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UNICYCLIC GRAPHS SATISFY HARARY'S CONJECTURE

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Ulam in [7] has conjectured that any graph G with $p \geq 3$ nodes is uniquely reconstructable from its collection of subgraphs $G_i = G - v_i$, $i = 1, 2, \ldots p$. This conjecture has been proved for various finite graphs including regular, Eulerian, unicyclic, separable, trees and cacti. Since Ulam's conjecture seems difficult to prove or disprove, some authors have tried to strengthen the conjecture [3]. One of these stronger conjectures is Harary's conjecture [2].

A graph G with $p \geq 4$ points can be reconstructed uniquely from its set of non-isomorphic subgraphs $G_i = G - v_i$.

Manvel [3] showed that the number of lines and the connectivity may be determined from the set of non-isomorphic subgraphs. It is easy to show that regular and Eulerian graphs satisfy this conjecture. In [4] Manvel showed that trees also satisfy it; we now prove that Harary's conjecture holds for unicyclic graphs.

THEOREM. *Unicyclic graphs are reconstructable from their set of non-isomorphic subgraphs* $\{G_i\}$.

Proof. See [1] for further details. First examine the situation in which no rooted tree around the cycle has more than 2 nodes. In [5] Manvel proved that if $\delta(G) \leq 3$, then the degree sequence of G may be determined from $\{G_i\}$. $\delta(G)$ is the minimum degree of any vertex in G. From this result we can determine the degree sequence of G as well as the neighbourhood degree sequence of any deleted vertex. Thus Manvel's method [6] of proving that unicyclic graphs satisfy Ulam's conjecture is immediately applicable to our situation.

Now we assume that some of the rooted trees have more than two nodes. We find τ, the collection of rooted trees $\{T_i\}$ in G. Let $\{G'_i\}$ be the set of non-isomorphic connected unicyclic subgraphs. Select a G'_j with the maximum number of trees of order 2 (if there are any); G'_j has the same number of non-trivial trees as G.

If we have some trees of order 2, then we can easily determine the number of trees of order 2 and can determine τ immediately. We now assume that there is no tree with two nodes, and there are at least two non-trivial rooted trees in G.

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Let $l(s)$ denote the maximum (minimum) order of all the trees. We now consider the following three cases:

Case 1. $l \geq s+2$

In this case, consider a G'_j which contains a rooted tree of order $s-1$. Thus v_j, the deleted node is a leaf of one of G's trees of order s. From G'_j we can find all trees of G which have order $\geq s+1$. Assume there are r trees of order l. For finding the trees of order s consider G'_k which contains $r-1$ trees of order l. Add all the trees of order s from G'_k to τ, thereby completing τ.

Case 2. $l=s+1$

In this case, as in case 1, we can find all trees of order l. Now for finding the trees of order s, consider one of the trees of order l and delete one of its leaves and call the resulting tree T. Then find G'_i which has a maximum number of trees isomorphic to T. This G'_i has as its rooted trees of order s exactly those of G, plus an extra copy of T, thus completing τ.

Case 3. $l=s$

Select two non-isomorphic G'_i and G'_k so that they have a different collection of rooted trees of order l (if no such subgraphs exist, then all the rooted trees are the same). Make the set τ_i from G'_i by taking all the nonisomorphic trees of order l together with their multiplicity. Similarly, construct τ_k from G'_k. τ is found as follows: if the tree T appears e times in τ_k and f times in τ_i, then T appears $\max(e,f)$ in τ.

We now consider the situation in which G has only one non-trivial tree T of order >2. The degree d of the root v_1 of T will be one greater than the number of components of the unique disconnected acyclic G_i. If $d \geq 4$, then we can reconstruct T by finding the subtrees $\{T'_i\}$ of v_1 from the subtrees $\{T''_i\}$ of v_1 in the $\{G'_i\}$. For finding $\{T'_i\}$ we can apply the technique used in Case 2.

If $d=3$ then as Manvel [6] mentioned, we can find the point v_2 of degree at least 3 which is closest to v_1 in T (if there is no such point then T is just a path). As before, we can then determine the subtrees of v_2 in G.

Now we have found τ in all cases. To attach the trees in τ to the cycle, Manvel's technique for reconstructing unicyclic graphs [6] from the set of vertex deleted subgraphs is exactly applicable.

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