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ON SEMI-PRIMARY SELF-INJECTIVE RINGS AND FAITH'S QUASI-FROBENIUS CONJECTURE

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Abstract A long-standing conjecture of Faith in ring theory states that a left self-injective semi-primary ring A is necessarily a quasi-Frobenius ring. We propose a new method for approaching this conjecture, and prove it under some mild conditions; we show that if the simple A-modules are at most countably generated over a subring of the centre of A, then the conjecture holds. Also, the conjecture holds for $\mathbb{K}$ algebras A over sufficiently large fields, i.e. if the cardinality of $\mathbb{K}$ is larger than the dimension of the simple A-modules (or of $A/\mathrm{Jac}(A)$). This effectively proves the conjecture in many situations, and we obtain several previously known results on this problem as a consequence.

Keywords: quasi-Frobenius rings; QF rings; Faith conjecture; self-injective ring

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1. Introduction and preliminaries

In classical ring theory, among the rings that are of interest and are intensively studied in the literature are the left and right self-injective rings. Left self-injective rings that are also artinian form another important class of rings called quasi-Frobenius (QF) rings. There are many equivalent definitions of these rings, and they have an intrinsic symmetry: a ring is QF if it is right self-injective and right semi-artinian, or, equivalently, noetherian or artinian on one side and injective on one side. Classical results also include those of Faith and Walker (see [**1**, Chapter 7, § 25]) stating that such rings are characterized by the fact that all right (equivalently, all left) injective (equivalently, projective) modules are projective (injective). These rings are important generalizations of Frobenius algebras, retaining their categorical properties; examples include group algebras of finite groups, Hopf algebras and certain cohomology rings (the cohomology of a compact oriented manifold is a Frobenius algebra). Moreover, such rings are important in many fields of mathematics, from representation theory, category theory and homological algebra to topology and coding theory.

Perhaps one of the most interesting questions regarding QF rings, and also in ring theory in general, is the following, known in the literature as Faith's QF conjecture.

Conjecture 1.1 (Faith). *A left self-injective semi-primary ring is QF.*

We refer the reader to [**1**, **6**, **14**] for general module and ring theory related to Faith's conjecture above.

We recall that a ring A is called semi-primary if the Jacobson radical $\mathrm{Jac}(A)$ of A is nilpotent and the ring A is semi-local, i.e. $A/\mathrm{Jac}(A)$ has finite length (equivalently, it is semi-simple). Much work has been dedicated to this problem over the years [**3**–**5**, **9**, **16**–**21**]; we also refer the reader to the recent survey [**7**], which contains a comprehensive account of the history and known results on QF rings and related topics.

In this paper we present a positive result for this conjecture under some mild restrictions. We show that, for a ring A with the property that its quotient modulo the Jacobson radical $A/\mathrm{Jac}(A)$ is at most countably generated over some subring of the centre of A, the conjecture holds. This covers the situation when A is an algebra over a field $\mathbb{K}$ and $\dim(A/\mathrm{Jac}(A)) \leqslant \aleph_0$, so it also includes, for example, the important situation when $A/\mathrm{Jac}(A)$ is not only semi-simple but also finite dimensional. For algebras over a field $\mathbb{K}$ we show that the conjecture holds as long the cardinality of $\mathbb{K}$ is at least equal to that of the dimension of the A-simple modules (equivalently, at least equal to $\dim(A/\mathrm{Jac}(A))$). The following theorem summarizes the main results of this paper.

Theorem 1.2. *Let R be a commutative ring, and let A be an R-algebra that is semi-primary and left self-injective. The following then hold.*

(i) *If $A/\mathrm{Jac}(A)$ is at most countably generated over R (equivalently, the simple A-modules are at most countably generated over R), then A is QF.*

(ii) *If R is a field, and $|R| > \dim_R(A/\mathrm{Jac}(A))$ (equivalently, the simple A-modules have dimension less than $|R|$), then A is QF.*

In particular, these can be applied if the conditions hold when $R = C(A)$, the centre of A.

As a consequence of our method and results, we also obtain short proofs of two other results of [**13**] and [**15**], stating that Faith's conjecture is true for countable dimensional algebras, and also for rings A for which $|A/J| \leqslant \aleph_0$ or $|A/J| < |A|$.

Our approach is based on the fact that, in general, for a QF ring there is a duality between left and right modules of finite length (finitely generated), and, in good part, the QF property is a reflection of how close $\mathrm{Hom}(\cdot, A)$ is to being a duality between these two categories. For this reason, we prefer a more categorical language rather than working with elements and ideals (although some propositions and proofs can be restated in terms of left and right annihilators of certain sets). Given a left self-injective semi-primary ring A, our method is based on finding a certain left A-module M that has a simple socle S, and M/S is semi-simple and has the largest possible length among such Ms. This module embeds in A if ${}_AA$ is injective, and its dual $\mathrm{Hom}(M, A)$ will be a quotient of A_A,

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allowing us to compare the left and right regular A-modules, and certain invariants for A such as left or right length, and to obtain that they must obey certain restrictions.

For the sake of completeness, we recall a few basic facts, most of which are easy to see and are well known in the literature. Let A be a ring and denote by $J = \mathrm{Jac}(A)$ its Jacobson radical. If A is semi-local, i.e. A/J is semi-simple, then an A-module is semi-simple if and only if it is cancelled by J. Indeed, every simple module is cancelled by J, and if $JM = 0$, then M has an A/J-module structure; since A/J is semi-simple, M is semi-simple as both A/J and the A-module are (the lattices of A-submodules and A/J-submodules of M coincide in this case). The Jacobson radical of an A-module in this case is $\mathrm{Jac}(M) = JM$. If A is semi-primary, and n is such that $J^n = 0 \neq J^{n-1}$, then A is semi-artinian with a Loewy series of length $n-1$, since J^k/J^{k+1} is semi-simple for all k. Write $A = \bigoplus_e Ae$, a sum of indecomposable left A-modules; such a decomposition obviously exists because A/J has finite length, and each Ae is obtained for some indecomposable (primitive) idempotent e (A is semi-perfect in this case). Note that if A is left self-injective, then each indecomposable Ae has a simple socle: indeed, if we have a non-trivial decomposition of the socle $s(Ae) = M \oplus N$, then we can find $E(M)$, $E(N)$ injective hulls of M, N contained in Ae, and we obtain that $Ae = E(M) \oplus E(N)$ is a non-trivial decomposition, which is a contradiction.

Note also that if A is left self-injective semi-primary, for each simple left A-module S, the right A-module $\mathrm{Hom}(S, A)$ is simple. First, note that it is non-zero. For this, we look at the isomorphism types of indecomposable modules Ae; these are projective and local, and are the cover of some simple left A-module. They are isomorphic if and only if their respective 'tops' are isomorphic. The number of isomorphism types of these simple modules that occur at the top of some Ae equals the number t of isomorphism types of simple A-modules. Moreover, since the indecomposable modules Ae are also injective with a simple socle, we see that they are isomorphic if and only if their socle is isomorphic. This shows that the distinct type of isomorphism of simples occurring as a socle of some Ae is also t, and so each simple S must appear as a socle of some Ae (i.e. it embeds in A). This shows that $\mathrm{Hom}(S, A) \neq 0$ for each simple left A-module S. Now, if $f, g \in \mathrm{Hom}(S, A)$, and $f \neq 0$, then $f\colon S \to A$ is a mono, and, since ${}_AA$ is injective, there exists some $h\colon A \to A$ such that $h \circ f = g$. Let c be such that $h(x) = xc$ for all $x \in A$. Then, $f(x)c = g(x)$ shows that $f \cdot c = g$ in $\mathrm{Hom}(S, A)$. This shows that $\mathrm{Hom}(S, A)$ is generated by any $f \neq 0$, so it is simple. In particular, since each simple module embeds in A, which is left injective, it follows that A is an injective cogenerator of the category of left A-modules, i.e. it is a left pseudo-Frobenius (PF) ring. We summarize this in the following proposition, most of which is known (see the references above).

Proposition 1.3. *Let A be a semi-primary and left self-injective ring. The following then hold.*

(i) $\mathrm{Hom}(A, S)$ *is a simple right A-module for each simple left A-module S.*

(ii) *A is a left PF ring.*

It is easy to see that the same conclusions follow in the case when A is semi-local, left semi-artinian and left self-injective.

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2. The main result

Let $\mathcal{S}$ be a set of representatives for the simple left A-modules, let $t = |\mathcal{S}|$ and let $A/J \cong \bigoplus_{S\in\mathcal{S}} S^{n_S}$. Let $\Sigma = s({}_AA)$ be the left socle of A. It is easy to see that this is an A-sub-bimodule of A. Note that, since each indecomposable module Ae has a simple socle, we have that length(Σ) equals the number of terms in the indecomposable decomposition $A = \bigoplus_e Ae$, which equals length$({}_AA/J)$ since each indecomposable Ae is local. Let $\Sigma = \bigoplus_{S\in\mathcal{S}} S^{p_s}$. We then have that $\sum_{S\in\mathcal{S}} p_S = \sum_{S\in\mathcal{S}} n_S$.

Proposition 2.1. *If A is a left self-injective semi-primary ring, then the set $\{\mathrm{Hom}(S, A) \mid S \in \mathcal{S}\}$ is a set of representatives for the simple right A-modules. In particular,* $\mathrm{Hom}(S, A)$, $\mathrm{Hom}(L, A)$ *are non-isomorphic for non-isomorphic $S, L \in \mathcal{S}$.*

Proof. Since A is left injective, the monomorphism $0 \to \Sigma \to A$ gives rise to the epimorphism of right A-modules $\mathrm{Hom}(A, A) \to \mathrm{Hom}(\Sigma, A) \to 0$. Since $\mathrm{Hom}(S, A)$ is simple for each $S \in \mathcal{S}$, we see that $\mathrm{Hom}(\Sigma, A) = \bigoplus_{S\in\mathcal{S}} \mathrm{Hom}(S, A)^{p_S}$ has length equal to

$$\mathrm{length}(\Sigma) = \sum_{S\in\mathcal{S}} p_S = \sum_{S\in\mathcal{S}} n_S = \mathrm{length}({}_AA/J).$$

But, by the classical Wedderburn–Artin theorem, $\mathrm{length}_A(A/J) = \mathrm{length}(A/J)_A$. Since $\mathrm{Hom}(\Sigma, A)$ is semi-simple, the kernel of $A \to \mathrm{Hom}(\Sigma, A)$ contains J, and furthermore, since $\mathrm{length}(A/J) = \mathrm{length}(\mathrm{Hom}(\Sigma, A))$, we obtain $A/J \cong \mathrm{Hom}(\Sigma, A)$ as right A-modules. This shows that all types of isomorphism of right A-modules are found among components of $\mathrm{Hom}(\Sigma, A)$, and this proves the statement. □

We note that the above proof further shows that there is an exact sequence of right A-modules

$$0 \to J \to A \to \mathrm{Hom}(\Sigma, A) \to 0.$$

But it is immediate to see that this means that $\{a \in A \mid \Sigma a = 0\} = J$, i.e. $\mathrm{ann}(\Sigma_A) = J$. In particular, this shows that Σ is also semi-simple as a right A-module, i.e. the left socle of A is contained in the right socle. It is not hard to see that they are equal. In fact, it is known that the left and right socles of a left PF-ring coincide [**12**, Theorem 6], and if A is semi-primary with the same left and right socle, then it is easy to show that the left and right Loewy series of A coincide [**2**, Proposition 2.1] (see also [**13**, Lemma 3.7]). We denote these terms of the Loewy series by Σ_k, with $\Sigma_1 = \Sigma$ being the socle of A.

For a left A-module M, we define, for short, $M^* = \mathrm{Hom}(M, A)$; this is a right A-module. The following proposition is key to our application.

Proposition 2.2. *Let A be a semi-primary left self-injective ring and let M be a left A-module such that there exists an exact sequence $0 \to S \to M \to L^{(\alpha)} \to 0$, with S, L simple modules, and assume that $S = s(M)$ is the socle of M ($L^{(\alpha)}$ denotes the coproduct of α copies of L). Then M^* is a local right module with unique maximal submodule $S^\perp = \{f \in \mathrm{Hom}(M, A) \mid f|_S = 0\}$ that is semi-simple isomorphic to the direct product $(L^*)^\alpha = \prod_\alpha L^*$ of α copies of L^*.*

Proof. We have an exact sequence $0 \to (L^*)^\alpha \to M^* \to S^* \to 0$; it is easy to see that the kernel of the morphism $M^* = \mathrm{Hom}(M, A) \to S^* = \mathrm{Hom}(S, A)$ is $S^\perp$. Hence, $S^\perp \cong (L^*)^\alpha$, which is right semi-simple since it is cancelled by J. Now, since M has a simple socle, and its socle embeds in A, which is injective, it follows that M embeds in A. We now note that M^* is generated by any $f \notin S^\perp$, which obviously shows that M^* is a local module with unique maximal submodule $S^\perp$. Indeed, such an f must be a monomorphism because S is essential in M, and given any other $h\colon M \to A$, by the injectivity of ${}_AA$ there exists $g \in \mathrm{Hom}(A, A)$ such that $g \circ f = h$. If $c \in A$ is such that $g(x) = xc$ for all $x \in A$, then we have $h = f \cdot c$ in M^*. Hence, $f \cdot A = M^*$. □

Note that the fact that M^* is local can also be proved by embedding M in some indecomposable Ae for an indecomposable idempotent e, and then, by applying the exact functor $\mathrm{Hom}(\cdot, A)$, one obtains an epimorphism $\mathrm{Hom}(Ae, A) = eA \to M^*$, and so M^* is local because eA is.

Assume that A is left self-injective and semi-primary. Let α be the largest cardinality for which there is a left module M with a simple socle and such that $M/s(M) \cong L^\alpha$ for some simple module L. Such a cardinality obviously exists, since any such module embeds in A because ${}_AA$ is injective, and there are only finitely many simple types of simple A-modules. Note that if Σ_2 is the second socle of A, then $\alpha \leqslant \mathrm{length}(\Sigma_2/\Sigma)$. We note that, if α is infinite, this is an equality. Indeed, if, for each simple module S, L, we denote by $\alpha_{S,L} = [(E(S)/S){:}L]$ the multiplicity of L in the second socle of the injective hull $E(S)$ of S, then $\alpha = \max_{S,L \in \mathcal{S}} \alpha_{S,L}$. Therefore,

$$\alpha \leqslant \sum_{S,L \in \mathcal{S}} \alpha_{S,L} \leqslant t^2\alpha = \alpha$$

if α is infinite (since $t = |\mathcal{S}|$ is finite). We note also that if Σ_k denotes the kth socle of A, then $\mathrm{length}(\Sigma_k/\Sigma_{k-1}) \leqslant \alpha$. This follows by induction on k: if this is true for k, then there exists an embedding $\Sigma_k/\Sigma_{k-1} \hookrightarrow A^{(\alpha)}$ (since ${}_AA$ is an injective cogenerator), and therefore we have $\mathrm{length}(\Sigma_{k+1}/\Sigma_k) \leqslant \mathrm{length}(\Sigma_1/\Sigma_0)^{(\alpha)} = \alpha \times \alpha = \alpha$, since α is an infinite cardinal. Thus, we may think of α as the length of ${}_AA$ when A is not left artinian; it is either finite if A is left artinian, or if it is infinite, it equals the length of the semi-simple module Σ_2/Σ_1. Note that, in order to show that A is QF, it is enough to show that α is finite.

We now show that if α is infinite, it also equals the length of J/J^2. We adopt the following convention, if no other reference is used: if λ is a cardinal, we write $\beta \leqslant \lambda$ either if λ is infinite and β is less or equal to λ, or if λ and β are both finite. Thus, we do not distinguish between finite cardinals; this is natural since, to prove that A is QF given that A is semi-primary and left self-injective, it is enough to show that A has finite length as a left or right module. The above statement about J/J^2 is a consequence of the following lemma.

Lemma 2.3. *Let A be a semi-local ring, let M be an A-module, let N be an A-submodule of M such that M/N is semi-simple, and let λ be an infinite cardinal.*

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(i) *If* length$(M/N) \leqslant \lambda$ *and* length$(N/JN) \leqslant \lambda$, *then we have* length$(M/JM) \leqslant \lambda$ *and* length$(JM/JN) \leqslant \lambda$.

(ii) *Assume that M has finite Loewy length and* length$(M/N) \leqslant \lambda$. *Then, if the lengths of consecutive quotients in the Jacobson filtration of N are at most λ (i.e.* length$(J^kN/J^{k+1}N) \leqslant \lambda$ *for all k), the same holds for the terms of M.*

Proof. (i) Since M/N is semi-simple, we have that $JM \subseteq N$, and so $JN \subseteq JM \subseteq N \subseteq M$ and N/JN and M/JM are semi-simple. Thus, N/JM is semi-simple and length$(N/JM) \leqslant$ length$(N/JN) \leqslant \lambda$ and length$(JM/JN) \leqslant$ length$(N/JN) \leqslant \lambda$, so length$(M/JM) \leqslant$ length$(M/N) +$ length$(N/JM) \leqslant \lambda + \lambda = \lambda$.

(ii) This follows by induction on the Loewy length of M. The initial step is obvious: for the induction part note that the Loewy length of JM is one less than that of M, and length$(JM/JN) \leqslant \lambda$, so we can use the induction hypothesis for the pair $JN \subseteq JM$, and get that the quotients in the Jacobson filtration of JM have length $\leqslant \lambda$; since length$(M/JM) \leqslant \lambda$ by (i), we get the desired statement. □

We now note that we can apply the previous proposition inductively to get that the terms of the Jacobson filtration of Σ_k all have length less than α in the case when α is infinite, or have finite length otherwise. For $\Sigma_n = A$ we get length$(J/J^2) \leqslant \alpha$. The fact that length$(J/J^2) \geqslant \alpha$ can be obtained by the dual argument; however, in what follows, we only need that length$(J/J^2) \leqslant \alpha$. In agreement with the above convention for cardinals, for an arbitrary module M over a ring R, we say that M is λ-generated if it has a system of generators of cardinality λ if λ is infinite, or that it is finitely generated if λ is finite. We can now give the following theorem.

Theorem 2.4. *Let R be a commutative ring, and let A be an R-algebra. Assume that A/J is λ-generated as an R-module. If A is left self-injective semi-primary, then α is finite or $\alpha < \lambda$. In particular, if A/J is countably generated over A, then A is QF.*

Proof. Assume that $\alpha \geqslant \lambda$, so we may assume that α is infinite. Since A/J is a finite direct sum of simple (left or right) modules, we get that each simple left module and each simple right module is λ-generated. We have seen that length$(J/J^2) \leqslant \alpha$, so J/J^2 is generated by $\lambda \times \alpha = \max\{\alpha, \lambda\} = \alpha$ elements, i.e. it is α-generated. Consider a left module $M \subset A$ as in Proposition 2.2; thus, $s(M) = S$ and $M/s(M) = L^{(\alpha)}$, with S, L simple left A-modules. Its 'dual' M^* is a quotient of A, and it is thus not difficult to see that the module $(L^*)^\alpha$ is a direct summand of J/J^2 (since J/J^2 is a semi-simple right A-module). Therefore, $(L^*)^\alpha$ is α-generated. We have an isomorphism of right A-modules, A/J-modules and R-modules $(L^*)^\alpha \cong (\Delta_L)^\alpha$, where $\Delta_L = \mathrm{End}_A(L)$; Δ_L is a division algebra, since L is a simple A-module. But, for any division ring D, the vector space D^β has dimension at least 2^β if β is infinite (see, for example, [**11**, Chapter IX] for a more precise statement; this also follows from the Fichtenholz–Kantorovich–Hausdorff theorem on the existence of 2^X independent subsets in the boolean algebra of the subsets of a set X, or of independent elements in more general boolean algebras; see, for example, [**8**]). This shows that the right R-module $(L^*)^{(2^\alpha)}$ is a direct summand in $(L^*)^\alpha$, so $(L^*)^{(2^\alpha)}$

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is α-generated. This leads to a contradiction, since each of the α generators has only finitely many non-zero components corresponding to summands of $(L^*)^{(2^\alpha)}$, and thus can generate only a part of $(L^*)^{(2^\alpha)}$ of A-length α.

The last statement follows since if $\lambda = \aleph_0$, then α is finite; in that case, A has left finite length, so it is left artinian, and so is QF. □

There is an analogue and a strengthening of this for an algebra over a field. If A is a $\mathbb{K}$-algebra, let $d = \dim_{\mathbb{K}}(A/J)$. For each simple left module S, let $\Delta_S = \mathrm{End}({}_A S)$ and let $d_S = \dim_{\mathbb{K}}(S)$, so $d = \sum_S d_S \cdot n_S^2$; it is also standard to see that $\mathrm{End}(S^*) = \Delta_S^{\mathrm{op}}$ for each simple left module S (because A/J is semi-simple).

Let $|X|$ denote the cardinality of a set X. We can now prove the following theorem.

Theorem 2.5. *Let A be a left self-injective semi-primary algebra over a field $\mathbb{K}$. Assume that A is infinite dimensional. Then, either A is left artinian (so it is QF), or the following hold:*

(i) $|K| \leqslant \dim_{\mathbb{K}}(A/J)$,

(ii) $\alpha < \dim_{\mathbb{K}}(A/J)$.

Proof. By the above considerations, for each simple left module S, L there exists a module $M \subset A$ such that $s(M) = S$ and $M/s(M) = L^{(\alpha_{S,L})}$ (note that $\alpha_{S,L}$ can be 0, which is the case precisely when $\mathrm{Ext}^1_A(L, S) = 0$). By Proposition 2.2, $M^* = \mathrm{Hom}(M, A)$ is a quotient of A and $(L^*)^{\alpha_{S,L}}$ is contained in M^*. Thus, we have that

$$\dim_{\mathbb{K}}(A) \geqslant \dim_{\mathbb{K}}(M^*) \geqslant \dim_{\mathbb{K}}(L^*)^{\alpha_{S,L}} = \dim_{\mathbb{K}}(\Delta_L^{\mathrm{op}})^{\alpha_{S,L}}.$$

But it is known that $\dim_\Delta(\Delta^\beta) = |\Delta|^\beta$ for any division algebra Δ and infinite cardinality β (see, for example, [**11**, Chapter IX]), so, since $|\Delta_L^{\mathrm{op}}| = |\Delta_L| = |\mathbb{K}| \times d_L$, we get $\dim(A) \geqslant (d_L \times |\mathbb{K}|)^{\alpha_{S,L}}$ whenever $\alpha_{S,L}$ is infinite. We see that $\dim_{\mathbb{K}}(\Sigma_2/\Sigma) \geqslant \dim_{\mathbb{K}}(\Sigma_k/\Sigma_{k-1})$ by induction on k as above; indeed, first note that $\dim_{\mathbb{K}}(\Sigma_2/\Sigma) \geqslant \alpha$, and, since there exists an embedding $\Sigma_k/\Sigma_{k-1} \hookrightarrow A^{(\alpha)}$, we get an embedding $A/\Sigma_{k-1} \hookrightarrow A^{(\alpha)}$. This implies that $\Sigma_{k+1}/\Sigma_k \hookrightarrow (\Sigma_2/\Sigma)^{(\alpha)}$, so

$$\dim(\Sigma_{k+1}/\Sigma_k) \leqslant \dim(\Sigma_2/\Sigma) \times \alpha \leqslant \dim((\Sigma_2/\Sigma)) \times \dim(\Sigma_2/\Sigma) = \dim(\Sigma_2/\Sigma),$$

because $\dim(\Sigma_2/\Sigma)$ is infinite when $\dim(A)$ is infinite. Since A is infinite dimensional and $\dim_{\mathbb{K}}(\Sigma_2/\Sigma) \geqslant \dim_{\mathbb{K}}(\Sigma_k/\Sigma_{k-1})$ for all k, we get that $\dim_{\mathbb{K}}(A) = \dim(\Sigma_2/\Sigma)$.

We have that $\Sigma_2/\Sigma = \bigoplus_{S,L} L^{(p_S \times \alpha_{S,L})}$; thus, $\dim_{\mathbb{K}}(A) = \sum_{S,L \in \mathcal{S}} d_L \times \alpha_{L,S} \times p_S = \max\{d_S, \alpha_{S,L} \mid S, L \in \mathcal{S}\}$ if A is infinite dimensional (p_S are finite numbers). Thus, $\dim_{\mathbb{K}}(A) = \max\{d, \alpha\}$.

Hence, we get

$$\max\{d, \alpha\} \geqslant (d_L \times |\mathbb{K}|)^{\alpha_{S,L}}$$

for each S, L for which $\alpha_{S,L}$ is infinite. In particular, this shows that $\max\{d, \alpha\} \geqslant |\mathbb{K}|^{(\alpha_{S,L})}$ for all $S, L \in \mathcal{S}$ for which $\alpha_{S,L}$ is infinite. Therefore, if we assume A is not left artinian, then α is infinite, and we get $\max\{d, \alpha\} \geqslant |\mathbb{K}|^\alpha$. Since $\alpha < |\mathbb{K}|^\alpha$, this shows that

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$d = \max\{d, \alpha\} = |\mathbb{K}|^\alpha$. Thus, $|\mathbb{K}| \leqslant d$ and $\alpha < d$ (since $\alpha < 2^\alpha$). In particular, we see that d is infinite (otherwise both d and α would be finite and we would get $\dim_{\mathbb{K}}(A) < \infty$), so $d = \dim_{\mathbb{K}}(A/J)$ in this case; this ends the proof. □

Corollary 2.6. *Let A be a semi-primary left self-injective $\mathbb{K}$-algebra. If either $|\mathbb{K}| > \dim_{\mathbb{K}}(A/\operatorname{Jac}(A))$ or $\dim_{\mathbb{K}}(A/\operatorname{Jac}(A))$ is at most countable, then A is QF.*

This finishes the proof of Theorem 1.2 on Faith's conjecture.

We note several other results in the literature that can be obtained as corollaries from here. The following can be obtained as a corollary of a result in [**15**], stating that a self-injective algebra of at most countable dimension is QF; nevertheless, the proofs of [**15**] use some further assumptions on A, such that the cardinality of A is regular, and also make use of the generalized continuum hypothesis (see also the erratum of [**15**]). It is, however, also a direct consequence of our results, as follows.

Corollary 2.7. *A semi-primary left self-injective algebra of countable dimension is QF.*

We note another application of the above method.

Corollary 2.8. *If A is a left self-injective semi-primary ring of infinite cardinality, then $|A/J| = |A|$.*

Proof. Let c be the largest cardinality of a simple left A-module; we obviously have that $c = |A/J| \leqslant |A|$. As before, using Proposition 2.2 we find the right module M^* with the socle $(L^*)^\alpha$ (where L is a left simple A-module) has cardinality at least 2^α, and is a quotient of A, so $|A| \geqslant 2^\alpha$. On the other hand, the cardinality of the modules Σ_k/Σ_{k-1} is less than $c \times \alpha = \max\{c, \alpha\}$. Since $\Sigma_n = A$ for some n, this shows that $|A| = \max\{c, \alpha\}$. If α is finite, obviously $|A| = c = |A/J|$; otherwise, since $\max\{c, \alpha\} \geqslant 2^\alpha$, and $\alpha < 2^\alpha$, we must have $c > \alpha$. In particular, $|A| = c = |A/J|$. □

The following is known from [**13**, Corollary 3.10]. We note that it also follows shortly as a consequence of our results.

Corollary 2.9.

(1) *A left self-injective semi-primary ring A with $|A/J| \leqslant \aleph_0$ is QF.*

(2) *A left self-injective semi-primary ring A with $|A/J| < |A|$ is QF.*

Proof. (1) follows by Theorem 1.2 (i) since A is at most countably generated over the prime subring of A (i.e the subring generated by $1 \in A$); (2) follows easily from the previous corollary. □

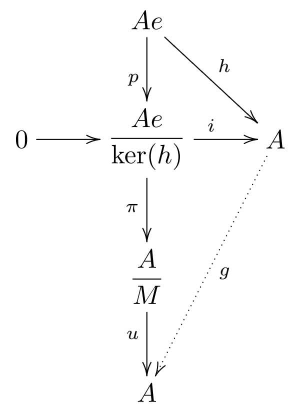
We now note another fact that is relevant in the context of Faith's QF conjecture; it shows that the duality exists at least at the level of simple modules (this is Proposition 2.6 from the preliminary preprint version [**10**]).

Proposition 2.10. *Let A be a left self-injective semi-primary ring. The following then hold.*

(i) *Each right A-module eA has a simple socle. Consequently, the right socle of A (which coincides with Σ) is finitely generated, and has the same left and right lengths.*

(ii) $\mathrm{Hom}(T, A)$ *is a simple left A-module for each right simple A-module T.*

Proof. (i) We have that $eA = \mathrm{Hom}(Ae, A)$. Let M be the unique maximal submodule of Ae. We show that $M^{\perp} = \{f : Ae \to A \mid f|_M = 0\} \subset eA$ is essential in eA. Let $0 \neq h : Ae \to A$. Then, $\ker(h) \neq Ae$, so $\ker(h) \subseteq M$, and thus we have the following commutative diagram:



Here, p and π are the canonical projections, $h = i \circ p$ is the canonical decomposition and u is a non-zero morphism from A/M to A, which exists since we know that all isomorphism types of simple modules embed in A. Since A is injective, the above diagram is completed commutatively by $g : A \to A$, $g(x) = xc$ for $x \in A$. Let $f = u \circ \pi \circ p$; then, obviously, $f \neq 0$, $f \in M^{\perp}$ and $g \circ h = f$, i.e. $h \cdot c = f$ in $\mathrm{Hom}(Ae, A)$. This shows that $M^{\perp} \cap hA \neq 0$ whenever $h \neq 0$, which in turn shows that $M^{\perp}$ is essential in eA. Moreover, it is easy to see that $M^{\perp} \cong (Ae/M)^*$ by dualizing the exact sequence $0 \to M \to Ae \to Ae/M \to 0$, so $M^{\perp}$ is simple. Thus, $eA = \mathrm{Hom}(Ae, A)$ has a simple (essential) socle.

(ii) We have already noted that each simple right A-module T is of the form $\mathrm{Hom}(S, A)$ for a simple left A-module. But, since there exists an epimorphism $A \to S$, by duality we get a monomorphism of right A-modules $0 \to T = \mathrm{Hom}(S, A) \to A$. This shows that $\mathrm{Hom}(T, A) \neq 0$. If $f \in \mathrm{Hom}(T, A)$, then $\mathrm{Im}(f)$ is contained in the right socle of A, which is Σ, so $J \cdot \mathrm{Im}(f) = 0$; therefore, for $a \in J$, $x \in T$, $(a \cdot f)(x) = a \cdot f(x) = 0$. Thus, $J \cdot \mathrm{Hom}(T, A) = 0$, so $\mathrm{Hom}(T, A)$ is a semi-simple left A-module. Moreover, there exist an indecomposable projective right module eA and an epimorphism $eA \to T \to 0$ that yields the sequence $0 \to \mathrm{Hom}(T, A) \to Ae$. But, since e is indecomposable, Ae is indecomposable and has a simple socle, and, as $\mathrm{Hom}(T, A)$ is semi-simple and non-zero, it follows that $\mathrm{Hom}(T, A)$ is the socle of Ae and is simple. □

We note that a possible procedure for proving this conjecture for other cases is the following. For a semi-primary left self-injective ring A, consider the Loewy series $0 \subset \Sigma_1 \subset \cdots \subset \Sigma_k \subset \cdots \Sigma_n = A$ of A; this is the same to the left and to the right. The first term has the same left and right length, as shown before. If this is true for all the factors in the Loewy series, that is, if the length of Σ_k/Σ_{k-1} is the same as left and right modules, one could apply the above procedure of Proposition 2.2 and Theorem 2.2 to obtain a positive answer to Faith's conjecture (in fact it is enough to show that Σ_1/Σ_0 has the same left and right length). Specifically, let M be a module like in Proposition 2.2 of maximal (infinite) length α modulo its socle; as before, one then sees that M^*/M^*J has A-length greater than α (it is semi-simple of length at least 2^α). This is again a contradiction to the fact that the left and right lengths of $\Sigma_{n-1}/\Sigma_{n-2}$ coincide. On the other hand, this also shows that if a counter-example to this conjecture exists, then some Σ_k/Σ_{k-1} has different left and right lengths. We thus note that the problem is essentially about comparing the left and right lengths of some bimodules over certain division algebras D_1, D_2, which in general is a difficult problem. We point out that, in general, it is possible for a bimodule over a division ring to have quite different left and right lengths. Indeed, one need only consider the field $\mathbb{F} = \mathbb{K}(X_i \mid i \in I)$ for an arbitrary infinite family of variables, and for $\mathbb{K}$ a field. Since $\mathbb{F}(Y) \cong \mathbb{F}$, there exists an non-surjective embedding $\varphi\colon \mathbb{F} \to \mathbb{F}(Y) \cong \mathbb{F}$. Then, the bimodule ${}_{\varphi(\mathbb{F})}\mathbb{F}_{\mathbb{F}}$ with the left structure given via φ and the right structure given by usual multiplication has right dimension 1 and left dimension at least the cardinality of I.

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