\epsilon(-z+\frac{1}{2},\pi_1)} = d_0 - 2d_1 z + \cdots ,$$
$$\frac{L(2z,\pi_1\times\pi_1)}{L(2z+1,\pi_1\times\pi_1)\epsilon(2z,\pi_1\times\pi_1)} = -1 + 2h_1 z + \cdots ,$$
$$\frac{L(2z+1,\omega_0)}{L(2z+2,\omega_0)\epsilon(2z+1,\omega_0)} = \frac{\frac{1}{2}b_{-1}}{z} + b_0 + 2b_1 z + \cdots ,$$
$$\frac{L(z+\frac{1}{2},\pi_1)}{L(z+\frac{3}{2},\pi_1)\epsilon(z+\frac{1}{2},\pi_1)} = d_0 + 2d_1 z + \cdots ,$$
$$R(\tau\sigma\tau,\Lambda,\pi) = R(\tau\sigma\tau,\beta_3,\pi) + 2Nz + \cdots ,$$
$$R(\sigma,\Lambda,\pi) = I + 2Pz + \cdots ,$$
$$\phi(\Lambda) = \phi(\beta_3) + 2zD\phi(\beta_3) + \cdots ,$$

$$\begin{aligned} R(\sigma\tau\sigma\tau,\Lambda,\pi) &= R(\sigma,\tau\sigma\tau\Lambda,\tau\sigma\tau\pi)R(\tau\sigma\tau,\Lambda,\tau\sigma\tau\pi) \\ &= R(\tau\sigma\tau,\beta_3,\pi) + 2z\Big(N + P\big(R(\tau\sigma\tau,\Lambda,\pi)\big)\Big), \\ R(\sigma\tau\sigma,\Lambda,\pi) &= R(\sigma\tau,\sigma\Lambda,\sigma\pi)R(\sigma,\Lambda,\pi) \\ &= R(\sigma\tau,\beta_3,\pi) + 2z\Big(N_1 + P_1\big(R(\sigma\tau,\beta_3,\pi)\big)\Big). \end{aligned}$$

Note that $R(\tau\sigma\tau,\Lambda,\pi)$, $R(\sigma,\Lambda,\pi)$ and ϕ are functions of $\frac{1}{2}\langle\Lambda,\beta_1^\vee\rangle$, so in the notation for Proposition 6.3, it was in terms of $z-\frac{1}{2}$, whereas in the notation for Proposition 7.3, it is in terms of $2z$.

$$\begin{aligned} \mathrm{Res}_{\beta_3} M^3(\sigma\tau,\Lambda,\pi)\phi &= \frac{L(\frac{1}{2},\pi_1)\big(-\frac{1}{2}c_2(F)\big)\,R(\sigma\tau,\beta_3,\pi)\phi}{L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)} \\ &= -\frac{1}{2}b_{-1}d_0R(\sigma\tau,\beta_3,\pi)\phi, \end{aligned}$$

$$\begin{aligned} \mathrm{Res}_{\beta_3} M^3(\sigma\tau\sigma,\Lambda,\pi)\phi &= \frac{(-1)L(\frac{1}{2},\pi_1)\big(\frac{1}{2}c_2(F)\big)\,R(\sigma\tau\sigma,\beta_3,\pi)\phi}{L(\frac{3}{2},\pi_1)L(2,\omega_0)\epsilon(\frac{1}{2},\pi_1)} \\ &= -\frac{1}{2}b_{-1}d_0R(\sigma\tau,\beta_3,\pi)\phi, \end{aligned}$$

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$$\begin{aligned}\operatorname{Res}_{\beta_3} M^3(\tau\sigma\tau,\Lambda,\pi)\phi &= -\frac{1}{2}b_{-1}^2 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3) - \frac{1}{2}b_{-1}^2 d_0^2 N\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1}^2 d_0 d_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1}^2 d_0 d_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1} b_0 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1} b_0 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &= -\frac{1}{2}b_{-1}^2 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3) - \frac{1}{2}b_{-1}^2 d_0^2 N\phi(\beta_3),\end{aligned}$$

$$\begin{aligned}\operatorname{Res}_{\beta_3} M^3(\sigma\tau\sigma\tau,\Lambda,\pi)\phi &= \frac{1}{2}b_{-1}^2 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3) + \frac{1}{2}b_{-1}^2 d_0^2 N\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1}^2 d_0^2 PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1}^2 d_0^2 h_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1}^2 d_0 d_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1}^2 d_0 d_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1} b_0 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1} b_0 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &= \frac{1}{2}b_{-1}^2 d_0^2 R(\tau\sigma\tau,\beta_3,\pi)D\phi(\beta_3) + \frac{1}{2}b_{-1}^2 d_0^2 N\phi(\beta_3)\\ &\quad + \frac{1}{2}b_{-1}^2 d_0^2 PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)\\ &\quad - \frac{1}{2}b_{-1}^2 d_0^2 h_1 R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\end{aligned}$$

$$\begin{aligned}(**)\quad \operatorname{Res}_{\beta_3}\operatorname{Res}_{S_3} A(\phi,\phi',\Lambda) &= -a_{-1}b_{-1}d_0\langle R(\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_1)\rangle\\ &\quad + \frac{1}{2}a_{-1}b_{-1}^2 d_0^2\langle PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle\\ &\quad - \frac{1}{2}a_{-1}b_{-1}^2 d_0^2 h_1\langle R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle. \ \blacksquare\end{aligned}$$

Remark 7.4 Since $\sigma\tau\beta_3 = -\beta_1 = -1\beta_1 + 0\beta_2$, $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_3} A(\phi,\phi',\Lambda)$ is not square integrable.

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8 Along S_4

$M(w,\Lambda,\pi)$ has a pole on S_4 only when $\omega_{\pi_1} = \omega_0$, $L(\frac{1}{2},\pi_1) \neq 0$. From Table 1, $M(w,\Lambda,\pi)$ has a pole when $w = \tau\sigma, \sigma\tau\sigma, \tau\sigma\tau, \sigma\tau\sigma\tau$. For $\Lambda = 2zf_2+f_1 = f_1+2zf_2$, $\langle\Lambda,\beta_1^\vee\rangle = -2z+1$, $\langle\Lambda,\beta_2^\vee\rangle = 2z$ and $\langle\Lambda,\beta_3^\vee\rangle = 2z+1$. Then

Lemma 8.1

$$M^4(\tau\sigma,\Lambda,\pi)\phi = \frac{L(-z+\frac{1}{2},\pi_1\times\tilde{\pi}_2)R(\tau\sigma,\Lambda,\pi)\phi}{L(-z+\frac{3}{2},\pi_1\times\tilde{\pi}_2)},$$

$$M^4(\sigma\tau\sigma,\Lambda,\pi)\phi = \frac{L(-z+\frac{1}{2},\pi_1\times\tilde{\pi}_2)L(z+\frac{1}{2},\pi_1\times\pi_2)R(\sigma\tau\sigma,\Lambda,\pi)\phi}{L(-z+\frac{3}{2},\pi_1\times\tilde{\pi}_2)L(z+\frac{3}{2},\pi_1\times\pi_2)\epsilon\bigstar_{11}},$$

$$M^4(\tau\sigma\tau,\Lambda,\pi)\phi = \frac{L(z,\pi_2)L(2z,\omega_{\pi_2})L(z+\frac{1}{2},\pi_1\times\pi_2)R(\tau\sigma\tau,\Lambda,\pi)\phi}{L(z+1,\pi_2)L(2z+1,\omega_{\pi_2})L(z+\frac{3}{2},\pi_1\times\pi_2)\epsilon\bigstar_{12}},$$

$$M^4(\sigma\tau\sigma\tau,\Lambda,\pi)\phi = \frac{L\bigstar_9 R(\sigma\tau\sigma\tau,\Lambda,\pi)\phi}{L\bigstar_{10}\epsilon\bigstar_{13}},$$

where

$$\epsilon\bigstar_{11} = \epsilon\Big(-z+\frac{1}{2},\pi_1\times\tilde{\pi}_2\Big)\epsilon\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big),$$

$$\epsilon\bigstar_{12} = \epsilon(z,\pi_2)\epsilon(2z,\omega_{\pi_2})\epsilon\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big),$$

$$\epsilon\bigstar_{13} = \epsilon\Big(-z+\frac{1}{2},\pi_1\times\tilde{\pi}_2\Big)\epsilon(z,\pi_2)\epsilon(2z,\omega_{\pi_2})\epsilon\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big),$$

$$L\bigstar_9 = L\Big(-z+\frac{1}{2},\pi_1\times\tilde{\pi}_2\Big)L(z,\pi_2)L(2z,\omega_{\pi_2})L\Big(z+\frac{1}{2},\pi_1\times\pi_2\Big),$$

$$L\bigstar_{10} = L\Big(-z+\frac{3}{2},\pi_1\times\tilde{\pi}_2\Big)L(z+1,\pi_2)L(2z+1,\omega_{\pi_2})L\Big(z+\frac{3}{2},\pi_1\times\pi_2\Big).$$

Proposition 8.2 *$M^4(w,\Lambda,\pi)$ does not have a pole at f_1, i.e. at $z=0$.*

Proof Direct observation. Note that if $\omega_{\pi_2} = \omega_0$, then there is a cancellation of poles of $L(2z,\omega_{\pi_2})$ and $L(2z+1,\omega_{\pi_2})$. ■

Theorem 8.3 *The sum of the non-square integrable residues is zero.*

Proof By the calculations in the proofs of Proposition 6.3 and Proposition 7.3 we have

$$(*)\qquad \begin{aligned}\mathrm{Res}_{\beta_3}\,\mathrm{Res}\,S_2A(\phi,\phi',\Lambda) &= a_{-1}b_{-1}d_0\langle R(\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_1)\rangle\\ &\quad-\frac{1}{2}a_{-1}b_{-1}^2d_0^2\langle PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle\\ &\quad+\frac{1}{2}a_{-1}b_{-1}^2d_0^2h_1\langle R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle\end{aligned}$$

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$$(**)\quad \mathrm{Res}_{\beta_3}\,\mathrm{Res}_{S_3}\,A(\phi,\phi',\Lambda) = -a_{-1}b_{-1}d_0\langle R(\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_1)\rangle$$
$$+\frac{1}{2}a_{-1}b_{-1}^2d_0^2\langle PR(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle$$
$$-\frac{1}{2}a_{-1}b_{-1}^2d_0^2h_1\langle R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3),\phi'(\beta_3)\rangle.$$

So they cancel each other out when we add them. ∎

9 Main Result

In conclusion, we have proved the following:

Proposition 9.1 *The following contribute to the residual spectrum*

$$L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_M,\quad M\simeq \mathrm{GL}_2\times\mathrm{GL}_2\,.$$

(i) $\pi=\pi_1\otimes\pi_1$, *where* $\pi_1\simeq\tilde\pi_1$, $L(\frac{1}{2},\pi_1)\neq 0$, *and* $\omega_{\pi_1}=\omega_0$ *at* $\Lambda=\gamma$;
(ii) $\pi=\pi_1\otimes\pi_2$, *where* $\pi_1\not\simeq\pi_2$, $\omega_{\pi_1}=\omega_0$, $\omega_{\pi_2}=\omega_0$, $L(\frac{1}{2},\pi_1)\neq 0$, $L(\frac{1}{2},\pi_2)\neq 0$ *at* $\Lambda=\beta_3$;
(iii) $\pi=\pi_1\otimes\pi_1$, *where* $\pi_1\simeq\tilde\pi_1$ *and* $\omega_{\pi_1}\neq\omega_0$ *at* $\Lambda=\beta_4$.

The residual spectrum is spanned by

(i) $R(\sigma\tau\sigma\tau,\gamma,\pi)\phi(\gamma)$;
(ii) $R(\tau\sigma\tau,\beta_3,\pi)\phi(\beta_3)$;
(iii) $R(\sigma\tau\sigma,\beta_4,\pi)\big(I+R(\tau,\beta_4,\pi)\big)\,\phi(\beta_4)=$

$$\bigotimes_v R(\sigma\tau\sigma,\beta_4,\pi_v)\Big(\bigotimes_v\phi_v+\bigotimes_v R(\tau,\beta_4,\pi_v)\phi_v\Big).$$

We need to analyze the image of intertwining operators

$$R(\sigma\tau\sigma\tau,\gamma,\pi_v)\colon I(\gamma,\pi_v)\to I(-\gamma,\pi_v),$$
$$R(\tau\sigma\tau,\beta_3,\pi_v)\colon I(\beta_3,\pi_v)\to I(-\beta_3,\pi'_v),$$

where $\pi'_v=\pi_{2v}\otimes\pi_{1v}$ and $R(\sigma\tau\sigma,\beta_4,\pi_v)\colon I(\beta_4,\pi_v)\longrightarrow I(-\beta_4,\pi_v)$.

Case (i) deals with $R(\sigma\tau\sigma\tau,\gamma,\pi_v)$. Note that $\sigma\tau\sigma\tau$ is the longest element in the Weyl group of the parabolic subgroup P. Hence the image of the intertwining operator $R(\sigma\tau\sigma\tau,\gamma,\pi_v)$ is the Langlands' quotient $J(\gamma,\pi_v)$ of $I(\gamma,\pi_v)$ when π_v is tempered. If π_v is nontempered, let $\pi_v=\pi(\mu|\ |^r,\mu|\ |^{-r})$ with $0<r<\frac{1}{2}$. Then by inducing in stages, $I(\gamma,\pi_v)=\mathrm{Ind}_B^G\,\mu|\ |^{\frac{3}{2}+r}\otimes\mu|\ |^{\frac{3}{2}-r}\otimes|\ |^{\frac{1}{2}+r}\otimes\mu|\ |^{\frac{1}{2}-r}$. Note that $\frac{3}{2}+r>\frac{3}{2}-r>\frac{1}{2}+r>\frac{1}{2}-r$. So it is in the Langlands' situation from the Borel subgroup. Hence, the image of $R(\sigma\tau\sigma\tau,\gamma,\pi_v)$ is the unique quotient of $I(\gamma,\pi_v)$. Let $J(\gamma,\pi)=\bigotimes_v J(\gamma,\pi_v)$.

In Case (ii), we consider by inducing in stages,

$$I(\beta_3,\pi_v)=\mathrm{Ind}_{\mathrm{GL}_4}^{\mathrm{Sp}_8}|\mathrm{det}|^{\frac{1}{2}}\otimes(\pi_{1v}\otimes\pi_{2v}).$$

If $\pi_{1v} \otimes \pi_{2v}$ is tempered, then the image of $R(\tau\sigma\tau, \beta_3, \pi_{1v})$ is the Langlands' quotient $J(\beta_3, \pi_v)$ of $I(\beta_3, \pi_v)$. If $\pi_{1v} \otimes \pi_{2v}$ is not tempered, as in the above, the image of $R(\tau\sigma\tau, \beta_3, \pi_v)$ is the unique quotient of $I(\beta_3, \pi_v)$. We denote it by $J(\beta_3, \pi_v)$. Let $J(\beta_3, \pi) = \bigotimes_v J(\beta_3, \pi_v)$.

In Case (iii), we consider by inducing in stages, namely, we use the fact that

$$I(\beta_4, \pi) = \mathrm{Ind}_P^G |\det| \otimes (\pi_1 \otimes \mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_1),$$

where $P = MN$, $M \simeq \mathrm{GL}_2 \times \mathrm{Sp}_4$. Here $R(\tau, \beta_4, \pi)$ is the self-intertwining operator for the induced representation $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_1$. Hence we need to analyze $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v}$ for each v.

Proposition 9.2 ([32]) *If π_{1v} is supercuspidal, then $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v}$ is reducible iff $\pi_{1v} \simeq \tilde{\pi}_{1v}$ and $\omega_{\pi_{1v}} \neq 1$. If it is reducible, then it is the sum of two inequivalent representations.*

Let us write

$$\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v} = \begin{cases} \pi_{+,v} \oplus \pi_{-,v}, & \text{if } \mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v} \text{ is reducible, where} \\ & \pi_{+,v} \text{ is generic with respect to } \psi_v, \\ \pi_{\cdot,v}, & \text{otherwise.} \end{cases}$$

As we remarked in Remark 5.4, if $\pi_1 \simeq \tilde{\pi}_1$, $\omega_{\pi_1} \neq \omega_0$, π_1 is a monomial cuspidal representation. Hence it is known that all π_{1v}s are tempered and π_{1v} cannot be a Steinberg representation. However, for the sake of completeness, we indicate what happens when π_{1v} is either the Steinberg representation, or a non-tempered representation.

Proposition 9.3 ([5]) *If $\pi_{1v} = \pi(\mu|\,|^{\frac{1}{2}}, \mu|\,|^{-\frac{1}{2}})$ with $\mu^2 = 1$, or $\pi_{1v} = \pi(\mu|\,|^r, \mu|\,|^{-r})$, $0 < r < \frac{1}{2}$, $\mu^2 = 1$, then $\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v}$ is always irreducible.*

Proposition 9.4 ([13]) *If $\pi_{1v} = \pi(\mu, \nu)$, then*

$$\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v} = \begin{cases} \textit{sum of four mutually inequivalent irreducible unitary representations,} & \textit{if } \mu = \mu^{-1}, \nu = \nu^{-1}, \ \mu \neq 1, \nu \neq 1, \mu \neq \nu, \\ \textit{sum of two inequivalent irreducible unitary representations,} & \textit{if } \nu = \mu = \mu^{-1}, \mu \neq 1, \\ & \textit{or } \mu = \mu^{-1}, \mu \neq 1, \nu = 1, \\ & \textit{or } \mu = 1, \nu = \nu^{-1}, \nu \neq 1 \\ \textit{irreducible,} & \textit{otherwise.} \end{cases}$$

We denote

$$\mathrm{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v} = \begin{cases} \pi_{+,+,v} \oplus \pi_{+,-,v} \oplus \pi_{-,+,v} \oplus \pi_{-,-,v}, & \text{if } \mu = \mu^{-1}, \nu = \nu^{-1}, \ \mu \neq 1, \nu \neq 1, \mu \neq \nu, \\ \pi_{+,v} \oplus \pi_{-,v}, & \text{if } \nu = \mu = \mu^{-1}, \mu \neq 1, \\ & \text{or } \mu = \mu^{-1}, \mu \neq 1, \nu = 1, \\ & \text{or } \mu = 1, \nu = \nu^{-1}, \nu \neq 1, \\ \pi_{\cdot,v}, & \text{otherwise.} \end{cases}$$

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Let us define

$$\begin{cases} \pi_{+,+,v} = \pi_{+,v}, \quad \pi_{+,-,v} = \pi_{-,v}, & \text{if } \nu = \mu = \mu^{-1}, \mu \neq 1, \text{ or } \mu = \mu^{-1}, \\ \pi_{-,+,v} = 0, \quad \pi_{-,-,v} = 0, & \mu \neq 1, \nu = 1, \text{ or } \mu = 1, \nu = \nu^{-1}, \\ & \nu \neq 1, \\ \pi_{+,+,v} = \pi_{\cdot,v}, \quad \pi_{+,-,v} = 0, & \text{if } \mu = 1, \nu = 1, \text{ or } \mu = \nu^{-1}, \mu^2 \neq 1, \\ \pi_{-,+,v} = 0, \quad \pi_{-,-,v} = 0, & \end{cases}$$

where $\pi_{+,+,v}$ is generic with respect to ψ_v. Similarly, if π_{1v} is supercuspidal, set

$$\begin{cases} \pi_{+,+,v} = \pi_{+,v}, \pi_{+,-,v} = \pi_{-,v}, & \text{if } \operatorname{Ind}_{\mathrm{GL}_2}^{\mathrm{Sp}_4} \pi_{1v} \text{ is reducible}, \\ \pi_{-,+,v} = 0, \pi_{-,-,v} = 0, & \\ \pi_{+,+,v} = \pi_{\cdot,v}, \pi_{+,-,v} = 0, \pi_{-,+,v} = 0, & \text{otherwise.} \\ \pi_{-,-,v} = 0, & \end{cases}$$

Let $\epsilon(\pi_{+,+,v}) = 1$, $\epsilon(\pi_{+,-,v}) = -1$, $\epsilon(\pi_{-,+,v}) = -1$ and $\epsilon(\pi_{-,-,v}) = 1$. Observe that for almost all v, $\pi_{+,+,v}$ is spherical, $\pi_{+,v}$ is spherical and $\pi_{\cdot,v}$ is spherical for their respective cases.

If $\rho_v \in \{\pi_{+,+,v}, \pi_{+,-,v}, \pi_{-,+,v}, \pi_{-,-,v}\}$, let $\epsilon(\rho_v)$ be the corresponding sign. Then

$$\begin{aligned} I(\beta_4, \pi_v) = \operatorname{Ind}_{\mathrm{GL}_2 \times \mathrm{Sp}_4}^{\mathrm{Sp}_8} |\det| \otimes (\pi_{1v} \otimes \pi_{+,+,v}) \\ \oplus \operatorname{Ind}_{\mathrm{GL}_2 \times \mathrm{Sp}_4}^{\mathrm{Sp}_8} |\det| \otimes (\pi_{1v} \otimes \pi_{+,-,v}) \\ \oplus \operatorname{Ind}_{\mathrm{GL}_2 \times \mathrm{Sp}_4}^{\mathrm{Sp}_8} |\det| \otimes (\pi_{1v} \otimes \pi_{-,+,v}) \\ \oplus \operatorname{Ind}_{\mathrm{GL}_2 \times \mathrm{Sp}_4}^{\mathrm{Sp}_8} |\det| \otimes (\pi_{1v} \otimes \pi_{-,-,v}). \end{aligned}$$

Let $J_{\pm,\pm,v}$ be the Langlands' quotients of $\operatorname{Ind}_{\mathrm{GL}_2 \times \mathrm{Sp}_4}^{\mathrm{Sp}_8} |\det| \otimes (\pi_{1v} \otimes \pi_{\pm,\pm,v})$, respectively. By Langlands' classification theorem, the common image of the intertwining operators $R(\sigma\tau\sigma, \beta_4, \pi_v)$ and $R(\sigma\tau\sigma\tau, \beta_4, \pi_v)$ is the direct sum of $J_{\pm,\pm,v}$. Let $J_v = \{J_{+,+,v}, J_{+,-,v}, J_{-,+,v}, J_{-,-,v}\}$. Let $\epsilon(\rho_v)$ be the corresponding sign for $\rho_v \in J_v$, namely, we set $\epsilon(J_{\cdot,\cdot,v}) = \epsilon(\pi_{\cdot,\cdot,v})$. So from R-group theory [14],

$$R(\tau, \beta_4, \pi_v)\phi_v = \begin{cases} \phi_v & \text{for } \phi_v \in \pi_{+,+,v} \text{ or } \phi_v \in \pi_{-,-,v}, \\ -\phi_v & \text{for } \phi_v \in \pi_{+,-,v} \text{ or } \phi_v \in \pi_{-,+,v}. \end{cases}$$

Then we define $J(\pi)$ to be the collection

$$J(\pi) = \{\Pi = \otimes \Pi_v \mid \Pi_v \in J_v, \text{ for all } v, \Pi_v = J_{+,+,v} \text{ for almost all } v, \textstyle\prod_v \epsilon(\Pi_v) = 1\}.$$

We note that $\prod_v \epsilon(\Pi_v)$ is well-defined and independent of the choice of ψ. Here if $\prod_v \epsilon(\Pi_v) = -1$, then

$$(\text{constant}) \bigotimes_v R(\sigma\tau\sigma, \beta_4, \pi_v)\Big(\bigotimes_v \phi_v + \bigotimes_v R(\tau, \beta_4, \pi_v)\phi_v\Big)$$

is zero. So $\operatorname{Res}_{\beta_4} \operatorname{Res}_{S_1} A(\phi, \phi', \Lambda) = 0$.

Theorem 9.5

$$L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A})\big)_M = \Big(\bigoplus_{\pi} J(\gamma,\pi)\Big) \oplus \Big(\bigoplus_{\pi} J(\beta_3,\pi)\Big) \oplus J(\pi),$$

where

- *In the first sum,* $\pi = \pi_1 \otimes \pi_1$*,* π_1 *runs through cuspidal representations of* GL_2 *with the trivial central character such that* $L(\frac{1}{2},\pi_1) \neq 0$.
- *In the second sum,* $\pi = \pi_1 \otimes \pi_2$*,* $\pi_1 \not\simeq \pi_2$*,* $\omega_{\pi_1} = \omega_0$*,* $\omega_{\pi_2} = \omega_0$*,* $L(\frac{1}{2},\pi_1) \neq 0$*,* $L(\frac{1}{2},\pi_2) \neq 0$.
- *In the third summand,* $\pi = \pi_1 \otimes \pi_1$*,* π_1 *runs through self-contragredient monomial cuspidal representations of* GL_2.

Remark 9.6 The fact that the point β_3 contributes to the residual spectrum is new, compared to the result in [17]. We can explain this, using a similar conjecture made in [16]. According to the conjecture in [16], the residual spectrum coming from the Levi subgroup $M = \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{Sp}_8$, is parametrized by the following three cases:

Case (i) $\pi = \pi_1 \otimes \pi_1$, and the distinguished unipotent orbit in $\mathrm{Sp}_4(\mathbb{C})$, where π_1 is a cuspidal representation of GL_2 with the trivial central character such that $L(\frac{1}{2},\pi_1) \neq 0$. (This means that the Eisenstein series attached to $\pi_1, \mathrm{GL}_2 \subset \mathrm{Sp}_4$, has a pole at $s = \frac{1}{2}$.) In that case, the point $\gamma = 3f_1 + f_2$ contributes to the residual spectrum. The conjectural Arthur parameter is as follows: Let $\phi\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ be the conjectural Langlands' parameter for π_1, where L_F is the hypothetical group. Then together with the distinguished unipotent orbit (4) in $\mathrm{Sp}_4(\mathbb{C})$, considered as a distinguished unipotent orbit in $\mathrm{GL}_4(\mathbb{C})$, it gives an Arthur parameter $\psi\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_8(\mathbb{C})$, attached to the residual spectrum of GL_8, namely, the quotient of Ind $|\det|^{\frac{3}{2}}\pi_1 \otimes |\det|^{\frac{1}{2}}\pi_1 \otimes |\det|^{-\frac{1}{2}}\pi_1 \otimes |\det|^{-\frac{3}{2}}\pi_1$. Then ψ factors through $O_8(\mathbb{C}) \subset \mathrm{SO}_9(\mathbb{C})$, and the resulting one is the desired Arthur parameter $\psi\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_9(\mathbb{C})$.

Case (ii) $\pi = \pi_1 \otimes \pi_2$, and the distinguished unipotent orbit in $\mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$, where π_1, π_2 are cuspidal representations of GL_2 with the trivial central character such that $\pi_1 \not\simeq \pi_2$, and $L(\frac{1}{2},\pi_i) \neq 0$ for $i = 1,2$. In this case, the point $\beta_3 = f_1 + f_2$ contributes to the residual spectrum. The conjectural Arthur parameter is as follows: Let $\phi_i\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ be the conjectural Langlands' parameter for π_i, $i = 1,2$. Then together with the distinguished unipotent orbit (2) in $\mathrm{Sp}_2(\mathbb{C})$, considered as a distinguished unipotent orbit in $\mathrm{GL}_2(\mathbb{C})$, it gives an Arthur parameter $\psi_i\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$, attached to the residual spectrum of GL_4, namely, the quotient of Ind $|\det|^{\frac{1}{2}}\pi_i \otimes |\det|^{-\frac{1}{2}}\pi_i$, $i = 1,2$. Then ψ_i factors through $O_4(\mathbb{C})$, and $\psi_1 \oplus \psi_2\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to O_4(\mathbb{C}) \oplus O_4(\mathbb{C}) \subset \mathrm{SO}_9(\mathbb{C})$ is the desired Arthur parameter.

Case (iii) $\pi = \pi_1 \otimes \pi_1$, and the distinguished unipotent orbit in $O_4(\mathbb{C})$, where π_1 is a self-contragredient monomial cuspidal representation of GL_2. (This means that the Eisenstein series attached to $\pi_1, \mathrm{GL}_2 \subset \mathrm{Sp}_4$, has no pole for $\Re s > 0$.) In this case, the point $\beta_4 = 2f_1$ contributes to the residual spectrum. The conjectural Arthur parameter is as follows: Let $\phi\colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ be the conjectural

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Langlands' parameter for π_1. Let (3,1) be the distinguished unipotent orbit in $O_4(\mathbb{C})$. Then 3 gives an Arthur parameter $\psi_1 \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_6(\mathbb{C})$, attached to the residual spectrum of GL_6, namely, the quotient of Ind $|\det|\,\pi_1 \otimes \pi_1 \otimes |\det|^{-1}\pi_1$. Also 1 gives the Arthur parameter $\psi_2 = \phi$. Then ψ_1, ψ_2 factor through $O_6(\mathbb{C})$, $O_2(\mathbb{C})$, respectively. Then $\psi_1 \oplus \psi_2 \colon L_F \times \mathrm{SL}_2(\mathbb{C}) \to O_6(\mathbb{C}) \oplus O_2(\mathbb{C}) \subset \mathrm{SO}_9(\mathbb{C})$ is the desired Arthur parameter.

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