

Normal forms for strong magnetic systems on surfaces: trapping regions and rigidity of Zoll systems

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Abstract. We prove a normal form for strong magnetic fields on a closed, oriented surface and use it to derive two dynamical results for the associated flow. First, we show the existence of invariant tori and trapping regions provided a natural non-resonance condition holds. Second, we prove that the flow cannot be Zoll unless (i) the Riemannian metric has constant curvature and the magnetic function is constant, or (ii) the magnetic function vanishes and the metric is Zoll. We complement the second result by exhibiting an exotic magnetic field on a flat two-torus yielding a Zoll flow for arbitrarily weak rescalings.

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1. Introduction

Let M be a closed, oriented surface. A *magnetic system* on M is a pair (g, b) , where g is a Riemannian metric on M and $b : M \rightarrow \mathbb{R}$ is a function, which we refer to as the magnetic function. A (g, b) -geodesic is a curve $\gamma : \mathbb{R} \rightarrow M$ which is parametrized by arc length and solves the equation

$$\kappa_\gamma(t) = b(\gamma(t)) \quad \text{for all } t \in \mathbb{R}, \quad (1.1)$$

where κ_γ is the geodesic curvature of γ .

The term ‘magnetic system’ refers to the fact that a solution of (1.1) describes a trajectory of a particle γ with unit charge and speed under the effect of the Lorentz force

generated by a stationary magnetic field. To fix ideas, if M is embedded in the euclidean three-dimensional space \mathbb{R}^3 and $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a magnetic field in the ambient space, then g is the restriction of the euclidean metric on M and b is the inner product of B with the unit normal to M in \mathbb{R}^3 .

The tangent lifts $(\gamma, \dot{\gamma})$ of (g, b) -geodesics yield the trajectories of a flow $\Phi_{(g,b)} : SM \times \mathbb{R} \rightarrow SM$ on the unit sphere bundle SM , whose dynamical properties have been the subject of intensive research since the seminal work of Arnold in the early 1960s [1]; see [12] for a survey. In the present paper, we will study two important aspects of the dynamics of $\Phi_{(g,b)}$.

The first one is the existence of trapping regions for the flow $\Phi_{(g,b)}$ when the magnetic function is *strong*. Here ‘strong’ means that $b = \epsilon^{-1}b_1$ for some non-vanishing function b_1 and small positive number ϵ . Such a study was initiated by Castilho in [10] and relies on the existence of invariant tori via the Moser twist theorem [17]. We describe this first aspect in detail in §1.2. The second one is the existence of magnetic systems (g, b) that produce a very simple dynamics, namely that their flow $\Phi_{(g,b)}$ induces a free circle action on SM , up to reparametrization. We call such systems *Zoll*. A trivial example of a Zoll system is given by $(g_{\text{con}}, b_{\text{con}})$, where g_{con} has constant Gaussian curvature K_{con} and b_{con} is a constant magnetic function such that $b_{\text{con}}^2 + K_{\text{con}} > 0$. In this case, $(g_{\text{con}}, b_{\text{con}})$ -geodesics are boundaries of geodesic balls and, hence, the system is Zoll. Purely Riemannian Zoll systems, namely those with $b = 0$, are possible only on $M = S^2$. An infinite-dimensional family of such examples was constructed in [20] among spheres of revolution and in [13] for the general case.

Key to both problems above is a Hamiltonian normal form for the flow $\Phi_{(g, \epsilon^{-1}b_1)}$. In order to describe it, we first need to see how to associate a Hamiltonian flow on the tangent bundle TM to a general magnetic flow $\Phi_{(g,b)}$. We refer to [6] for the proof of the statements below.

Let μ be the area form given by the metric g and the orientation on M . Moreover, let $\lambda \in \Omega^1(TM)$ be the Hilbert form of g :

$$\lambda_{(q,v)} \cdot \xi = g_q(v, d\pi \cdot \xi) \quad \text{for all } (q, v) \in TM, \xi \in T_{(q,v)}TM,$$

where $\pi : TM \rightarrow M$ is the canonical projection, $\pi(q, v) = q$. Notice that λ is the pull-back of the Liouville one-form on T^*M by means of the metric g . We define the symplectic form $\omega_{(g,b)} \in \Omega^2(TM)$ on TM via

$$\omega_{(g,b)} := d\lambda - \pi^*(b\mu)$$

and the kinetic Hamiltonian function via

$$H_g : TM \rightarrow \mathbb{R}, \quad H_g(q, v) = \frac{1}{2}|v|_q^2 \quad \text{for all } (q, v) \in TM,$$

where $|\cdot|$ is the norm associated with g . We denote by $X_{H_g}^{\omega_{(g,b)}}$ the Hamiltonian vector field of H_g with respect to $\omega_{(g,b)}$. Let $S_rM := \{(q, v) \in TM \mid |v|_q = r\}$ be the sphere bundle of radius $r > 0$. Then S_rM is invariant under the Hamiltonian flow and $X_{H_g}^{\omega_{(g,b)}}|_{S_rM}$ is a nowhere-vanishing section of the characteristic distribution $\ker(\omega_{(g,b)}|_{S_rM}) \subset T(S_rM)$.

We can now state the promised relationship between the magnetic and the Hamiltonian flows. Denote by $\Gamma : \mathbb{R} \rightarrow M$ a curve with constant speed $r = |\dot{\Gamma}| > 0$. Then $(\Gamma, \dot{\Gamma})$ is a

trajectory of the Hamiltonian flow on $S_r M$ if and only if the arc-length reparametrization $\gamma := \Gamma(\cdot/r)$ is a $(g, (1/r)b)$ -geodesic.

Applying the above argument with $r = 1$ and $b = \epsilon^{-1}b_1$, we see that $\Phi_{(g, \epsilon^{-1}b_1)}$ -geodesics are the integral curves of the characteristic distribution

$$\ker(\omega_\epsilon|_{SM}) \subset T(SM), \quad \omega_\epsilon := \epsilon d\lambda - \pi^*(b_1\mu) = \epsilon\omega_{(g, \epsilon^{-1}b_1)}.$$

1.1. *A Hamiltonian normal form.* We will find now a Hamiltonian normal form for $\Phi_{(g, \epsilon^{-1}b_1)}$ over an open set $U \subset M$ on which b_1 is nowhere vanishing and the circle bundle $\pi : SU \rightarrow U$ admits a section. Such a normal form can be seen as the Hamiltonian upgrade of a vector-field normal form for $\Phi_{(g, \epsilon^{-1}b_1)}$ which is due to Arnold [3, Theorems 2 and 3] and on which we will comment more at the end of this subsection.

Thus, let $W : U \rightarrow SU$ be a section and write $\theta : SU \rightarrow \mathbb{T}$ for the angle between a unit tangent vector and W , measured according to the Riemannian metric and the given orientation on M . We refer to θ as the *angular function* associated with W .

THEOREM 1.1. *Let (g, b_1) be a magnetic system on a closed, oriented surface M . Let $U \subset M$ be an open set on which b_1 does not vanish and such that $\pi : SU \rightarrow U$ admits a section $W : U \rightarrow SU$. Denote with $\theta : SU \rightarrow \mathbb{T}$ the angular function associated with W . Then, for every open set U' such that $\overline{U'} \subset U$, there exist $\epsilon_0 > 0$ and an isotopy of embeddings $\Psi_\epsilon : SU' \rightarrow SU$, $\epsilon \in [0, \epsilon_0)$, with Ψ_0 being the standard inclusion $SU' \hookrightarrow SU$, such that*

$$\Psi_\epsilon^* \omega_\epsilon|_{SM} = d(H_\epsilon d\theta) - \pi^*(b_1\mu),$$

where $H_\epsilon : SU' \rightarrow \mathbb{R}$ is a path of functions such that

$$H_\epsilon = -\frac{\epsilon^2}{2b_1 \circ \pi} + o(\epsilon^2). \tag{1.2}$$

If b_1 is constant, then we can choose Ψ_ϵ so that

$$H_\epsilon = -\frac{\epsilon^2}{2b_1} - \frac{\epsilon^4}{(2b_1)^3} K \circ \pi + o(\epsilon^4), \tag{1.3}$$

where $K : M \rightarrow \mathbb{R}$ is the Gaussian curvature of g .

Remark 1.2. There is an analogous normal form over open sets U of M , whose tangent bundle is not trivial. This happens exactly when $U = M$ and $M \neq \mathbb{T}^2$. In this situation, $\omega_\epsilon|_{SM} = d\alpha_\epsilon$ for a path of contact forms α_ϵ on SM . Integrating a suitable vector field Z_ϵ , one finds a path of diffeomorphisms $\Psi_\epsilon : SM \rightarrow SM$ such that

$$\Psi_\epsilon^* \alpha_\epsilon = (1 - H_\epsilon)\alpha_0,$$

where H_ϵ is of the form (1.2) or (when b_1 is a non-zero constant) (1.3).

The normal form up to order ϵ^2 contained in equation (1.2) was already proved by Castilho in [10, Theorem 3.1] and used to show the existence of invariant tori under certain assumptions on b_1 ; see §1.2 for more details. A similar normal form was also developed by Raymond and Vũ Ngọc in [19] to study the semi-classical limit of magnetic systems.

Our main contribution in Theorem 1.1 is, therefore, to push Castilho’s normal form to order ϵ^4 when the magnetic function b_1 is constant; see equation (1.3). We shall mention that our method of proof is slightly different from his, as we construct the vector field Z_ϵ generating the isotopy Ψ_ϵ at once using Moser’s method [18]: Z_ϵ is the unique vector field contained in the distribution \mathcal{H} tangent to the level sets of θ satisfying

$$\omega_\epsilon(Z_\epsilon, \cdot) = -\lambda - dc_\epsilon \quad \text{on } \mathcal{H}$$

for some function $c_\epsilon : SU \rightarrow \mathbb{R}$ that we will suitably choose.

Let us now comment on the first dynamical implications of Theorem 1.1. This result tells us that we can read off $(g, \epsilon^{-1}b_1)$ -geodesics as trajectories of a time-dependent Hamiltonian flow on U' . To this purpose, let us write by $H_{\epsilon,\theta} : U' \rightarrow \mathbb{R}$, $\theta \in \mathbb{T}$, the function $H_{\epsilon,\theta}(q) := H_\epsilon(e^{i\theta}W(q))$, where $e^{i\theta} : SU \rightarrow SU$ is the fiberwise rotation by angle θ . We define the θ -dependent vector field $X_{H_{\epsilon,\theta}}$ on U' by

$$b_1\mu(X_{H_{\epsilon,\theta}}, \cdot) = -dH_{\epsilon,\theta} \tag{1.4}$$

and the vector field X_{H_ϵ} on SU' by

$$X_{H_\epsilon} = \tilde{X}_{H_{\epsilon,\theta}} + \partial_\theta, \tag{1.5}$$

where $\tilde{X}_{H_{\epsilon,\theta}}$ denotes the lift of $X_{H_{\epsilon,\theta}}$ tangent to the level sets of the angular function. If $\Phi^t_{H_\epsilon}$ is the flow of X_{H_ϵ} on SU' , then for all $\theta_0 \in \mathbb{T}$ and $\theta_1 \in \mathbb{R}$,

$$\varphi_\epsilon^{\theta_0,\theta_1} := \pi \circ \Phi^{\theta_1}_{H_\epsilon} \circ e^{i\theta_0}W$$

is the flow of $X_{H_{\epsilon,\theta}}$ on U' and

$$\theta(\Phi^{\theta_1}_{H_\epsilon}(e^{i\theta_0}W)) = \theta_0 + \theta_1. \tag{1.6}$$

We see that X_{H_ϵ} spans the kernel of $-\pi^*(b_1\mu) + d(H_\epsilon d\theta)$. Therefore, Ψ_ϵ sends trajectories of the flow Φ_{H_ϵ} to tangent lifts of $(g, \epsilon^{-1}b_1)$ -geodesics, up to reparametrization. More precisely, let $z \in SU'$ and consider $\eta_z : [0, \theta_\infty) \rightarrow SU'$ to be the maximal solution of the flow Φ_{H_ϵ} with $\eta_z(0) = z$. Let $\gamma_{\Psi_\epsilon(z)} : \mathbb{R} \rightarrow M$ be the $(g, \epsilon^{-1}b_1)$ -geodesic with $\dot{\gamma}_{\Psi_\epsilon(z)}(0) = \Psi_\epsilon(z)$. There exists an increasing function $t : [0, \theta_\infty) \rightarrow [0, \ell_\infty)$ such that

$$\dot{\gamma}_{\Psi_\epsilon(z)}(t(\theta)) = \Psi_\epsilon(\eta_z(\theta)) \quad \text{for all } \theta \in [0, \theta_\infty). \tag{1.7}$$

From Formulae (1.2) and (1.3) for H_ϵ , we see that $(g, \epsilon^{-1}b_1)$ -geodesics:

- (i) follow for a long time non-degenerate level sets of b_1 with a drift velocity proportional to $\epsilon^2|d(b_1^{-2})|$;
- (ii) follow for a long time non-degenerate level sets of K , if b_1 is a non-zero constant, with a drift velocity proportional to $\epsilon^4b_1^{-4}|dK|$.

Such a dichotomy follows already from Arnold’s normal form for vector fields cited above [3], which shows that the magnetic function b_1 or the Gaussian curvature K , if b_1 is constant, are *adiabatic invariants* for the flow $\Phi_{(g,\epsilon^{-1}b_1)}$. Roughly speaking, a quantity I is an adiabatic invariant if, for some $\alpha > 0$ and any threshold $\delta > 0$, there exists an $\epsilon_\delta > 0$ such that for all $\epsilon < \epsilon_\delta$ the change of I along solutions with parameter ϵ is smaller than δ for a time interval of length $\epsilon^{-\alpha}$ (see [2, §52] for more details). We remark, however, that

the normal form for vector fields is not enough for the two applications considered in this paper and we crucially need the Hamiltonian normal form contained in Theorem 1.1.

We conclude this subsection by observing that the theory of adiabatic invariants for magnetic fields in three-dimensional euclidean space plays an important role in plasma physics as it can be applied to confine charged particles in some region of space; see [16] and the references therein. We refer to [8] for a mathematical treatment of magnetic adiabatic invariants in euclidean space and to [14] for their applications to the spectrum of the magnetic Laplacian.

1.2. *Application I: trapped motions.* Throughout this subsection, we denote by (g, b_1) a magnetic system on a closed, oriented surface M such that $b_1 : M \rightarrow (0, \infty)$ is positive. We also denote by μ the area form given by g and the orientation on M . To write the statements in a more compact form, we define the function

$$\zeta : M \rightarrow \mathbb{R}, \quad \zeta = \begin{cases} K & \text{if } b_1 \text{ is constant,} \\ b_1^{-2} & \text{otherwise.} \end{cases} \tag{1.8}$$

Moreover, we will denote by L_{c_0} a non-empty connected component of a regular level set $\{\zeta = c_0\}$ for some $c_0 \in \mathbb{R}$. In particular, L_{c_0} is an embedded circle belonging to some family of embedded circles $c \mapsto L_c$ with $c \in (c_0 - \delta_0, c_0 + \delta_0)$ for a suitable $\delta_0 > 0$. There are action-angle coordinates $(I, \varphi) \in (I_-, I_+) \times \mathbb{T}$ around L_{c_0} for the symplectic form μ such that $\zeta = \tilde{\zeta} \circ I$ for some function $\tilde{\zeta} : (I_0 - \delta'_0, I_0 + \delta'_0) \rightarrow (c_0 - \delta_0, c_0 + \delta_0)$ with $\tilde{\zeta}(I_0) = c_0$ and some $\delta'_0 > 0$. The circle L_{c_0} is said to be *non-resonant* if

$$\frac{d^2 \tilde{\zeta}}{dI^2}(I_0) \neq 0. \tag{1.9}$$

Combining the Hamiltonian normal form in Theorem 1.1 with the Moser twist theorem [17], one can show that $(g, \epsilon^{-1}b_1)$ -geodesics are trapped in neighborhoods of non-resonant circles.

THEOREM 1.3. *Let L_{c_0} be non-resonant and let U be a neighborhood of L_{c_0} . Then there exist $\epsilon_0 > 0$ and a neighborhood $U_1 \subset U$ of L_{c_0} such that, for every $\epsilon \in (0, \epsilon_0)$, all $(g, \epsilon^{-1}b_1)$ -geodesics with starting point in U_1 remain in U for all times.*

Following Castilho [10, Corollary 1.3], one can use Theorem 1.3 to prove trapping of $(g, \epsilon^{-1}b_1)$ -geodesics around circles L of minima (or maxima) for ζ . The key idea is that L is approximated on either sides by non-resonant circles.

COROLLARY 1.4. *Let $L \subset M$ be an embedded circle. Suppose that L is an isolated critical set of ζ consisting of local minima or maxima. Then, for all neighborhoods U of L , there exist $\epsilon_0 > 0$ and a neighborhood $U_1 \subset U$ of L such that, for all $\epsilon \in (0, \epsilon_0)$, all $(g, \epsilon^{-1}b_1)$ -geodesics with starting point in U_1 remain in U for all times.*

We now want to give some criteria for the existence of non-resonant circles. One classical idea is to look for such circles around a non-degenerate local minimum (maximum) point $q_* \in M$ of ζ by means of the Birkhoff normal form. The Birkhoff normal form tells

us that ζ can be written in a suitable Darboux chart centered at q_* as

$$\zeta(q) = \zeta(q_*) \pm \frac{1}{2}\sqrt{\det \text{Hess } \zeta(q_*)} r^2 + a(q_*)r^4 + o(r^4),$$

where r is the radial coordinate and $a(q_*) \in \mathbb{R}$ is a real number. Here $\det \text{Hess } \zeta(q_*)$ is the determinant of the Hessian of ζ in Darboux coordinates with respect to μ , which coincides with the determinant of the Hessian with respect to g . Since $I = \frac{1}{2}r^2$ is the action variable in Darboux coordinates, we see that

$$\lim_{I \rightarrow 0} \frac{d^2 \tilde{\zeta}}{dI^2} \neq 0 \tag{1.10}$$

as soon as $a(q_*) \neq 0$. In this case, small circles L_c around q_* will be non-resonant. We mention here an equivalent way to check (1.10) using coordinates (x, y) around q_* where $\zeta = \zeta(q_*) + \frac{1}{2}r^2$, which exist thanks to the Morse lemma. Let ρ be the unique function satisfying $\mu = \rho dx \wedge dy$. We will see in Lemma 4.7 that (1.10) holds if and only if the Laplacian $\Delta\rho(q_*)$ of ρ at q_* in the (x, y) -coordinates does not vanish.

On the other hand, when the local minimum (maximum) point q_* is isolated but degenerate, we will see that the existence of non-resonant circles accumulating at q_* automatically follows.

COROLLARY 1.5. *Let $q_* \in M$ be an isolated local minimum (maximum) point for ζ which is either degenerate or is non-degenerate and satisfies $a(q_*) \neq 0$, equivalently $\Delta\rho(q_*) \neq 0$. Then, for all neighborhoods U of q_* , there exist $\epsilon_0 > 0$ and a neighborhood $U_1 \subset U$ of q_* such that, for all $\epsilon \in (0, \epsilon_0)$, all $(g, \epsilon^{-1}b_1)$ -geodesics with initial starting point in U_1 remain in U for all times.*

We now present two other situations in which non-resonant circles can be found. The first one uses a saddle point for ζ .

THEOREM 1.6. *Suppose that ζ has a non-degenerate saddle critical point $q_* \in M$ such that there are no critical values of ζ in the interval $(\zeta(q_*), \zeta(q_*) + \delta)$ for some $\delta > 0$. Then there exists a sequence of embedded circles L_{c_n} such that L_{c_n} is non-resonant and*

$$\lim_{n \rightarrow \infty} \text{dist}(q_*, L_{c_n}) = 0.$$

The second one works when the closed, oriented surface M is the two-sphere and the function ζ has exactly one minimum and one maximum point, which, by Corollary 1.5, can be both assumed to be non-degenerate.

THEOREM 1.7. *Suppose that $M = S^2$ and that ζ has one non-degenerate minimum at $q_{\min} \in S^2$, one non-degenerate maximum at $q_{\max} \in S^2$ and no other critical point. If*

$$\sqrt{\det \text{Hess } \zeta(q_{\min})} \neq \sqrt{\det \text{Hess } \zeta(q_{\max})},$$

then there is a non-resonant circle L_c . Here the Hessian is taken with respect to the metric g .

1.3. *Application II: rigidity and flexibility of Zoll systems.* The only known Zoll pairs (g, b) , which are different from the trivial examples $(g_{\text{con}}, b_{\text{con}})$, or the purely Riemannian examples with $b = 0$ on S^2 , were constructed in [4]. These exotic Zoll pairs are defined on $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ with angular coordinates (x, y) . Their metric g is flat and their magnetic function b depends on the x -variable only.

Unlike the trivial and the purely Riemannian examples, such Zoll pairs do not remain Zoll if we rescale b by an arbitrary constant $r > 0$. Therefore, we would like to understand if there are examples of magnetic systems (g, b) such that $(g, r^{-1}b)$ is Zoll for different values of r or, more generally, for values of r belonging to some given set. This corresponds to asking that the Hamiltonian flow of H_g on the twisted tangent bundle $(TM, \omega_{(g,b)})$ is Zoll at several energy levels $\frac{1}{2}r^2$. To better handle this question, we introduce the following definition.

Definition 1.8. Let \mathcal{M} be a family of magnetic systems on a closed, oriented surface M and let R be a subset of $(0, \infty)$. We say that \mathcal{M} is *Zoll-rigid at R* provided the following holds: if $(g, b) \in \mathcal{M}$ is a magnetic system such that $(g, r^{-1}b)$ is Zoll for every $r \in R$, then g has constant curvature and b is constant, or $M = S^2$, g is a Zoll metric and b is identically zero. If \mathcal{M} is not Zoll-rigid at R , we say that \mathcal{M} is *Zoll-flexible at R* .

In [5], the first rigidity phenomena were discovered:

- (i) magnetic systems on \mathbb{T}^2 are Zoll-rigid at any set R which accumulates to zero and infinity;
- (ii) magnetic systems on surfaces with genus at least two and with Mañé critical value of the universal cover equal to c are Zoll-rigid at any $R \subset (0, \sqrt{2c})$ which accumulates to $\sqrt{2c}$.

In this paper we push the study of rigidity further and prove the following statement.

THEOREM 1.9. *Magnetic systems on a closed, oriented surface M are Zoll-rigid at any set R which accumulates to zero. Namely, if (g, b_1) is a magnetic system on M such that $(g, \epsilon_n^{-1}b_1)$ is Zoll for some sequence $\epsilon_n \rightarrow 0$, then: either $b_1 = 0$, $M = S^2$ and g is a Zoll metric, or b_1 is a non-zero constant and g is a metric of constant curvature.*

The proof of Theorem 1.9 hinges on the normal form proved in Theorem 1.1. Roughly speaking, if b_1 is not constant, we can suppose, up to changing the sign, that b_1 has a positive maximum and apply the normal form to the regions $U = U_1 = \{b_1 > \delta_1\}$ and $U' = U_2 = \{b_1 > \delta_2\}$ for some positive $\delta_1 < \delta_2$ in the interval $(\min b, \max b)$. Using a theorem of Ginzburg [11], we can find U_3 such that there exists $q_0 \in U_3$ with the following property: the flow line of Φ_{H_ϵ} starting at $W(q_0)$ has period 2π . If now $(g, \epsilon^{-1}b_1)$ is Zoll, a topological argument shows that for all $q \in U_3$ the flow line of Φ_{H_ϵ} starting at $W(q)$ has period 2π . However, since b_1 is not constant, the normal form tells us that there must be $q_1 \in U_3$ such that the flow line of Φ_{H_ϵ} starting at $W(q_1)$ drifts with speed of order ϵ^2 along a regular level set of b_1 inside U_3 . Hence, this flow line cannot close up in time 2π if ϵ is small enough. Thus, $(g, \epsilon^{-1}b_1)$ cannot be Zoll for ϵ small enough when b_1 is not constant. If now b_1 is a non-zero constant but the Gaussian curvature K of g is non-constant, we

can apply a similar argument to the regions $U_1 = \{K > \delta_1\}$ and $U_2 = \{K > \delta_2\}$ for some $\delta_1 < \delta_2$ in the interval $(\min K, \max K)$ to get that $(g, \epsilon^{-1}b_1)$ cannot be Zoll for ϵ small.

Remark 1.10. For the restricted case of rotationally invariant systems a slight variant of Theorem 1.9 was proved by Kudryavtseva and Podlipaev in [15], where rigidity is established provided all orbits are periodic (a condition which is weaker than being Zoll) for every sufficiently small speed (whereas the Zoll condition in Theorem 1.9 only holds on a sequence of speeds converging to zero). It is natural to expect that such a statement should hold also for general magnetic systems on surfaces using the techniques of the present paper. As observed in [15], results such as Theorem 1.9 can be thought of as magnetic analogues of a classical result of Bertrand [9] asserting that the only attracting central forces for which every bounded orbit is periodic are the harmonic and the gravitational ones [2, §8D].

An intriguing problem is to understand exactly which sets guarantee rigidity and which ones flexibility. The threshold between the two behaviors can be subtle as the next result shows. We first define the sets

$$R_* := \{\xi \in (0, \infty) \mid J_1(\xi) = 0\}, \quad R_{\mathbb{N}} := \bigcup_{k \in \mathbb{N}} \frac{1}{k} R_*,$$

where J_1 is the first Bessel function. The set R_* is discrete, bounded away from zero, unbounded from above and asymptotic to the arithmetic progression $\{\frac{3}{4}\pi + 2\pi k \mid k \in \mathbb{N}\}$. In particular, $R_{\mathbb{N}}$ is a countable dense subset of $(0, \infty)$.

THEOREM 1.11. *Let \mathcal{M} be the set of rotationally invariant magnetic systems on \mathbb{T}^2 with average 1. Namely $(g, b) \in \mathcal{M}$ if and only if $g = dx^2 + a(x)^2 dy^2$ and $b = b(x)$ for some functions $a, b : \mathbb{T} \rightarrow \mathbb{R}$ such that $\int_{\mathbb{T}} ab dx = \int_{\mathbb{T}} a dx$. The family \mathcal{M} is:*

- (a) Zoll-rigid at every set R which is unbounded from above and not contained in $R_{\mathbb{N}}$;
- (b) Zoll-flexible at each of the sets $(1/k)R_*$, $k \in \mathbb{N}$.

Remark 1.12. By [7], there are no rotationally invariant Zoll systems on \mathbb{T}^2 with zero average. Thus, up to rescaling b by a constant, there is no loss of generality in assuming that the average is 1.

Theorem 1.11 is rather surprising and represents the first instance where the structure of the space of Zoll magnetic systems is influenced by the topology of the surface. Indeed, since the Mañé critical value of the universal cover for magnetic systems with non-zero average is infinite, we can reformulate Theorem 1.11(b) by saying that magnetic systems on \mathbb{T}^2 with average 1 are Zoll-flexible for a particular set accumulating at the Mañé critical value of the universal cover. This is in sharp contrast with the rigidity for surfaces with genus at least two described in the item (ii) above [5].

1.4. Structure of the paper. In §2 we present some classical facts about the geometry of surfaces that serve as preliminaries to the Hamiltonian normal form contained in Theorem 1.1 which will be proved in §3. In §4 we establish the existence of trapping regions and

prove Theorem 1.3 and its Corollaries 1.4 and 1.5. The criteria contained in Theorem 1.6 and Theorem 1.7 for the existence of non-resonating circles will also be discussed there. §5 deals with the proof of Theorem 1.9 about the Zoll-rigidity of strong magnetic fields, while §6 shows Theorem 1.11 about the rigidity versus flexibility behavior for rotationally symmetric magnetic fields on the two-torus.

2. Preliminaries from the differential geometry of surfaces

In this section, we recall some facts from the Riemannian geometry of surfaces that will be useful later on. Let $U \subset M$ be an open set and assume that we have a section $W : U \rightarrow SU$ of $\pi : SU \rightarrow U$. We denote by $e^{i\theta} : SU \rightarrow SU$ the flow of fiberwise rotations and use the shorthand $(\cdot)^\perp := e^{i\pi/2}(\cdot)$. We denote by ∂_θ the vector field generating the flow of rotations. Moreover, we denote by $\mathcal{H} := \ker d\theta \subset T(SU)$ the distribution tangent to the level sets of θ . We have

$$T(SU) = \mathcal{H} \oplus (\mathbb{R} \cdot \partial_\theta).$$

If $u \in TU$, we denote by \tilde{u} the unique element in \mathcal{H} such that $d\pi[\tilde{u}] = u$.

A key object in our computations will be the one-form $\alpha \in \Omega^1(U)$ given by

$$\alpha = g(\nabla.W, W^\perp), \tag{2.1}$$

where ∇ is the Levi-Civita connection of g . The rotations commute with the connection, so that for every $\theta \in \mathbb{T}$ we have

$$\alpha = g(\nabla.W, W^\perp) = g(\nabla.(e^{i\theta}W), (e^{i\theta}W)^\perp). \tag{2.2}$$

Let $\hat{\lambda}$ be the pull-back on SM of λ . Our first task is to compute $d\hat{\lambda}$ on SU in terms of α .

LEMMA 2.1. *We have*

$$(d\hat{\lambda})_v = (\pi^*\alpha + d\theta) \wedge g(d\pi[\cdot], v^\perp) \text{ for all } v \in SU.$$

Proof. Let $v = e^{i\theta}W(q)$ for some $\theta \in \mathbb{T}$ and $q \in U$. The formula for $(d\hat{\lambda})_v$ has to be checked for the pairs $\tilde{u}_1, \tilde{u}_2 \in \mathcal{H}_v$ and $\partial_\theta, \tilde{u} \in \mathcal{H}$. Let $u_i := d\pi[\tilde{u}_i]$ for $i = 1, 2$. We can suppose without loss of generality that u_1, u_2 are defined in a neighborhood of q and satisfy $[u_1, u_2] = 0$ there. We compute first

$$\begin{aligned} &\tilde{u}_1(g(u_2, e^{i\theta}W)) \\ &= g(\nabla_{u_1}u_2, v) + g(u_2, \nabla_{u_1}(e^{i\theta}W)) = g(\nabla_{u_1}u_2, v) + g(u_2, v^\perp)g(e^{i\theta}W^\perp, \nabla_{u_1}(e^{i\theta}W)) \\ &= g(\nabla_{u_1}u_2, v) + g(u_2, v^\perp)g(W^\perp, \nabla_{u_1}W) \\ &= g(\nabla_{u_1}u_2, v) + g(u_2, v^\perp)\alpha(u_1), \end{aligned}$$

where in the second equality we used the fact that $v^\perp = e^{i\theta}W^\perp$ and the identity

$$g(\nabla.(e^{i\theta}W), e^{i\theta}W) = \frac{1}{2}d|e^{i\theta}W|^2 = \frac{1}{2}d(1) = 0,$$

and the identity (2.2) in the third equality.

The value of $d\hat{\lambda}(\tilde{u}_1, \tilde{u}_2)$ is the antisymmetrization of the expression above since $[u_1, u_2] = 0$. Taking into account that ∇ is symmetric, we obtain

$$(d\hat{\lambda})_v(\tilde{u}_1, \tilde{u}_2) = g(u_2, v^\perp)\alpha(u_1) - g(u_2, v)\alpha(u_1) = \pi^*\alpha \wedge g(d\pi[\cdot], v^\perp)(\tilde{u}_1, \tilde{u}_2).$$

For the pair $\partial_\theta, \tilde{u}$, we similarly get

$$\begin{aligned} (d\hat{\lambda})_v(\partial_\theta, \tilde{u}) &= \partial_\theta(g(u, e^{i\theta}W)) - 0 = g(u, \partial_\theta(e^{i\theta}W)) = g(u, e^{i\theta}W^\perp) \\ &= g(u, v^\perp) = d\theta \wedge g(d\pi[\cdot], v^\perp)(\partial_\theta, \tilde{u}). \end{aligned} \quad \square$$

We now link the Gaussian curvature $K : U \rightarrow \mathbb{R}$ of g to the function

$$f : SU \rightarrow \mathbb{R}, \quad f(v) := \alpha(v)^2 + d(\alpha(e^{i\theta(v)}W^\perp)) \cdot v. \tag{2.3}$$

LEMMA 2.2. *For every $v \in SU$, we have*

$$K(\pi(v)) = f(v) + f(v^\perp). \tag{2.4}$$

In particular, there exists a function $a : SU \rightarrow \mathbb{R}$ such that

$$\frac{1}{2}K \circ \pi = f + \partial_\theta a. \tag{2.5}$$

Proof. For every fixed $\theta \in \mathbb{T}$, we first check the formula

$$[e^{i\theta}W, e^{i\theta}W^\perp] = -\alpha(e^{i\theta}W)e^{i\theta}W - \alpha(e^{i\theta}W^\perp)e^{i\theta}W^\perp.$$

It is enough to prove the result for $\theta = 0$. Using the symmetry of ∇ , we get

$$g([W, W^\perp], W) = g(\nabla_W W^\perp, W) + g(\nabla_{W^\perp} W, W) = -g(\nabla_W W, W^\perp) + 0 = -\alpha(W)$$

and analogously $g([W, W^\perp], W^\perp) = -\alpha(W^\perp)$.

To compute the Gaussian curvature, we use the classical identity $K\mu = d\alpha$, so that

$$K(\pi(v)) = d\alpha(v, v^\perp). \tag{2.6}$$

Let us now set $\theta := \theta(v)$. Then the vector fields $e^{i\theta}W$ and $e^{i\theta}W^\perp$ extend the vectors v and v^\perp on the whole U . Therefore, we can expand the right-hand side of (2.6) using (2.2):

$$\begin{aligned} d\alpha(v, v^\perp) &= d(\alpha(e^{i\theta}W^\perp)) \cdot v - d(\alpha(e^{i\theta}W)) \cdot v^\perp - \alpha([e^{i\theta}W, e^{i\theta}W^\perp]) \\ &= d(\alpha(e^{i\theta}W^\perp)) \cdot v + d(\alpha(e^{i(\theta+\pi/2)}W^\perp)) \cdot v^\perp + \alpha(v)^2 + \alpha(v^\perp)^2 \\ &= f(v) + f(v^\perp), \end{aligned}$$

so that we arrive at (2.4). Fixing $q \in U$ and averaging the identity (2.4) over $\pi^{-1}(q)$, we get

$$K(q) = \frac{1}{2\pi} \int_0^{2\pi} (f(e^{i\theta}W(q)) + f(e^{i(\theta+\pi/2)}W(q)))d\theta = \frac{1}{\pi} \int_0^{2\pi} f(e^{i\theta}W(q))d\theta.$$

It follows that the function

$$f_1 : SU \rightarrow \mathbb{R}, \quad f_1 := \frac{1}{2}K \circ \pi - f,$$

has zero average along the circle $\pi^{-1}(q)$ for all $q \in U$. Because of this fact, the function

$$a : SU \rightarrow \mathbb{R}, \quad a(v) := \int_0^{\theta(v)} f_1(e^{i\theta'} W(\pi(v))) d\theta'$$

is well defined and it is immediate to check that $\partial_\theta a = f_1$, as required. □

3. The construction of the normal form

This section is entirely dedicated to the proof of Theorem 1.1. Recalling that $\hat{\lambda}$ is the restriction to SM of λ , we write

$$\hat{\omega}_\epsilon := \epsilon d\hat{\lambda} - \pi^*(b_1\mu)$$

for the pull-back of ω_ϵ to SM .

We consider $U \subset M$, an open set on which we have a section W of $\pi : SU \rightarrow U$ with associated angular function θ . We further assume that b_1 is nowhere vanishing on U . We take an open set U' such that $\overline{U'} \subset U$. Then there exists a positive number $r > 0$ such that any curve $\gamma : [t_0, t_1] \rightarrow M$ with $\gamma(t_0) \in U'$ and length less than r is entirely contained in U .

Let $\Psi_\epsilon : SU' \rightarrow SU$ denote an isotopy for ϵ in some interval $[0, \epsilon_0]$ such that Ψ_0 is the standard inclusion $SU' \hookrightarrow SU$. Up to shrinking ϵ_0 further, by the definition of $r > 0$ we see that Ψ_ϵ is obtained by integrating an ϵ -dependent vector field Z_ϵ on SU . We aim at finding Z_ϵ such that

$$\Psi_\epsilon^* \hat{\omega}_\epsilon = -\pi^*(b_1\mu) + d(H_\epsilon d\theta) \quad \text{on } SU' \tag{3.1}$$

for some $H_\epsilon : SU \rightarrow \mathbb{R}$ with $H_0 = 0$, satisfying either (1.2), or (1.3) if b_1 is constant on U .

Taking the derivative of (3.1) in ϵ and using that $\partial_\epsilon \hat{\omega}_\epsilon = \hat{\lambda}$, we see that (3.1) is equivalent to

$$\Psi_\epsilon^*(d\hat{\lambda} + \mathcal{L}_{Z_\epsilon} \hat{\omega}_\epsilon) = d(h_\epsilon d\theta), \quad h_\epsilon := \partial_\epsilon H_\epsilon.$$

Thanks to the Cartan formula, this equation can be rewritten as

$$d[\Psi_\epsilon^*(\hat{\lambda} + \iota_{Z_\epsilon} \hat{\omega}_\epsilon) - h_\epsilon d\theta] = 0. \tag{3.2}$$

We require now that Z_ϵ belong to the horizontal distribution \mathcal{H} tangent to the level sets of θ . By taking the pull-back by $(\Psi_\epsilon)^{-1}$ on both sides, we see that (3.2) is solved if there exists a function $c_\epsilon : SU \rightarrow \mathbb{R}$ with

$$\hat{\lambda} + \iota_{Z_\epsilon} \hat{\omega}_\epsilon + dc_\epsilon = h_\epsilon \circ \Psi_\epsilon^{-1} d\theta. \tag{3.3}$$

This equation can be decomposed in the horizontal and vertical components at $v \in SU$:

$$\hat{\omega}_\epsilon|_{\mathcal{H}}(Z_\epsilon, \cdot) = -\hat{\lambda}|_{\mathcal{H}} - d|_{\mathcal{H}}c_\epsilon, \tag{3.4a}$$

$$h_\epsilon \circ \Psi_\epsilon^{-1}(v) = \epsilon g(Z_\epsilon, v^\perp) + \partial_\theta c_\epsilon(v). \tag{3.4b}$$

For any function c_ϵ , equation (3.4a) determines uniquely the vector field Z_ϵ and hence the isotopy Ψ_ϵ , since $\hat{\omega}_\epsilon|_{\mathcal{H}}$ is a non-degenerate bilinear form for ϵ small enough. Then equation (3.4b) determines h_ϵ , and hence the function H_ϵ , uniquely.

We expand Z_ϵ and h_ϵ to the first order in ϵ :

$$Z_\epsilon = Z_0 + \epsilon Z'_\epsilon, \quad h_\epsilon = h_0 + \epsilon h'_\epsilon.$$

We choose $c_\epsilon = \epsilon^2 c'_\epsilon$ for some function c'_ϵ to be determined later. Then we evaluate equations (3.4a) and (3.4b) at $\epsilon = 0$. From (3.4a), we get

$$\hat{\omega}_0|_{\mathcal{H}}(Z_0, \cdot) = -\hat{\lambda}|_{\mathcal{H}}.$$

By the definitions of $\hat{\omega}_0$ and $\hat{\lambda}$, this equation can be rewritten as

$$-b_1 g(d\pi[Z_0^\perp], \cdot) = -g(v, \cdot) \quad \text{on } \mathcal{H},$$

from which it follows that

$$Z_0 = -b_1^{-1} \tilde{v}^\perp.$$

Second, from (3.4b), we get that

$$h_0 = 0.$$

Dividing (3.4a) and (3.4b) by ϵ , we arrive at the equivalent equations

$$\hat{\omega}_\epsilon|_{\mathcal{H}}(Z'_\epsilon, \cdot) = b_1^{-1} d\hat{\lambda}|_{\mathcal{H}}(\tilde{v}^\perp, \cdot) - \epsilon d|_{\mathcal{H}}c'_\epsilon, \tag{3.5a}$$

$$h'_\epsilon \circ \Psi_\epsilon^{-1}(v) = -b_1^{-1} + \epsilon g(d\pi[Z'_\epsilon], v^\perp) + \epsilon \partial_\theta c'_\epsilon(v). \tag{3.5b}$$

Evaluating (3.5a) at $\epsilon = 0$ and using the expression for $d\hat{\lambda}$ obtained in Lemma 2.1, we find the equation

$$-b_1 g(d\pi[Z_0^\perp], \cdot) = b_1^{-1} \alpha(v^\perp) g(v^\perp, \cdot) - b_1^{-1} \alpha,$$

from which it follows that

$$g(d\pi[Z_0^\perp], v) = b_1^{-2} \alpha(v), \quad g(d\pi[Z_0^\perp], v^\perp) = 0,$$

which is equivalent to

$$Z'_0 = -b_1^{-2} \alpha(v) \tilde{v}^\perp.$$

Evaluating (3.5b) at $\epsilon = 0$, we get

$$h'_0 = -b_1^{-1},$$

which yields the desired expansion of H_ϵ to the second order in ϵ .

Let us now go to higher order under the hypothesis that b_1 is a non-zero constant. We write

$$Z'_\epsilon = Z'_0 + \epsilon Z''_\epsilon, \quad h'_\epsilon = -b_1^{-1} + \epsilon h''_\epsilon, \quad c'_\epsilon = c'_0 + \epsilon c''_\epsilon$$

and substitute these expressions in (3.5a) and (3.5b). After dividing by ϵ , we get the new set of equations

$$\hat{\omega}_\epsilon|_{\mathcal{H}}(Z''_\epsilon, \cdot) = b_1^{-2} \alpha(v) d\hat{\lambda}|_{\mathcal{H}}(\tilde{v}^\perp, \cdot) - d|_{\mathcal{H}}c''_\epsilon - \epsilon d|_{\mathcal{H}}c'_\epsilon, \tag{3.6a}$$

$$h''_\epsilon \circ \Psi_\epsilon^{-1}(v) = b_1^{-2} \alpha(v) + \epsilon g(d\pi[Z''_\epsilon], v^\perp) + \partial_\theta c''_\epsilon(v) + \epsilon \partial_\theta c'_\epsilon(v). \tag{3.6b}$$

We now choose $c'_0(v) = b_1^{-2} \alpha(v)$, so that

$$b_1^{-2} \alpha(v) + \partial_\theta c'_0(v) = 0.$$

In this case, evaluating (3.6b) at $\epsilon = 0$, we get $h''_0 = 0$. Thus, we can substitute $h''_\epsilon = \epsilon h'''_\epsilon$ in (3.6b) and, dividing this equation by ϵ , we find that

$$h'''_\epsilon \circ \Psi_\epsilon^{-1}(v) = g(d\pi[Z''_\epsilon], v^\perp) + \partial_\theta c''_\epsilon(v).$$

Evaluating at $\epsilon = 0$, we have the formula

$$h'''_0(v) = g(d\pi[Z''_0], v^\perp) + \partial_\theta c''_0(v). \tag{3.7}$$

In order to determine the first summand on the right, we evaluate (3.6a) at $\epsilon = 0$ and insert \tilde{v} :

$$-b_1 g(d\pi[Z''_0^\perp], v) = -b_1^{-2}[\alpha(v)^2 + d(\alpha(e^{i\theta(v)}W^\perp)) \cdot v].$$

Here we have used that b_1 is constant. Recalling the definition of the function f from (2.3), we arrive at

$$g(d\pi[Z''_0], v^\perp) = -b_1^{-3} f(v).$$

Let $a : SU \rightarrow \mathbb{R}$ be the function given by Lemma 2.2 and choose $c''_\epsilon = c''_0 = -b_1^{-3}a$. Thanks to (2.5) and the fact that b_1 is constant, we see that (3.7) is equivalent to

$$h'''_0 = -b_1^{-3} f - b_1^{-3} \partial_\theta a = -\frac{1}{2} b_1^{-3} K.$$

Thus,

$$h_\epsilon = -\epsilon b_1^{-1} - \frac{1}{2} \epsilon^3 b_1^{-3} K$$

and the formula for H_ϵ follows. This finishes the proof of Theorem 1.1. □

4. Trapping regions for the magnetic flow

We start by recalling the celebrated twist theorem of Moser [17]. We give here the version for flows which can be easily deduced from the original one for mappings.

Consider the standard symplectic form $\omega = dI \wedge d\varphi$ on the open annulus $(I_-, I_+) \times \mathbb{T}$ with coordinates (I, φ) and let

$$H_{\epsilon,\theta} = h_0(\epsilon) + \epsilon^k h_1 \circ I + o(\epsilon^k) \tag{4.1}$$

be a Hamiltonian periodically depending on the time θ . Here $h_0(\epsilon)$ is a real constant, k is a positive integer and $h_1 : (I_-, I_+) \rightarrow \mathbb{R}$ is a function. We call Φ_{H_ϵ} the Hamiltonian flow on the set $(I_-, I_+) \times \mathbb{T}^2$, where the second angular coordinate is given by the periodic time θ .

THEOREM 4.1. (The Moser twist theorem [17]) *Suppose that for some $I_0 \in (I_-, I_+)$, we have*

$$\frac{d^2 h_1}{dI^2}(I_0) \neq 0.$$

Then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$, there is a two-dimensional torus $\mathcal{T}_0^\epsilon \subset (I_-, I_+) \times \mathbb{T}^2$ which is C^1 -close to the torus $\{I_0\} \times \mathbb{T}^2$ and is invariant under Φ_{H_ϵ} . In particular, \mathcal{T}_0^ϵ divides $(I_-, I_+) \times \mathbb{T}^2$ into two open regions both invariant under the flow.

Remark 4.2. Applying the Moser twist theorem to $I_0^- \in (I_-, I_0)$ and $I_0^+ \in (I_0, I_+)$ arbitrarily close to I_0 (which can also be taken to be non-resonant), we find ϵ_0 such that for $\epsilon \in [0, \epsilon_0]$ all the trajectories of Φ_{H_ϵ} starting in the region \mathcal{U}' between the tori $\mathcal{T}_{I_0^-}^\epsilon$ and $\mathcal{T}_{I_0^+}^\epsilon$ will remain in \mathcal{U}' for all times.

Let us see how we can apply the Moser twist theorem for local perturbations of an autonomous Hamiltonian $H_1 : M \rightarrow \mathbb{R}$ on some symplectic surface (M, ω) . Suppose that L_{c_0} is a connected component of a regular level set $\{H_1 = c_0\}$ which is diffeomorphic to a circle. Then there is a tubular neighborhood U of L_{c_0} foliated by embedded circles $c \mapsto L_c$ for $c \in (c_-, c_+)$ so that $H_1(L_c) = c$. By [2, §50], there are action-angle coordinates on the neighborhood of L_{c_0} foliated by the circles L_c . The action variable I can be written as a composition $\tilde{I} \circ H_1$, where \tilde{I} is the strictly monotonically increasing function given by

$$\tilde{I}(c) = \int_{\mathcal{A}_{c_0,c}} \omega$$

and $\mathcal{A}_{c_0,c}$ denotes the annular region between L_{c_0} and L_c . We have that

$$\frac{d\tilde{I}}{dc} = \int_{L_c} \eta,$$

where η is any one-form on U satisfying $\omega = \eta \wedge dH_1$. For example, we can take

$$\eta = \omega \left(\frac{1}{|\nabla H_1|^2} \nabla H_1, \cdot \right), \tag{4.2}$$

where the gradient and the norm are taken with respect to an arbitrary metric. With this choice, we see that there exists a positive constant $\delta > 0$ not depending on c such that

$$\frac{d\tilde{I}}{dc} \geq \delta \int_{L_c} |\nabla H_1|^{-1} ds, \tag{4.3}$$

where ds is the arc length of L_c .

If \tilde{h}_1 is the inverse function of \tilde{I} , we conclude that $H_1 = \tilde{h}_1 \circ \tilde{I}$ and

$$\text{for all } e = \tilde{I}(c), \quad \frac{d^2 \tilde{h}_1}{de^2}(e) \neq 0 \iff \frac{d^2 \tilde{I}}{dc^2}(c) \neq 0.$$

Definition 4.3. We say that L_c is *non-resonant* if $d^2 \tilde{I}/dc^2(c) \neq 0$.

Remark 4.4. Two simple conditions to show the existence of non-resonant circles are as follows. First, if $d\tilde{I}/dc$ is not constant on (c_-, c_+) , then there exists a non-resonant circle L_c with $c \in (c_-, c_+)$. Second, if

$$\lim_{c \rightarrow c_+} \frac{d\tilde{I}}{dc} = +\infty,$$

then there is a sequence $(c_n) \subset (c_-, c_+)$ with the property that $c_n \rightarrow c_+$ and L_{c_n} is a non-resonant circle. A similar statement holds for c_- instead of c_+ .

Remark 4.5. The non-resonance condition for a circle L_c is a property that is invariant upon multiplying the symplectic form by a non-zero constant. In view of the applications to magnetic fields, we need to consider the case in which ω also depends on H_1 in a certain way. More specifically, suppose that H_1 is positive and that $\omega = H_1^{-1}\omega_1$ for some symplectic form ω_1 . Then

$$H_1^{-1}\omega_1 = \eta \wedge dH_1 \iff \omega_1 = \eta \wedge d(\frac{1}{2}H_1^2).$$

This means that

$$\frac{d\tilde{I}_\omega}{dc}(c) = \frac{d\tilde{I}_{\omega_1}}{dc_1}\left(\frac{1}{2}c^2\right),$$

where we put the symplectic form in subscript to distinguish the two cases. Differentiating in c , we get

$$\frac{d^2\tilde{I}_\omega}{dc^2}(c) = c \frac{d^2\tilde{I}_{\omega_1}}{d^2c_1}\left(\frac{1}{2}c^2\right),$$

so that a level set of H_1 is non-resonant with respect to $H_1^{-1}\omega_1$ if and only if it is non-resonant as a level set of $\frac{1}{2}H_1^2$ with respect to ω_1 .

Let us consider a family of time-dependent Hamiltonians on U having the form

$$H_{\epsilon,\theta} = h_0(\epsilon) + \epsilon^k H_1 + o(\epsilon^k),$$

where $h_0(\epsilon)$ is some constant, k is a positive integer and the dependence on the time $\theta \in \mathbb{T}$ is periodic. Using the action-angle coordinates on U around L_{c_0} described above, we can bring the Hamiltonian into the form (4.1). If now L_{c_0} is a non-resonant circle, the following statement follows directly from the Moser twist theorem (Theorem 4.1) and the subsequent Remark 4.2: for all neighborhoods \mathcal{U}' of L_{c_0} , there exist an $\epsilon_0 > 0$ and a neighborhood $\mathcal{U}'' \subset \mathcal{U}'$ of L_{c_0} with the property that for all $\epsilon \in [0, \epsilon_0]$, every trajectory of Φ_{H_ϵ} starting in $\mathcal{U}'' \times \mathbb{T}$ will stay in $\mathcal{U}' \times \mathbb{T}$ for all times.

We can now use the above discussion to prove trapping for strong magnetic fields.

Proof of Theorem 1.3. Suppose that $L = L_{c_0}$ is a non-resonant circle for $\zeta = b_1^{-2}$ with respect to the symplectic form μ on M . By Remark 4.5, L is a non-resonant circle for $H_1 := -\frac{1}{2}b_1^{-1}$ with respect to $\omega = b_1\mu$. Let U be an arbitrary neighborhood of L . Upon shrinking it, we can assume that U is a tubular neighborhood of L so that SU admits an angular function θ . We apply Theorem 1.1 to U and a further tubular neighborhood U' of L with $\bar{U}' \subset U$: there exists ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$, there is a Hamiltonian $H_{\epsilon,\theta} = h_0(\epsilon) + \epsilon^2 H_1 + o(\epsilon^2)$ with $h_0(\epsilon) = 0$ such that the Hamiltonian flow Φ_{H_ϵ} is conjugated via the map Ψ_ϵ to the magnetic flow $\Phi_{(g,\epsilon^{-1}b_1)}$, up to time reparametrization. Applying the Moser twist theorem, we see that, upon shrinking ϵ_0 , there is a neighborhood U'' of L in U' such that for all $\epsilon \in (0, \epsilon_0)$, a trajectory of Φ_{H_ϵ} starting in SU'' will stay in SU' for all times. This means that all $(g, \epsilon^{-1}b_1)$ -geodesics with initial velocity vector in $\Psi_\epsilon(SU'')$ will stay in $\pi(\Psi_\epsilon(SU'))$ for all times. Upon shrinking ϵ_0 , we see that there exists a neighborhood U''' of L with $SU''' \subset \Psi_\epsilon(SU'')$ and that $\pi(\Psi_\epsilon(SU'')) \subset U$. This means that any $(g, \epsilon^{-1}b_1)$ -geodesic starting in $U_1 := U'''$ will stay inside U for all times.

This finishes the proof of Theorem 1.3 for the function $\zeta = b_1^{-2}$. The proof of Theorem 1.3 when b_1 is a positive constant and the function ζ is the Gaussian curvature K is analogous and is left to the reader. \square

We now give some criteria for the existence of non-resonant circles for H_1 . The first criterion is due to Castilho [10, Corollary 1.3]. We present the short proof here for the convenience of the reader.

LEMMA 4.6. *Suppose that H_1 attains a minimum or maximum at L_{c_0} and that there are no other critical points of H_1 in a neighborhood of L (for example, L_{c_0} is a Morse–Bott component for H_1). Then there are non-resonant circles on either side of L_{c_0} accumulating at L_{c_0} .*

Proof. Let us suppose without loss of generality that c_0 is the minimum of H_1 . Consider a family of circles $c \mapsto L_c$ contained in regular level sets of H_1 staying on one side of L_{c_0} for $c \in (c_0, c_1)$. In particular, $L_c \rightarrow L_{c_0}$ uniformly as $c \rightarrow c_0$. It is enough to show that

$$\lim_{c \rightarrow c_0} \frac{d\tilde{I}}{dc} = +\infty. \tag{4.4}$$

By the inequality (4.3), we get the lower bound

$$\frac{d\tilde{I}}{dc} = \delta \int_{L_c} |\nabla H_1|^{-1} ds \geq \delta (\max_{L_c} |\nabla H_1|)^{-1} \text{length}(L_c).$$

As c tends to c_0 , $\text{length}(L_c)$ is bounded away from zero, while $\max_{L_c} |\nabla H_1|$ converges to zero. The limit in (4.4) follows. \square

Proof of Corollary 1.4. We recall that by Remark 4.5, a circle L is non-resonant for $H_1 := -\frac{1}{2}b_1^{-1}$ with respect to $\omega = b_1\mu$ if and only if it is non-resonant for $\zeta = b_1^{-2}$ with respect to μ . Let U be an arbitrary neighborhood of the critical circle L_{c_0} . We can assume that U is a tubular neighborhood of L . We apply Theorem 1.1 to U and a neighborhood U' of L such that $\bar{U}' \subset U$ and get a diffeomorphism Ψ_ϵ and a function $H_{\epsilon,\theta} = h_0(\epsilon) + \epsilon^2 H_1 + o(\epsilon^2)$ with $h_0(\epsilon) = 0$ for all $\epsilon \in [0, \epsilon_0]$. By Lemma 4.6, there are non-resonant circles L_- and L_+ for H_1 inside U' on either side of L_{c_0} . Applying the Moser twist theorem (Theorem 4.1) to L_- and L_+ , we see that, for a possibly smaller ϵ_0 , there is a neighborhood U'' of L such that for all $\epsilon \in [0, \epsilon_0]$ every trajectory of Φ_{H_ϵ} with initial condition in SU'' remains in SU' for all times. Taking an even smaller ϵ_0 , we get $\pi(\Psi_\epsilon(SU')) \subset U$ and we find a neighborhood U''' of L such that $SU''' \subset \Psi_\epsilon(SU'')$. Thus, any $(g, \epsilon^{-1}b_1)$ -geodesic starting in $U_1 := U'''$ will be contained in U for all times.

The proof of Corollary 1.4 for $\zeta = b_1^{-2}$ is completed. For $\zeta = K$ and b_1 a positive constant, the proof is analogous and we omit it. \square

Let us analyze the situation in a neighborhood of an isolated minimum or maximum $q_* \in U$ for H_1 . Let $c_0 := H_1(q_*)$, so that we have a family of circles L_c for c in some interval (c_0, c_+) (or (c_-, c_0)) converging uniformly to q_* as $c \rightarrow c_0$.

LEMMA 4.7. Let $q_* \in M$ be an isolated local minimum or maximum for H_1 . If q_* is degenerate, then

$$\lim_{c \rightarrow c_0} \frac{d\tilde{I}}{dc} = +\infty. \tag{4.5}$$

If q_* is non-degenerate, then

$$\lim_{c \rightarrow c_0} \frac{d\tilde{I}}{dc} = \frac{2\pi}{\sqrt{\det \text{Hess } H_1(q_*)}}, \quad \lim_{c \rightarrow c_0} \frac{d^2\tilde{I}}{dc^2} = \pi \Delta\rho(q_*), \tag{4.6}$$

where the Hessian is taken in Darboux coordinates around q_* , and ρ is a function with the property that $\omega = \rho \, dx \wedge dy$ in coordinates (x, y) around q_* such that $H_1 = c_0 + \frac{1}{2}(x^2 + y^2)$.

Proof. We deal only with the case of a local minimum and leave the case of a local maximum to the reader. We make the preliminary remark that, up to an additive constant, we have

$$\tilde{I}(c) = \int_{\{H_1 \leq c\}} \omega,$$

where $\{H_1 \leq c\}$ is a sublevel set of H_1 in a neighborhood of q_* .

Assume first that q_* is degenerate. There exist Darboux coordinates (x, y) around q_* such that $H_1 = c_0 + \frac{1}{2}(ux)^2 + o(r^2)$ for some constant $u \geq 0$. In particular, we find a constant $C > 0$ such that

$$H_1 - c_0 \leq \frac{1}{2}(ux)^2 + \frac{1}{2}(Cr)^3.$$

We distinguish two cases.

If $u = 0$, then we readily see that

$$\{r^2 \leq C^{-2}2^{2/3}(c - c_0)^{2/3}\} \subset \{H_1 \leq c\}.$$

Therefore,

$$\begin{aligned} \frac{\tilde{I}(c)}{c - c_0} &= \frac{1}{c - c_0} \int_{\{H_1 \leq c\}} \omega \\ &\geq \frac{1}{c - c_0} \int_{\{r^2 \leq C^{-2}2^{2/3}(c - c_0)^{2/3}\}} \omega \\ &= \frac{\pi C^{-2}2^{2/3}(c - c_0)^{2/3}}{c - c_0} \\ &= \frac{\pi 2^{2/3}}{C^2(c - c_0)^{1/3}}. \end{aligned}$$

Since the last term diverges as c tends to c_0 , we get (4.5).

If $u \neq 0$, then

$$\{x^2 \leq u^{-2}(c - c_0), \quad y^2 \leq C^{-2}(c - c_0)^{2/3} - u^{-2}(c - c_0)\} \subset \{H_1 \leq c\}.$$

Therefore, arguing as above, we obtain that

$$\begin{aligned} \frac{\tilde{I}(c)}{c - c_0} &\geq \frac{u^{-1}(c - c_0)^{1/2}(C^{-2}(c - c_0)^{2/3} - u^{-2}(c - c_0))^{1/2}}{c - c_0} \\ &= \frac{1}{uC(c - c_0)^{1/6}} \left(1 - \frac{C^2(c - c_0)^{1/3}}{u^2} \right)^{1/2}. \end{aligned}$$

Since the rightmost term diverges as c tends to c_0 , we again get (4.5).

Let us assume now that q_* is non-degenerate. There are Darboux coordinates centered at q_* such that $H_1 = c_0 + \frac{1}{2}ur^2 + o(r^2)$, where $u := \sqrt{\det \text{Hess } H_1(q_*)}$. Then $\nabla H_1 = ur\partial_r + o(r)$ and

$$\frac{\nabla H_1}{|\nabla H_1|^2} = \frac{1}{ur}\partial_r + Y \quad \text{where } \lim_{r \rightarrow 0} r|Y| = 0.$$

We compute

$$\frac{d\tilde{I}}{dc} = \int_{L_c} \iota_{(\nabla H_1/|\nabla H_1|^2)}\omega = \int_{L_c} (\iota_{(1/ur)\partial_r}\omega + \iota_Y\omega) = \int_{L_c} \left(\frac{d\theta}{u} + \iota_Y\omega \right) = \frac{2\pi}{u} + \int_{L_c} \iota_Y\omega.$$

If ds is the arc length of L_c , then there exists $C_1 > 0$ such that $\iota_Y\omega|_{L_c} \leq C_1 ds$. From the expansion of H_1 at q_* , we also see that there exists $C_2 > 0$ such that $\text{length}(L_c) \leq C_2 \max_{L_c} r$. Thus,

$$\begin{aligned} \lim_{c \rightarrow c_0} \left| \int_{L_c} \iota_Y\omega \right| &\leq \lim_{c \rightarrow c_0} \max_{L_c} |Y| C_1 \text{length}(L_c) \leq \lim_{c \rightarrow c_0} \max_{L_c} |Y| C_1 C_2 \max_{L_c} r \\ &= C_1 C_2 \lim_{r \rightarrow 0} r|Y| = 0 \end{aligned}$$

and the first limit in (4.6) follows. For the second limit, we notice that in coordinates where $H_1 = c_0 + \frac{1}{2}r^2$, we have $\eta = \rho d\theta$. For $c > c_0$, we define r_c via the equation $c = c_0 + \frac{1}{2}r_c^2$. Thanks to Stokes' theorem, we have

$$\frac{d^2\tilde{I}}{dc^2} = \int_{L_c} \frac{1}{r} \iota_{\partial_r} d\eta = \int_0^{2\pi} \frac{\partial_r \rho(r_c, \theta)}{r_c} d\theta.$$

Since $\partial_r \rho(0, \theta) = d\rho \cdot e^{i\theta}$, we have $\int_0^{2\pi} \partial_r \rho(0, \theta) d\theta = 0$ and hence

$$\begin{aligned} \lim_{c \rightarrow c_0} \frac{d^2\tilde{I}}{dc^2} &= \lim_{c \rightarrow c_0} \int_0^{2\pi} \frac{\partial_r \rho(r_c, \theta) - \partial_r \rho(0, \theta)}{r_c} d\theta \\ &= \int_0^{2\pi} \partial_{rr}^2 \rho(0, \theta) d\theta \\ &= \int_0^{2\pi} \text{Hess } \rho(q_*)[e^{i\theta}, e^{i\theta}] d\theta \\ &= \pi \text{Trace}(\text{Hess } \rho(q_*)) \\ &= \pi \Delta \rho(q_*). \end{aligned} \quad \square$$

From the lemma above we deduce immediately the following result, which, together with Remark 4.4, yields Theorem 1.7.

LEMMA 4.8. *Suppose that $M = S^2$ and that H_1 has exactly one non-degenerate minimum at $q_{\min} \in S^2$ and one non-degenerate maximum at $q_{\max} \in S^2$ so that there exists a family of embedded circles $c \mapsto L_c, c \in (H_1(q_{\min}), H_1(q_{\max}))$. Assume that*

$$\sqrt{\det \text{Hess } \zeta(q_{\min})} \neq \sqrt{\det \text{Hess } \zeta(q_{\max})},$$

where the Hessian is taken in Darboux coordinates for μ . Then

$$\frac{d\tilde{I}}{dc} : (H_1(q_{\min}), H_1(q_{\max})) \rightarrow \mathbb{R}$$

is not constant. □

Proof of Corollary 1.5. Thanks to Remark 4.5, a circle L is non-resonant for $H_1 := -\frac{1}{2}b_1^{-1}$ with respect to $\omega = b_1\mu$ if and only if it is non-resonant for $\zeta = b_1^{-2}$ with respect to μ . Let U be an arbitrary neighborhood of q_* , which we can suppose to be diffeomorphic to a ball. We apply Theorem 1.1 to U and a neighborhood U' of q_* such that $\bar{U}' \subset U$. We obtain a diffeomorphism Ψ_ϵ and a function $H_{\epsilon,\theta} = h_0(\epsilon) + \epsilon^2 H_1 + o(\epsilon^2)$ with $h_0(\epsilon) = 0$ for every $\epsilon \in [0, \epsilon_0]$. By Lemma 4.7, there is a non-resonant circle L for H_1 inside U' . Using the Moser twist theorem (Theorem 4.1) on L , we can find, by choosing a smaller ϵ_0 , a neighborhood U'' of q_* with the property that, for all $\epsilon \in [0, \epsilon_0]$, every flow line of Φ_{H_ϵ} starting in SU'' stays in SU' forever. After shrinking ϵ_0 again, we have $\pi(\Psi_\epsilon(SU')) \subset U$ and there is a neighborhood U''' of q_* such that $SU''' \subset \Psi_\epsilon(SU'')$. We deduce that any $(g, \epsilon^{-1}b_1)$ -geodesic passing through $U_1 := U'''$ will stay in U forever. We have thus showed Corollary 1.5 for $\zeta = b_1^{-1}$. The proof for $\zeta = K$ when b_1 is a positive constant is completely analogous. □

The last criterion deals with a saddle point of H_1 and together with Remark 4.4 yields Theorem 1.6.

LEMMA 4.9. *Suppose that H_1 has a non-degenerate saddle critical point q_* such that there are no critical values of H_1 in the interval (c_0, c_1) for $c_0 := H_1(q_*)$ and some $c_1 > c_0$. Then there exists a family of circles $c \mapsto L_c, c \in (c_0, c_1)$, such that*

$$\lim_{c \rightarrow c_0} \text{dist}(q_*, L_c) = 0, \quad \lim_{c \rightarrow c_0} \frac{d\tilde{I}}{dc} = +\infty.$$

Proof. There is a chart $U' \rightarrow (-\delta, \delta)^2$ centered at q_* such that H_1 has the form $H_1 = c_0 + xy$ in the corresponding coordinates. Since there are no critical values in (c_0, c_1) , there is a smooth family of circles $c \mapsto L_c, c \in (c_0, c_1)$, with the property that

$$H_1(L_c) = c \quad \text{and} \quad L_c \cap U' = \{(c - c_0)\delta^{-1} < x < \delta, y = (c - c_0)x^{-1}\}.$$

We compute

$$\begin{aligned} \frac{d\tilde{I}}{dc} &\geq \delta \int_{L_c} |\nabla H_1|^{-1} ds \\ &\geq \delta \int_{L_c \cap U'} |\nabla H_1|^{-1} ds \end{aligned}$$

$$\begin{aligned}
 &= \delta \int_{(c-c_0)\delta^{-1}}^{\delta} r^{-1} \sqrt{1 + \frac{(c-c_0)^2}{x^4}} dx \\
 &\geq \delta \int_{(c-c_0)^{1/2}}^{\delta} r^{-1} dx \\
 &\geq \delta \int_{(c-c_0)^{1/2}}^{\delta} \frac{1}{2x} dx,
 \end{aligned}$$

where in the last inequality we used that $y = (c - c_0)x^{-1} \leq x$ if $x \geq \sqrt{c - c_0}$. It is now easy to see that the last integral diverges for c tending to c_0 . □

5. Zoll-rigidity for strong magnetic fields

In this section, we give a proof of Theorem 1.9 on the Zoll-rigidity for sets $R \subset (0, \infty)$ accumulating to 0. By definition of geodesic curvature, $(g, \epsilon^{-1}b_1)$ -geodesics and $(g, -\epsilon^{-1}b_1)$ -geodesics are the same curves parametrized in opposite directions. Thus, we can assume that

$$\int_M b_1 \mu \geq 0.$$

Let us first suppose that $b_1 : M \rightarrow \mathbb{R}$ is not constant. Then $\max b_1 > 0$ and we fix positive numbers $\delta_1 < \delta_2 < \delta_3$ in the interval $(\min b_1, \max b_1)$. We define $U_i := \{b_1 > \delta_i\}$ for $i = 1, 2$ and let U_3 be a connected component of $\{b_1 > \delta_3\}$ such that

$$U_3 \cap \{b_1 = \max b_1\} \neq \emptyset.$$

Since $\bar{U}_1 \neq M$, there exists a section W of $\pi : SU_1 \rightarrow U_1$. Thus, we can apply Theorem 1.1 with $U = U_1, U' = U_2$, which means that, up to shrinking ϵ_0 , we can find an isotopy of embeddings $\Psi_\epsilon : SU' \rightarrow SU$ for $\epsilon \in [0, \epsilon_0]$ such that

$$\Psi_\epsilon^* \hat{\omega}_\epsilon = -\pi^*(b_1 \mu) + d(H_\epsilon d\theta),$$

where θ is the angular function of W and $H_\epsilon : SU \rightarrow \mathbb{R}$ satisfies

$$H_\epsilon = \epsilon^2 H'_\epsilon, \quad H'_0 = -(2b_1)^{-1} \circ \pi.$$

Let us remember the definition of the vector fields $X_{H_\epsilon, \theta}$ and X_{H_ϵ} in (1.4) and (1.5) and of the associated flows φ_{H_ϵ} and Φ_{H_ϵ} . We have

$$X_{H_\epsilon, \theta} = \epsilon^2 X_{H'_\epsilon, \theta} \quad \text{for all } \theta \in \mathbb{T}, \quad \varphi_{H_\epsilon}^{\theta_0, \theta_1} = \varphi_{H'_\epsilon}^{\theta_0, \epsilon^2 \theta_1} \quad \text{for all } \theta_0 \in \mathbb{T}, \theta_1 \in \mathbb{R}. \quad (5.1)$$

Since $\bar{U}_3 \subset U_2$, there exists $\ell_0 > 0$ such that if $q : [0, 1] \rightarrow M$ is a curve with $q(0) \in U_3$ and length less than ℓ_0 , then $q([0, 1]) \subset U_2$. Up to shrinking ϵ_0 , there exists a constant $\tau_0 > 0$ such that $\varphi_{H'_\epsilon}^{\theta_0, \tau_1} : U_3 \rightarrow U_2$ is well defined for every $\tau_1 \in [0, \tau_0]$ and every $\epsilon \in [0, \epsilon_0]$. By (5.1), this implies that for every $\epsilon \in [0, \epsilon_0]$, the flow $\varphi_{H_\epsilon}^{\theta_0, \theta_1} : U_3 \rightarrow U_2$ is well defined for all $\theta_1 \in [0, \epsilon^{-2} \tau_0]$ and, hence, the flow $\Phi_{H_\epsilon}^{\theta_1} : SU_3 \rightarrow SU_2$ is also well defined for $\theta_1 \in [0, \epsilon^{-2} \tau_0]$. Up to shrinking ϵ_0 , we assume that

$$2\pi < \epsilon_0^{-2} \tau_0,$$

so that the set

$$Q_\epsilon := \{q \in U_3 \mid \varphi_{H_\epsilon}^{0,2\pi}(q) = q\}$$

is well defined.

The set Q_ϵ plays a central role in the proof of Theorem 1.9 and now we want to study it better. To this purpose, let us consider the following construction. Let $z : [0, T] \rightarrow SU_1$ be a curve. We define $\tilde{\theta}_z : [0, T] \rightarrow \mathbb{R}$ to be any real lift of the function $\theta \circ z : [0, T] \rightarrow \mathbb{T}$. We define

$$\Delta\theta_z := \tilde{\theta}_z(T) - \tilde{\theta}_z(0) = \int_0^T d\theta\left(\frac{d}{dt}z\right) dt.$$

In particular, if $z(T) = z(0)$, we see that $\Delta\theta_z = 2k\pi$ for some integer k . Let us specialize this construction further and consider for any $v \in SU_1$ the $(g, \epsilon^{-1}b_1)$ -geodesic $\gamma_v : \mathbb{R} \rightarrow M$ with $\dot{\gamma}(0) = v$. We define the set

$$S := \{(v, \ell) \in SU_1 \times (0, \infty) \mid \gamma_v([0, \ell]) \subset U_1\}.$$

Then the function

$$\Delta\theta : S \rightarrow \mathbb{R}, \quad \Delta\theta(v, \ell) := \Delta\theta_{\dot{\gamma}_v|_{[0,\ell]}} \tag{5.2}$$

is well defined and continuous since the function $d\theta((d/dt)\dot{\gamma}_v)$ is continuous in v .

LEMMA 5.1. *A point $q \in U_3$ belongs to Q_ϵ if and only if the $(g, \epsilon^{-1}b_1)$ -geodesic with initial velocity vector $\Psi_\epsilon(W(q))$ has some period ℓ , it is contained in $\Psi_\epsilon(SU_2)$ and $\Delta\theta(\Psi_\epsilon(W(q)), \ell) = 2\pi$.*

Proof. Let us take $q \in U_3$ and consider $\eta_{W(q)}$ to be the maximal solution of Φ_{H_ϵ} with $\eta_{W(q)}(0) = W(q)$ and $\gamma_{\Psi_\epsilon(W(q))}$ to be the $(g, \epsilon^{-1}b_1)$ -geodesic with $\dot{\gamma}_{\Psi_\epsilon(W(q))}(0) = \Psi_\epsilon(W(q))$, which are related by equation (1.7) and the time reparametrization $t : [0, \theta_\infty) \rightarrow [0, \ell_\infty)$. Let $\ell \in [0, \ell_\infty)$ and $\theta \in [0, \theta_\infty)$ be such that $t(\theta) = \ell$. Then $\dot{\gamma}(\ell) = \dot{\gamma}(0)$ if and only if $\eta_{W(q)}(\theta) = \eta_{W(q)}(0)$. In this case, $\Delta\theta(\Psi_\epsilon(W(q)), \ell) = 2k\pi$ for some $k \in \mathbb{Z}$. By (1.7), it follows that

$$2k\pi = \Delta\theta(\Psi_\epsilon(W(q)), \ell) = \Delta\theta_{\Psi_\epsilon(\eta_{W(q)})|_{[0,\theta]}} = \Delta\theta_{\eta_{W(q)}|_{[0,\theta]}} = \theta, \tag{5.3}$$

where we have used the facts that Ψ_ϵ is isotopic to the standard inclusion and that (1.6) holds. It is now immediate to check the equivalence in the statement of the lemma. \square

We now show that Q_ϵ is not empty for ϵ small enough. This argument goes back to [11] and we sketch here how to adapt it to our context.

LEMMA 5.2. *Let b_1 be non-constant and let U_1, U_2, U_3 be chosen as above. Up to shrinking ϵ_0 , the set Q_ϵ is not empty for all $\epsilon \in [0, \epsilon_0]$.*

Proof. Let $\epsilon \in [0, \epsilon_0]$. For every $q \in U_3$, let us define the curve

$$\bar{\eta}_q : [0, 2\pi] \rightarrow U_2, \quad \bar{\eta}_q(\theta) = \varphi_\epsilon^{0,\theta}(q) \quad \text{for all } \theta \in [0, 2\pi].$$

Up to shrinking ϵ_0 , there is a function $u_q : [0, 2\pi] \rightarrow T_q M$ such that $s \mapsto \exp_q(su_q(\theta))$, $s \in [0, 1]$, is the unique minimizing geodesic between q and $\bar{\eta}_q(\theta)$. We define the disc

$$\Gamma_q : [0, 1] \times [0, 2\pi] \rightarrow M, \quad \Gamma_q(s, \theta) := \exp_q(su_q(\theta))$$

for all $(s, \theta) \in [0, 1] \times [0, 2\pi]$.

Let us consider the following functional:

$$\mathbb{A}_\epsilon : U_3 \rightarrow \mathbb{R}, \quad - \int_{D^2} \Gamma_q^*(b_1\mu) + \int_0^{2\pi} H_{\epsilon,\theta}(\bar{\eta}_q(\theta)) d\theta.$$

Since $\bar{\eta}_q$ satisfies $b_1\mu(\dot{\bar{\eta}}_q, \cdot) = -dH_{\epsilon,\theta}$, we obtain

$$d_q \mathbb{A}_\epsilon \cdot \xi = \int_0^1 (b_1\mu)_{\Gamma_q(s,2\pi)}(\tilde{\xi}(s), \partial_s \Gamma_q(s, 2\pi)) ds \quad \text{for all } \xi \in T_q M,$$

where $\tilde{\xi} : [0, 1] \rightarrow TM$ is the unique Jacobi field along the geodesic $s \mapsto \Gamma_q(s, 2\pi)$ such that $\tilde{\xi}(0) = \xi$ and $\tilde{\xi}(1) = d_q\varphi_\epsilon^{0,2\pi}[\xi]$. We claim that $q \in Q_\epsilon$ if and only if $d_q \mathbb{A}_\epsilon = 0$. We know that $q \in Q_\epsilon$ if and only if $s \mapsto \Gamma_q(s, 2\pi) = q$ is constant. Thus, we see that $q \in Q_\epsilon \Rightarrow d_q \mathbb{A}_\epsilon = 0$. If $q \notin Q_\epsilon$, then $\partial_s \Gamma_q(0, 2\pi)$ is a non-zero element of $T_q M$ and we can find $\xi \in T_q M$ such that $b_1\mu(\xi, \partial_s \Gamma_q(0, 2\pi)) > 0$. Since $d\varphi_\epsilon^{0,2\pi}[\xi]$ and ξ are close, we also have that $(b_1\mu)_{\Gamma_q(s,2\pi)}(\tilde{\xi}(s), \partial_s \Gamma_q(s, 2\pi)) > 0$ for all $s \in [0, 1]$. Thus, $d_q \mathbb{A}_\epsilon \cdot \xi > 0$.

We now show that \mathbb{A}_ϵ has a critical point by checking that \mathbb{A}_ϵ has an interior maximum. Indeed, by (5.1), there exists $C > 0$ such that

$$|\dot{\bar{\eta}}_q| \leq C\epsilon^2, \tag{5.4}$$

which also implies that

$$|\text{dist}(q, \bar{\eta}_q(\theta))| \leq 2\pi C\epsilon^2 \quad \text{for all } \theta \in [0, 2\pi]. \tag{5.5}$$

Estimates (5.4) and (5.5) together yield $|d\Gamma_q| \leq C'\epsilon^2$ for some constant $C' > 0$, so that

$$- \int_{D^2} \Gamma_q^*(b_1\mu) = o(\epsilon^2).$$

On the other hand, using again (5.5), we see that

$$\int_0^{2\pi} H_{\epsilon,\theta}(\bar{\eta}_q(\theta))d\theta = \int_0^{2\pi} H_{0,\theta}(q)d\theta + o(\epsilon^2) = -\epsilon^2 2\pi(2b_1)^{-1}(q) + o(\epsilon^2).$$

Putting the two estimates together, we get

$$\mathbb{A}_\epsilon = \epsilon^2[-\pi b_1^{-1} + o(1)]. \tag{5.6}$$

By definition of U_3 , the function $-\pi b_1^{-1}$ attains the maximum $-\pi(\max b_1)^{-1}$ in the interior. By (5.6), the same is true for \mathbb{A}_ϵ . □

LEMMA 5.3. *Let b_1 be non-constant and let U_1, U_2, U_3 be chosen as above. Up to shrinking ϵ_0 , the set $U_3 \setminus Q_\epsilon$ is not empty for all $\epsilon \in (0, \epsilon_0]$.*

Proof. Let δ_4 be a regular value of b_1 in the interval $(\delta_3, \max b_1)$. By the definition of U_3 , there exists $q \in U_3$ with $b_1(q) = \delta_4$. Then we can find $\tau_1 > 0$ such that the curve

$$q'_0 : [0, \tau_1] \rightarrow U_3, \quad q'_0(\tau) := \varphi_{H'_0}^{0,\tau}(q)$$

is an embedding since $H'_0 = -b_1^{-1} \circ \pi$ and therefore q'_0 parametrizes a piece of the regular level set $\{b_1 = \delta_4\}$. Up to shrinking ϵ_0 , all curves

$$q'_\epsilon : [0, \tau_0] \rightarrow U_3, \quad q'_\epsilon(\theta) := \varphi_{H'_\epsilon}^{0,\tau}(q),$$

are embedded because $q'_\epsilon \rightarrow q'_0$ in the C^1 -topology as $\epsilon \rightarrow 0$. By (5.1), we have

$$q'_\epsilon(\epsilon^2\theta) = \varphi_{H'_\epsilon}^{0,\theta}(q),$$

so that $\theta \mapsto \varphi_{H'_\epsilon}^{0,\theta}(q)$ is an embedding for $\theta \in [0, \epsilon^{-2}\tau_1]$. Then $q \notin Q_\epsilon$ if $\epsilon \in (0, \epsilon_0]$ and we shrink ϵ_0 so that $2\pi \leq \epsilon_0^{-2}\tau_1$. □

LEMMA 5.4. *Let b_1 be non-constant and let U_1, U_2, U_3 be chosen as above. Up to shrinking ϵ_0 , for all $\epsilon \in (0, \epsilon_0]$ the following property holds: if the system $(g, \epsilon^{-1}b_1)$ is Zoll, then $Q_\epsilon = U_3$.*

Proof. The set Q_ϵ is closed. Moreover, it is not empty by Lemma 5.2. Thus, by the connectedness of U_3 , the lemma follows by showing that Q_ϵ is open whenever $(g, \epsilon^{-1}b_1)$ is Zoll. Since $(g, \epsilon^{-1}b_1)$ is Zoll, the function $\ell : U_3 \rightarrow \mathbb{R}$ associating to each $q \in U_3$ the period of the $(g, \epsilon^{-1}b_1)$ -geodesic $\dot{\gamma}_{\Psi_\epsilon(W(q))}$ with initial velocity $\Psi_\epsilon(W(q))$ is continuous. Let now $q_0 \in Q_\epsilon$. By Lemma 5.1, we have $\dot{\gamma}_{\Psi_\epsilon(W(q_0))}|_{[0,\ell(q_0)]}$ and $\Delta\theta(\Psi_\epsilon(W(q_0)), \ell(q_0)) = 2\pi$. By the continuity of ℓ , we can also find an open neighborhood B of q_0 inside U_3 such that

$$q \in B \Rightarrow \dot{\gamma}_{\Psi_\epsilon(W(q))} \subset \Psi_\epsilon(SU_2) \subset SU_1.$$

Thus, the map $q \mapsto \Delta\theta(\Psi_\epsilon(W(q)), \ell(q))$ is well defined on B , continuous and integer valued. This implies that

$$\Delta\theta(\Psi_\epsilon(W_\epsilon(q)), \ell(q)) = 2\pi \quad \text{for all } q \in B.$$

By Lemma 5.1, this implies that $B \subset Q_\epsilon$. Since $q_0 \in Q_\epsilon$ was arbitrary, we see that Q_ϵ is open. □

Combining Lemma 5.3 and Lemma 5.4, we see that if b_1 is not constant, then $(g, \epsilon^{-1}b_1)$ is not Zoll for any $\epsilon \in (0, \epsilon_0]$. This proves the first part of Theorem 1.9.

Let us now assume that b_1 is a non-zero constant, which we can take to be positive, and suppose that the Gaussian curvature $K : M \rightarrow \mathbb{R}$ of g is not constant. We want to show that also in this case there exists $\epsilon_0 > 0$ such that $(g, \epsilon^{-1}b_1)$ is not Zoll for any $\epsilon \in (0, \epsilon_0]$. The argument is very similar to the one given above for a non-constant function b_1 and we give here only a sketch.

Let us fix $\delta_1 < \delta_2 < \delta_3$ in the non-empty interval $(\min K, \max K)$. We define $U_i := \{K > \delta_i\}$ for $i = 1, 2$ and take U_3 to be a connected component of $\{K > \delta_3\}$ such that $U_3 \cap \{K = \max K\} \neq \emptyset$. We have a section W of $\pi : SU_1 \rightarrow U_1$ and we apply Theorem

1.1 to $U = U_1$ and $U' = U_2$. We get the Hamiltonian function

$$H_\epsilon = -\epsilon^2(2b_1)^{-1} + \epsilon^4 H''_\epsilon, \quad H''_0 = -(2b_1)^{-3} K.$$

Since b_1 is constant, we have

$$X_{H_{\epsilon,\theta}} = \epsilon^4 X_{H''_{\epsilon,\theta}}, \quad \varphi_{H_\epsilon}^{\theta_0,\theta_1} = \varphi_{H''_\epsilon}^{\theta_0,\epsilon^4\theta_1} \quad \text{for all } \theta_0 \in \mathbb{T}, \theta_1 \in \mathbb{R}. \tag{5.7}$$

Now Lemmas 5.1–5.4 hold for this new choice of U_1, U_2, U_3 . More precisely, in Lemma 5.2, we use (5.7) instead of (5.1) to get

$$\mathbb{A}_\epsilon = -\epsilon^2\pi b_1^{-1} + \epsilon^4[-2\pi(2b_1)^{-3}K + o(1)],$$

which has an interior minimum in U_3 because K has an interior maximum in U_3 . In Lemma 5.3, we take δ_4 to be a regular value of K in the interval $(\delta_3, \max K)$ and use again (5.7) instead of (5.1).

This finishes the argument for b_1 constant and K non-constant and proves Theorem 1.9.

6. Rotationally invariant Zoll systems

In this section, we prove that rotationally invariant Zoll magnetic systems with average 1 on the two-torus are Zoll-rigid at unbounded sets not contained in $R_{\mathbb{N}}$ but are Zoll-flexible at any of the unbounded discrete sets $\frac{1}{k}R_*$ for some $k \in \mathbb{N}$. To this end, we require some preliminaries, referring to [4] for the details. As usual, we write $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and consider the cylinder $\mathbb{T} \times \mathbb{R}$ with coordinates $x \in \mathbb{T}$ and $y \in \mathbb{R}$. A rotationally invariant magnetic system (g, b) on $\mathbb{T} \times \mathbb{R}$ is then given by

$$g = dx^2 + a^2(x) dy^2, \quad b = b(x),$$

where $a : \mathbb{T} \rightarrow (0, +\infty)$ and $b : \mathbb{T} \rightarrow \mathbb{R}$ are smooth functions. Taking the quotient by the translation $(x, y) \mapsto (x, y + 2\pi)$, we get a magnetic system on \mathbb{T}^2 which is Zoll at a speed r if and only if the corresponding system on the cylinder is Zoll at the speed r . Thus, it is enough to consider the magnetic system on the cylinder.

Up to rescaling the y -variable $(x, y) \mapsto (x, cy)$ and changing a to a/c , we can assume that a integrates to 2π over \mathbb{T} . Requiring the average of b to be 1 then amounts to having

$$\int_{\mathbb{T}} a(x)b(x) dx = 2\pi.$$

If r is a positive constant, the equations of motion of a $(g, r^{-1}b)$ -geodesic $(x, y) : \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$ whose tangent vector makes an angle $\theta : \mathbb{R} \rightarrow \mathbb{T}$ with ∂_x read

$$\begin{cases} \dot{x} = \cos \theta, \\ \dot{y} = \frac{\sin \theta}{a(x)}, \\ \dot{\theta} = r^{-1}b(x) - \frac{a'(x)}{a(x)} \sin \theta. \end{cases} \tag{6.1}$$

Being invariant under the flow of the vector field ∂_y , the system admits by Noether's theorem an additional first integral. Setting $\mathbb{T}_{2\pi r^{-1}} := \mathbb{R}/2\pi r^{-1}\mathbb{Z}$ and letting $B : \mathbb{T} \rightarrow \mathbb{T}$

be any function with $B'(x) = a(x)b(x)$, the first integral is given by

$$I : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}_{2\pi r^{-1}}, \quad I(x, \theta) = a(x) \sin \theta - r^{-1} B(x). \tag{6.2}$$

Let us now take $g = g_{\text{flat}}$, namely $a \equiv 1$. Then (6.1) and (6.2) reduce to

$$\begin{cases} \dot{x} = \cos \theta, \\ \dot{y} = \sin \theta, \\ \dot{\theta} = r^{-1} b(x), \end{cases} \quad I(x, \theta) = \sin \theta - r^{-1} B(x).$$

Let us suppose that $b > 0$. In this case, the functions $t \mapsto \theta(t)$ and $x \mapsto B(x)$ are strictly monotonically increasing and we can express x and y as functions of θ . In particular, if $I(x, y) = I_0$, then

$$x = B^{-1}(r(\sin \theta - I_0)), \quad \frac{dy}{d\theta} = r \sin \theta \cdot v(r(\sin \theta - I_0)),$$

where

$$v : \mathbb{T} \rightarrow \mathbb{R}, \quad v(u) := \frac{1}{b(B^{-1}(u))}.$$

From this formula, we see that x is 2π -periodic in θ , while there is a shift in the y -coordinate when θ goes from $-\pi/2$ to $\pi/2$ given by the *shift function* $\Delta_r : \mathbb{T}_{2\pi r^{-1}} \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} \Delta_r^b(I_0) &:= \int_{-\pi/2}^{\pi/2} \frac{dy}{d\theta} d\theta = r \int_{-\pi/2}^{\pi/2} \sin \theta \cdot v(r(\sin \theta - I_0)) d\theta \\ &= r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta \cdot v'(r(\sin \theta - I_0)) d\theta, \end{aligned} \tag{6.3}$$

where we used integration by parts. Clearly, $(g, r^{-1}b)$ is Zoll if and only if the shift function Δ_r vanishes identically. In order to check this condition, it is convenient to rewrite Δ_r slightly by making the change of variable $u = \sin \theta$:

$$\Delta_r^b(I_0) = r^2 \int_{-1}^1 \sqrt{1-u^2} \cdot v'(r(u - I_0)) du.$$

The function Δ_r^b has zero average and therefore it vanishes if and only if its Fourier coefficients $\widehat{\Delta}_r^b(k)$ for $k \in \mathbb{Z} \setminus \{0\}$ vanish. We claim that

$$\widehat{\Delta}_r^b(k) = -i\pi r J_1(kr) \cdot \widehat{v}(-k), \tag{6.4}$$

where $J_1 : (0, \infty) \rightarrow \mathbb{R}$ is the first Bessel function

$$J_1(x) = \frac{x}{\pi} \int_{-1}^1 \sqrt{1-u^2} \cdot e^{-ixu} du.$$

Indeed, we have

$$\begin{aligned}
 \widehat{\Delta}_r(k) &= r^2 \int_{\mathbb{T}_{2\pi r^{-1}}} \int_{-1}^1 \sqrt{1-u^2} \cdot v'(r(u-I_0))e^{-ikrI_0} du dI_0 \\
 &= r^2 \int_{-1}^1 \sqrt{1-u^2} \cdot \left(\int_{\mathbb{T}_{2\pi r^{-1}}} v'(r(u-I_0))e^{-ikrI_0} dI_0 \right) du \\
 &= r^2 \int_{-1}^1 \sqrt{1-u^2} \cdot \left(\int_{\mathbb{T}_{2\pi}} v'(s)e^{-ikr(u-r^{-1}s)} ds \right) du \\
 &= r^2 \int_{-1}^1 \sqrt{1-u^2} \cdot e^{-ikru} \cdot \left(\int_{\mathbb{T}_{2\pi}} v'(s)e^{iks} ds \right) du \\
 &= -r^2 ik \int_{-1}^1 \sqrt{1-u^2} \cdot e^{-ikru} \cdot \left(\int_{\mathbb{T}_{2\pi}} v(s)e^{iks} ds \right) du \\
 &= -i\pi r J_1(kr) \cdot \widehat{v}(-k).
 \end{aligned}$$

Let us recall the definition of the sets

$$R_* = \{\xi \in (0, \infty) \mid J_1(\xi) = 0\}, \quad R_{\mathbb{N}} = \bigcup_{k \in \mathbb{N}} \frac{1}{k} R_*.$$

From the properties of J_1 , we know that R_* is discrete, bounded from below and unbounded from above, so that $R_{\mathbb{N}}$ is countable and dense in $(0, \infty)$.

If $r \notin R_{\mathbb{N}}$, then the only real function v satisfying (6.4) for all $k \in \mathbb{Z}$ is the constant function. Therefore,

$$\text{for all } r \in (0, \infty) \setminus R_{\mathbb{N}}, \quad (g_{\text{flat}}, r^{-1}b) \text{ is Zoll} \iff b \text{ is constant.} \tag{6.5}$$

On the other hand, for every $k \in \mathbb{N}$, let us implicitly define $b_k : \mathbb{T} \rightarrow (0, \infty)$ with average 1 by specifying

$$v_k(u) := 1 + \epsilon \sin(ku) \quad \text{for some } \epsilon \in (-1, 1) \setminus \{0\}.$$

Then $\widehat{v}_k(k') = 0$ for $|k'| \neq 0, k$ and using (6.4) we conclude that $\widehat{\Delta}_r^{b_k} = 0$ if and only if $r \in (1/k)R_*$. In other words, we have the equivalence

$$(g_{\text{flat}}, r^{-1}b_k) \text{ is Zoll} \iff r \in \frac{1}{k} R_*. \tag{6.6}$$

We are now in position to prove Theorem 1.11.

Proof of Theorem 1.11. Let R be any subset of $(0, \infty)$ which is unbounded from above and not contained in $R_{\mathbb{N}}$. Let us suppose that (g, b) is a rotationally symmetric magnetic system on \mathbb{T}^2 with average 1 such that $(g, r^{-1}b)$ is Zoll for every $r \in R$. Then [5, Proposition 2.1] implies that $g = g_{\text{flat}}$ is a flat metric. From (6.5) applied to $r \in R \setminus R_{\mathbb{N}}$, we deduce that b is constant. This shows Statement (a). Statement (b) follows by considering the function b_k constructed above and using (6.6). □

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