THE DIRECT-CONNECTEDNESS FUNCTION IN THE RANDOM CONNECTION MODEL

SABINE JANSEN [iD],* **
LEONID KOLESNIKOV [iD],* *** AND
KILIAN MATZKE,* **** *Ludwig-Maximilians-Universität München*

Abstract

We investigate expansions for connectedness functions in the random connection model of continuum percolation in powers of the intensity. Precisely, we study the pair-connectedness and the direct-connectedness functions, related to each other via the Ornstein–Zernike equation. We exhibit the fact that the coefficients of the expansions consist of sums over connected and 2-connected graphs. In the physics literature, this is known to be the case more generally for percolation models based on Gibbs point processes and stands in analogy to the formalism developed for correlation functions in liquid-state statistical mechanics.

We find a representation of the direct-connectedness function and bounds on the intensity which allow us to pass to the thermodynamic limit. In some cases (e.g., in high dimensions), the results are valid in almost the entire subcritical regime. Moreover, we relate these expansions to the physics literature and we show how they coincide with the expression provided by the lace expansion.

Keywords: Ornstein–Zernike equation; random connection model; connectedness functions; Poisson process; percolation; graphical expansions; lace expansion

2020 Mathematics Subject Classification: Primary 60K35
Secondary 60G55; 82B43; 60D05

1. Introduction and main result

Perturbation analysis plays an important role in both stochastic geometry [14, Chapter 19] and statistical mechanics. For Gibbs point processes (grand-canonical Gibbs measures in statistical mechanics), quantities like factorial moment densities (also called correlation functions) are highly nontrivial functions of the intensity of the Gibbs point process itself (density) or the intensity of an underlying Poisson point process (activity). When interactions are pairwise, it is well known that the coefficients of these expansions are given by sums over geometric, weighted graphs. There is a vast literature addressing the convergence of these expansions; see, for example, [2, 16]. Some attempts have been made at exploiting power series expansions

from statistical mechanics for likelihood analysis of spatial point patterns in spatial statistics; see [19].

The physics literature provides similar power series expansions for connectedness functions in a class of percolation models driven by Gibbs point processes, the so-called random connection models (RCMs) [6]. The expansion coefficients for the pair-connectedness function can be written in terms of a sum of certain connected graphs (see (3.1)) and the coefficients for the direct-connectedness function in terms of a sum over certain 2-connected graphs (see (4.1)). The two functions are related via the Ornstein–Zernike equation (OZE) [20], an integral equation which is of paramount importance in physical chemistry and soft matter physics and which enters some approaches to percolation theory; see [25, Chapter 10]. For Bernoulli bond percolation on $\mathbb{Z}^d$, the OZE encodes a renewal structure and is used to prove Ornstein–Zernike behavior [4], a precise asymptotic formula for pair-connectedness functions in the subcritical regime that incorporates subleading corrections to the exponential decay. The OZE also appears as a by-product of lace expansions [10, Proposition 5.2].

The expansions for connectedness functions appearing in [6] are derived as a means of discussing the following question: is it possible to choose the notion of connectivity in such a way that the percolation transition, if it occurs at all, coincides with the phase transition in the sense of non-uniqueness of Gibbs measures? We remind the reader that the relationship between the two phenomena is rather subtle, and in general the corresponding critical parameters do not match; see [12] and references therein. To the best of our knowledge, the question above has not been fully answered for continuum systems, although Betsch and Last [1] were recently able to show that uniqueness of the Gibbs measure follows from the non-percolation of an associated RCM driven by a Poisson point process.

Moreover, the convergence of the expansions for connectedness functions has not been treated in a mathematically rigorous way, in stark contrast with the rich theory of cluster expansions. Even in the simplest case of the RCM driven by a Poisson point process that we consider in this paper, where activity and density coincide and are called the intensity, rigorous results for the expansion of connectedness functions barely exist: the first ones were obtained by Last and Ziesche in [15]. However Last and Ziesche do not prove that their expansions coincide with the physicists' expansion, and they do not prove quantitative bounds for the domain of convergence of the small-intensity expansion.

Our main result addresses graphical expansions of the direct-connectedness function in infinite volume. The results by Last and Ziesche [15], combined with our combinatorial considerations from Section 6.2, imply that the physicists' expansions have a positive radius of convergence; however, it is not our purpose to provide a quantitative bound for the latter. Instead, we perform first a re-summation, in finite volume, of the physicists' expansion. Although the re-summed expansion is no longer a power series in the intensity of the underlying Poisson point process, it has the (conjectured) advantage of converging in a bigger domain than the physicists' expansion. We provide quantitative bounds on the intensity that allow us to pass to the infinite-volume limit in the re-summed expansion of the direct-connectedness function. The proof uses the continuum BK inequality proved in [10].

In addition, we discuss the relationship of the physicists' and our expansion to the lace expansion for the continuum random connection model [10]. Roughly, the lace expansion could in theory be rederived from the graphical expansion by yet another re-summation step. In fact a notion of laces similar to the laces for the self-avoiding random walk [2, 22] already enters the proof of our main result on graphical expansions (see Section 4.3). Thus, contrary to what is stated in [9, Chapter 6.1], the denomination 'lace expansion' for percolation is not a

misnomer, at least for continuum systems. It is unclear, however, whether the discussion offers a new angle of attack on the intricate convergence problems in the theory of lace expansions.

Let us properly introduce the RCM and state our results. The RCM depends on two parameters, namely its *intensity* $\lambda \geq 0$ and the (measurable) *connection function* $\varphi : \mathbb{R}^d \to [0, 1]$, satisfying

$$0 < \int \varphi(x) \mathrm{d}x < \infty$$

as well as radial symmetry $\varphi(x) = \varphi(-x)$ for all $x \in \mathbb{R}^d$. The model is described informally as follows: the vertex set is taken to be a homogeneous Poisson point process (PPP) in $\mathbb{R}^d$ of intensity λ, denoted by η. For any pair $x, y \in \eta$, we add the edge $\{x, y\}$ with probability $\varphi(x - y)$ and independently of all other pairs. We refer to [10, 18] for a formal construction.

The RCM is an undirected simple random spatial graph and a standard model of continuum percolation. We denote it by ξ and we use $\mathbb{P}_\lambda$ to denote the corresponding probability measure. Its vertex set is $V(\xi) = \eta$, and we let $E(\xi)$ denote its edge set.

For $x \in \mathbb{R}^d$, we let ξ^x be the RCM augmented by the point x. In other words, the vertex set of ξ^x is $\eta \cup \{x\}$ and the edges are formed as described above. In particular, edges between x and points of η are drawn independently and according to φ. More generally, for a set of points $x_1, \ldots, x_k$, we let $\xi^{x_1, \ldots, x_k}$ be the RCM with vertex set $\eta \cup \{x_1, \ldots, x_k\}$ (also here, edges between deterministic points x_1, x_2 are drawn independently and according to φ).

We say that $x, y \in \eta$ are connected (and write $x \longleftrightarrow y$ in ξ) if there is a path from x to y in ξ. For $x \in \mathbb{R}^d$, we let $\mathscr{C}(x) = \mathscr{C}(x, \xi^x) = \{y \in \eta^x : x \longleftrightarrow y$ in $\xi^x\}$ be the cluster of x and define the *pair-connectedness* (or *two-point*) function $\tau_\lambda : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ to be

$$\tau_\lambda(x, y) := \mathbb{P}_\lambda\big(x \longleftrightarrow y \text{ in } \xi^{x, y}\big). \tag{1.1}$$

Thanks to the translation-invariance of the model, we have $\tau_\lambda(x, y) = \tau_\lambda(\mathbf{0}, x - y)$ (where $\mathbf{0}$ denotes the origin in $\mathbb{R}^d$), and we can also define τ_λ as a function $\tau_\lambda : \mathbb{R}^d \to [0, 1]$ with $\tau_\lambda(x) = \mathbb{P}_\lambda\big(\mathbf{0} \longleftrightarrow x$ in $\xi^{\mathbf{0}, x}\big)$.

We say that $x, y \in \eta$ are *2-connected* (or *doubly connected*) and write $x \Longleftrightarrow y$ in ξ if there are two paths from x to y that have only their endpoints in common (or if x and y are directly connected by an edge or if $x = y$). We define

$$\sigma_\lambda(x) := \mathbb{P}_\lambda\big(\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0}, x}\big).$$

Recall that the critical intensity for percolation is defined by

$$\lambda_c = \sup\{\lambda \geq 0 : \mathbb{P}_\lambda(|\mathscr{C}(\mathbf{0})| = \infty) = 0\}$$

and that the identity

$$\sup\{\lambda \geq 0 : \mathbb{P}_\lambda(|\mathscr{C}(\mathbf{0})| = \infty) = 0\} = \sup\{\lambda \geq 0 : \int \tau_\lambda(x) \mathrm{d}x < \infty\}$$

has been shown to hold true for connection functions ϕ that are nonincreasing in the Euclidean distance (see [17]). It is proved in [15] that for $\lambda < \lambda_c$, there exists a uniquely defined integrable and essentially bounded function $g_\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\tau_\lambda(x, y) = g_\lambda(x, y) + \lambda \int_{\mathbb{R}^d} g_\lambda(x, z) \tau_\lambda(z, y) \, \mathrm{d}(z), \quad x, y \in \mathbb{R}^d. \tag{1.2}$$

This equation is known as the Ornstein–Zernike equation (OZE), and g_λ is called the *direct-connectedness function*.

For two integrable functions $f, g : \mathbb{R}^d \to \mathbb{R}$, we recall the convolution $f * g$ to be given by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x) g(x - y) \mathrm{d}y.$$

We let $f^{*1} = f$ and $f^{*m} = f^{*(m-1)} * f$. Notice that we can interpret both the pair-connectedness function τ_λ and the direct-connectedness function g_λ as functions on $\mathbb{R}^d$, thanks to translation-invariance. The OZE then can be formulated as

$$\tau_\lambda = g_\lambda + \lambda(g_\lambda * \tau_\lambda). \tag{1.3}$$

Naturally, the question arises whether one can provide an explicit form for the direct-connectedness function g_λ. Unfortunately, an immediate probabilistic interpretation of g_λ is not known. One classical approach from the physics literature is to obtain explicit approximations for the solution g_λ of (1.2) by introducing complementary equations, known as closure relations, the choice of which depends on the specifics of the model considered. Different closure relations provide different explicit approximations for g_λ and thus also for the pair-connectedness function τ_λ, e.g., via a reformulation of the OZE (1.2) for the Fourier transforms of the connectedness functions. Most prominent are the Percus–Yevick closure relations [5, 25]; other examples can be found in [7]. Another approach [6] is to directly provide an independent definition of g_λ in terms of a graphical expansion and then argue that this expansion satisfies the OZE (1.2). We follow the spirit of the latter approach: our main result is a graphical expansion for the direct-connectedness function, with quantitative bounds on the domain of convergence.

Let

$$\lambda_* := \sup\left\{ \lambda \geq 0 : \sup_{x \in \mathbb{R}^d} \sum_{k \geq 1} \lambda^{k-1} \sigma_\lambda^{*k}(x) < \infty \right\}, \qquad \tilde{\lambda}_* := \sup\left\{ \lambda \geq 0 : \lambda \int \sigma_\lambda(x) \, \mathrm{d}x < 1 \right\}. \tag{1.4}$$

It is not hard to see that $\tilde{\lambda}_* \leq \lambda_* \leq \lambda_c$ using (1.5) below.

We can now state our main theorem. It provides (in general dimension) the first rigorous quantitative bounds on λ under which the direct-connectedness function admits a convergent graphical expansion.

Theorem 1.1. (Graphical expansion of the direct-connectedness function.) *For $\lambda < \lambda_*$, the direct-connectedness function $g_\lambda(x_1, x_2)$ is given by the expansion (4.24), which is absolutely convergent pointwise for all $(x_1, x_2) \in \mathbb{R}^{2d}$. Moreover, for $\lambda < \tilde{\lambda}_*$, the expansion (4.24) converges in the $L^1(\mathbb{R}^d, \mathrm{d}x_2)$-norm for all $x_1 \in \mathbb{R}^d$.*

The convergence results for the expansion (4.24) are proved in Theorem 4.1 and Theorem 4.2; the equality with the direct-connectedness function is proved in Section 5.

Last and Ziesche show that there is some $\lambda_0 > 0$ such that g_λ is given by a power series for $\lambda \in [0, \lambda_0)$. No quantitative bounds for λ_0 are provided, however. In Section 6.2, we discuss how to relate this expansion to our expression for g_λ. We now make several remarks on Theorem 1.1 and the quantitative nature of the bounds provided there.

- Since $0 \le \sigma_\lambda \le 1$, we can bound

$$\sum_{k \ge 1} \lambda^{k-1} \sigma_\lambda^{*k}(x) \le \sum_{k \ge 0} \left(\lambda \int \sigma_\lambda(x) \, dx \right)^k = \sum_{k \ge 0} \left(\mathbb{E}_\lambda \left[\left| \{ x \in \eta : \mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0}} \} \right| \right] \right)^k, \quad (1.5)$$

where the identity is due to the Mecke equation (2.1). This shows that $\tilde{\lambda}_* \le \lambda_*$ and that $\tilde{\lambda}_*$ is the point where the expected number of points in η that are 2-connected to the origin passes 1 (i.e., we have $\mathbb{E}_\lambda \left[\left| \{ x \in \eta : \mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0}} \} \right| \right] \ge 1$ for all $\lambda > \tilde{\lambda}_*$).

- The argument of the geometric series in (1.5) can be further bounded from above by

$$\lambda \int \tau_\lambda(x) dx = \mathbb{E}_\lambda \left[\left| \{ x \in \eta : \mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0}} \} \right| \right],$$

the expected cluster size (minus 1). A classical branching-process argument gives that $\tilde{\lambda}_* \ge 1/2$ (see, for example, [21, Theorem 3]).

- In high dimension, we have the following result, proven in [10]: under some additional assumptions on φ (see [10, Section 1.2]), there is an absolute constant c_0 such that

$$\lambda_c \int \sigma_{\lambda_c}(x) \, dx \le 1 + c_0/d$$

in sufficiently high dimension, or, for a class of *spread-out* models (closely related to Kac potentials in statistical mechanics; see [8]) with a parameter L,

$$\lambda_c \int \sigma_{\lambda_c}(x) \, dx \le 1 + c_0 L^{-d}$$

for all dimensions $d > 6$ (in the spread-out case, c_0 is independent of L but may depend on d). As σ_λ is nondecreasing in λ, this provides a bound for the whole subcritical regime. This also implies that for every $\varepsilon > 0$, there is d_0 (respectively, L_0) such that $\tilde{\lambda}_* \ge 1 - \varepsilon$ for all $d \ge d_0$ (respectively, $L \ge L_0$ and $d > 6$). As we also know that $\lambda_c \searrow 1$ as the dimension becomes large, this shows that in high dimension, $\tilde{\lambda}_*$ (and thus also λ_*) gets arbitrarily close to λ_c.

Outline of the paper. The paper proceeds as follows. We introduce most of our important notation in Section 2. This allows us to demonstrate some basic (and mostly well-known) central ideas in Section 3, where the two-point function is discussed in finite volume. Section 4 contains the main body of work for the proof of Theorem 1.1 (the convergence results). The remainder of Theorem 1.1 regarding the OZE is then proved in Section 5.

We discuss our results in Section 6. In particular, we point out where many of the formulas can be found in the physics literature (not rigorously proven) and allude to generalizations to Gibbs point processes. Moreover, we highlight the connection to two other expressions for the pair-connectedness function; in particular, we show how our expansions relate to the lace expansion. Lastly, we address other percolation models very briefly in Section 6.4.

2. Fixing notation

2.1. General notation

We let $[n] := \{1, \ldots, n\}$ and $[n]_0 := [n] \cup \{0\}$. For a set V, we write $\binom{V}{2} := \{E \subseteq V : |E| = 2\}$. For $I = \{i_1, i_2, \ldots, i_\kappa\} \subset \mathbb{N}$, let $\vec{x}_I = (x_{i_1}, \ldots, x_{i_\kappa})$. For compact intervals $[a, b] \subset \mathbb{R}$, we write
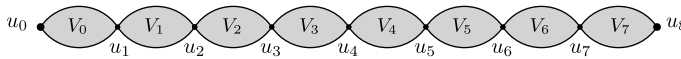
FIGURE 1. A schematic sketch of the pivot decomposition $(u_0, V_0, \ldots, V_7, u_8)$ of G, setting $x = u_0$ and $y = u_{k+1}$.

$\vec{x}_{[a,b]} = \vec{x}_I$ with $I = [a, b] \cap \mathbb{N}$. If $a = 1$, we write $\vec{x}_{[b]} = \vec{x}_{[1,b]}$. By some abuse of notation, we are going to interpret $\vec{x}_{[a,b]}$ both as an ordered vector and as a set.

If not specified otherwise, Λ denotes a bounded, measurable subset of $\mathbb{R}^d$.

2.2. Graph theory

We recall that a (simple) graph $G = (V, E) = (V(G), E(G))$ is a tuple with *vertex* set (or set of *points, sites, nodes*) V and *edge* set (or set of *bonds*) $E \subseteq \binom{V}{2}$. In this paper, we will always consider graphs with $V \subset \mathbb{R}^d$, and for $x, y \in \mathbb{R}^d$, an edge $\{x, y\}$ will sometimes be abbreviated xy.

If $xy \in E$, we write $x \sim y$ (and say that x and y are *adjacent*). We extend this notation and write $x \sim W$ for $x \in V$ and $W \subseteq V$ if there is $y \in W$ such that $x \sim y$; also, we write $A \sim B$ if there is $x \in A$ such that $x \sim B$. For $W \subseteq V$, we define the *W-neighborhood* $N_W(x) = \{y \in W : x \sim y\}$ and the *W-degree* of a vertex $x \in V$ as $\deg_W(x) = |N_W(x)|$, and we write $N(x) = N_V(x)$ as well as $\deg(x) = \deg_V(x)$. For two sets $A, B \subseteq V$, we write $E(A, B) = \{xy \in E(G) : x \in A, y \in B\}$.

Given a graph $G = (V, E)$ and $W \subseteq V$, we denote by $G[W] := (W, \{e \in E : e \subseteq W\})$ the subgraph of G *induced* by W. Given two simple graphs G, H, we let $G \oplus H := (V(G) \cup V(H), E(G) \cup E(H))$.

Connectivity. Given a graph G and two of its vertices $x, y \in V(G)$, we say that x and y are *connected* if there is a *path* between x and y—that is, a sequence of vertices $x = v_0, v_1, \ldots, v_k = y$ for some $k \in \mathbb{N}_0$ such that $v_{i-1} v_i \in E(G)$ for $i \in [k]$. We write $x \longleftrightarrow y$ in G or simply $x \longleftrightarrow y$. We call $\mathscr{C}(x) = \mathscr{C}(x; G) = \{y \in V(G) : x \longleftrightarrow y\}$ the *cluster* (or *connected component*) of x in G. If there is only one cluster in G, we say that G is connected.

For $x \longleftrightarrow y$ in G, we let $\mathsf{Piv}(x, y; G)$ denote the set of *pivotal* vertices for the connection between x and y. That is, $v \notin \{x, y\}$ is in $\mathsf{Piv}(x, y; G)$ if every path from x to y in G passes through v. We say that x is doubly connected to y in G (and write $x \Longleftrightarrow y$ in G) if $\mathsf{Piv}(x, y; G) = \varnothing$. We remark that in the physics literature, pivotal points are usually known as *nodal* points.

In the pathological case $x = y$, we use the convention $x \longleftrightarrow x$ in G and set $\mathsf{Piv}(x, x; G) = \varnothing$ for any graph G with $x \in V(G)$ (equivalently, $x \Longleftrightarrow x$ in G).

We observe that the pivotal points $\{u_1, \ldots, u_k\}$ can be ordered in a way such that every path from x to y passes through the pivotal points in the order $(u_1, \ldots, u_k)$. We define $\mathsf{PD}(x, y, G) = \mathsf{PD}(G)$ to be the *pivot decomposition* of G, that is, a partition of the vertex set V into a sequence, $(x, V_0, u_1, V_1, \ldots, u_k, V_k, y)$, where $(u_1, \ldots, u_k)$ are the ordered pivotal points and V_i is the (possibly empty) set of vertices that can be reached only by passing through u_i and that is still connected to x after the removal of u_{i+1}. See Figure 1.

Classes of graphs. Given a (locally finite) set $X \subset \mathbb{R}^d$, we let $\mathcal{G}(X)$ be the set of graphs with vertex set X. We let $\mathcal{C}(X)$ be the set of connected graphs on X. Moreover, for $x, y \in X$, we let $\mathcal{D}_{x,y}(X) \subseteq \mathcal{C}(X)$ be the set of non-pivotal graphs, i.e., the set of connected graphs such that $\mathsf{Piv}(x, y; G) = \varnothing$.

Given m *bags* $X_1, \ldots, X_m \subset \mathbb{R}^d$ with $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq m$, we let $\mathcal{G}(X_1, \ldots, X_m)$ denote the set of *m-partite* graphs on $X_1, \ldots, X_m$, i.e., the set of graphs G

with $V(G) = \cup_{i=1}^{m} X_i$ and $E(G[X_i]) = \varnothing$ for $i \in [m]$. Note that we allow bags to have (at most) one vertex in common, which is a slight abuse of the notation in graph theory, where m-partite graphs have disjoint bags.

The notion of ($\pm$)-graphs. We introduce a ($\pm$)-graph as a triple

$$G^{\pm} = (V(G), E^{+}(G), E^{-}(G)) = (V, E^{+}, E^{-}),$$

where V is the vertex set and $E^{+}, E^{-} \subseteq \binom{V}{2}$ are disjoint. In other words $G^{\pm}$ is a graph where every edge is of exactly one of two types (plus or minus). We set $E := E^{+} \cup E^{-}$ and associate to $G^{\pm}$ the two simple graphs $G^{|\pm|} := (V, E)$ and $G^{+} := (V^{+}, E^{+})$, where $V^{+} := \{x \in V : \exists e \in E^{+} : x \in e\}$ are the vertices incident to at least one ($+$)-edge.

We extend all the notions for simple graphs to ($\pm$)-graphs. In particular, given $X \subset \mathbb{R}^d$, we let $\mathcal{G}^{\pm}(X)$ be the set of ($\pm$)-graphs on X. Moreover, $\mathcal{C}^{\pm}(X)$ are the ($\pm$)-connected graphs on X, that is, the graphs such that $G^{|\pm|}$ is connected. Similarly, $\mathcal{C}^{+}(X) \subset \mathcal{C}^{\pm}(X)$ are the ($+$)-connected graphs, that is, those where G^{+} is connected and $V(G) = V^{+}$. For $x, y \in X$, we denote by $\mathcal{D}^{\pm}_{x,y}(X)$ the set of those ($\pm$)-connected graphs on X where $\mathsf{Piv}(x, y; G^{|\pm|}) = \varnothing$, and by $\mathcal{D}^{+}_{x,y}(X) \subset \mathcal{D}^{\pm}_{x,y}(X)$ the set of those ($\pm$)-connected graphs on X where $\mathsf{Piv}(x, y; G^{+}) = \varnothing$. We also define the ($\pm$)-pivot decomposition $\mathsf{PD}^{\pm}(x, y, G^{\pm}) = \mathsf{PD}^{\pm}(G^{\pm}) = \mathsf{PD}(G^{|\pm|})$ and the ($+$)-pivot decomposition $\mathsf{PD}^{+}(x, y, G^{\pm}) = \mathsf{PD}^{+}(G^{\pm}) = \mathsf{PD}(G^{+})$. Lastly, we write $x \xleftrightarrow{+} y$ if there is a path from x to y in E^{+}.

Given a ($\pm$)-graph G and a simple graph H, we define

$$G \oplus H := (V(G) \cup V(H), E^{+}(G), E^{-}(G) \cup E(H)).$$

Weights. Given a simple graph G, a ($\pm$)-graph H on $X \subset \mathbb{R}^d$, and the connection function φ, we define the weights

$$\mathbf{w}(G) := (-1)^{|E(G)|} \prod_{\{x,y\} \in E(G)} \varphi(x - y), \qquad \mathbf{w}^{\pm}(H) := (-1)^{|E^{-}(H)|} \prod_{\{x,y\} \in E(H)} \varphi(x - y).$$

2.3. The random connection model

The RCM ξ can be formally constructed as a point process, that is, a random variable taking values in the space of locally finite counting measures $(\mathbf{N}, \mathcal{N})$ on some underlying metric space $\mathbb{X}$. There are various ways to choose $\mathbb{X}$. One option is to let $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$ for an appropriate mark space $\mathbb{M}$ (see [18]); another way can be found in [10, 15]. In any case, one can reconstruct from ξ the point process η on $\mathbb{R}^d$ which makes up the vertex set of ξ. We treat η both as a counting measure and as a set, giving meaning to statements of the form $x \in \eta$.

If $e = \{x, y\}$ is an edge, then we write $\varphi(e) = \varphi(x - y)$. For a bounded set $\Lambda \subset \mathbb{R}^d$, we write $\eta_{\Lambda} = \eta \cap \Lambda$ and let ξ_{Λ} denote the RCM restricted to Λ, that is, $\xi[\eta_{\Lambda}]$. The two-point function restricted to Λ is defined as $\tau^{\Lambda}_{\lambda}(x, y) = \mathbb{P}_{\lambda}(x \longleftrightarrow y \text{ in } \xi^{x,y}_{\Lambda})$ for $x, y \in \Lambda$ and zero otherwise.

For $V \subset W$, there is a natural way to couple the models ξ^V and ξ^W, which is by deleting from ξ^W all points in $W \setminus V$ along with their incident edges. We implicitly assume throughout this paper that this coupling for different sets of added points is used.

The Mecke equation. Since it is used repeatedly throughout this paper, we state the Mecke equation, a standard tool in point process theory, in its version for the RCM (see

[15]). For $m \in \mathbb{N}$ and a measurable function $f : \mathbf{N} \times \mathbb{R}^{dm} \to \mathbb{R}_{\geq 0}$, the Mecke equation states that

$$\mathbb{E}_\lambda \left[\sum_{\vec{x}_{[m]} \in \eta^{(m)}} f(\xi, \vec{x}_{[m]}) \right] = \lambda^m \int \mathbb{E}_\lambda \big[f(\xi^{x_1, \dots, x_m}, \vec{x}_{[m]}) \big] \, d\vec{x}_{[m]}, \tag{2.1}$$

where $\eta^{(m)} = \big\{ \vec{x}_{[m]} \in \eta^m : x_i \neq x_j \text{ for } i \neq j \big\}$ are the pairwise distinct tuples.

Rescaling. It is a standard trick in continuum percolation to rescale space in order to normalize a quantity of interest, which is $\int \varphi(x) \, dx$ in our case. We refer to [18, Section 2.2]. As a consequence, we may without loss of generality assume that $\int \varphi(x) \, dx = 1$.

The BK inequality. We say that $A \in \mathcal{N}$ *lives on* Λ if $\mathbb{1}_A(\mu) = \mathbb{1}_A(\mu_\Lambda)$ for every $\mu \in \mathbf{N}$. We call an event $A \in \mathcal{N}$ *increasing* if $\mu \in A$ implies $\nu \in A$ for each $\nu \in \mathbf{N}$ with $\mu \subseteq \nu$. Let $\mathcal{R}$ denote the ring of all finite unions of half-open rectangles with rational coordinates. For two increasing events $A, B \in \mathcal{N}$ we define

$$A \circ B := \{\mu \in \mathbf{N} : \exists K, L \in \mathcal{R} \text{ s.t. } K \cap L = \varnothing \text{ and } \mu_K \in A, \ \mu_L \in B\}. \tag{2.2}$$

Informally, this is the event that A and B take place in spatially disjoint regions. It is proved in [10, Theorem 2.1] that for two increasing events A and B living on Λ, we have

$$\mathbb{P}_\lambda(A \circ B) \leq \mathbb{P}_\lambda(A) \mathbb{P}_\lambda(B).$$

The RCM on a fixed vertex set. Given some (finite) set $X \subset \mathbb{R}^d$ and a function $\varphi : \mathbb{R}^d \to [0, 1]$, we will often have to deal with the following random graph: its vertex set is X, and two vertices $x, y \in X$ are adjacent with probability $\varphi(x - y)$, independently of other pairs of vertices. This is simply the RCM conditioned to have the vertex set X. To highlight the difference from ξ, which depends on the PPP η, we denote this random graph by $\Gamma_\varphi(X)$. If $Y \subset X$, then we write $\Gamma_\varphi(Y)$ for $\Gamma_\varphi(X)[Y]$. Since there is no dependence on λ, we write $\mathbb{P}$ for the probability measure of the RCM with fixed vertex set.

3. Fixing ideas: the two-point function in finite volume

We use this section to put the definitions of Section 2 into action and to derive a power series expansion for τ_λ in finite volume. We start by motivating the introduction of $(\pm)$-graphs by linking them to the RCM Γ_φ.

Observation 3.1. (*Connection between $(\pm)$-graphs and probabilities.*) Let $X \subset \mathbb{R}^d$ be finite. Let $\mathfrak{P} \subseteq \mathcal{G}(X)$ be a graph property. Then

$$\sum_{\substack{G \in \mathcal{G}^\pm(X): \\ (V(G), E^+(G)) \in \mathfrak{P}}} \mathbf{w}^\pm(G) = \mathbb{P}\big(\Gamma_\varphi(X) \in \mathfrak{P}\big).$$

Proof. Note that

$$\mathbb{P}\big(\Gamma_\varphi(X) \in \mathfrak{P}\big) = \sum_{\substack{G \in \mathcal{G}(X): \\ G \in \mathfrak{P}}} \prod_{e \in E(G)} \varphi(e) \prod_{e \in \binom{X}{2} \setminus E(G)} (1 - \varphi(e)).$$

Expanding the factor $\prod_{e \in \binom{X}{2} \setminus E(G)} (1 - \varphi(e))$ into a sum proves the claim. $\qquad \square$

Note that the weight of a $(\pm)$-graph may also be calculated by taking the product over all its edges, with factors $\varphi(\cdot)$ and $-\varphi(\cdot)$ for edges in E^+ and E^-, respectively. Observation 3.1 motivates that the edges in E^+ correspond to the edges in the random graph Γ_φ.

Next we prove a power series expansion for τ_λ in terms of the intensity λ. The expansion (3.1) has already been given by Coniglio, De Angelis and Forlani [6, Equation (12)], who work in the more general context of Gibbs point processes but do not prove convergence. The proposition enters the proof of Proposition 5.1.

Notice that the coefficients of power series expansions like (3.1) are given by integrals with respect to the Lebesgue measure, and it is sufficient that the integrands be defined up to Lebesgue null sets for those integrals to be well-defined. Since vectors $\vec{x}_{[3,n+2]} \in \mathbb{R}^{dn}$ with fewer than n distinct entries constitute a Lebesgue null set, we can assume that for $x_1 \neq x_2$ only graphs with vertex sets of cardinality $n+2$ contribute to the nth coefficient in (3.1). The same considerations apply to all graphical expansions appearing from here on, including our main definition (4.6).

Proposition 3.1. (Graphical expansion for the two-point function.) *Consider the RCM restricted to a bounded measurable set $\Lambda \subset \mathbb{R}^d$, and let $x_1, x_2 \in \Lambda$. Then*

$$\tau_\lambda^\Lambda(x_1, x_2) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{G \in \mathcal{C}^\pm\left(\vec{x}_{[n+2]}\right): \\ x_1 \overset{+}{\longleftrightarrow} x_2}} \mathbf{w}^\pm(G) \, d\vec{x}_{[3,n+2]} \tag{3.1}$$

with

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left| \sum_{\substack{G \in \mathcal{C}^\pm\left(\vec{x}_{[n+2]}\right): \\ x_1 \overset{+}{\longleftrightarrow} x_2}} \mathbf{w}^\pm(G) \right| d\vec{x}_{[3,n+2]} \leq \exp\left\{2\lambda + \lambda|\Lambda|e^\lambda\right\} < \infty.$$

Note that Proposition 3.1 is valid for all intensities $\lambda \geq 0$. This situation is completely different from familiar cluster expansions [2], where the radius of convergence of relevant expansions is finite in finite volume as well.

The expansion (3.1) amounts to the physicists' expansion in powers of the activity. The expansion in powers of the density instead involves sums over a smaller class of graphs. For PPPs, activity and density are the same and the two expansions must coincide. In our context, we point out that the sum over graphs in (3.1) can be reduced to the sum over the subset of graphs in $\mathcal{C}^\pm$ that contain a $(+)$-path from x_1 to x_2 and that have no *articulation points* (with respect to x_1, x_2). To define articulation points, recall that a cut vertex leaves a connected graph disconnected upon its deletion. Now, an articulation point is a cut vertex that is not pivotal for the x_1–x_2 connection. It is not difficult to see that for fixed points $x_{[n+2]}$, the graphs with articulation points in the sum over graphs G in (3.1) exactly cancel out. This cancellation happens at fixed n and does not require any re-summations between graphs with different numbers of vertices.

The proof of Proposition 3.1 builds on yet another equivalent representation: in Equation (3.1) we can discard those graphs G for which G^+ is not connected and those for which not every $(-)$-edge has at least one endpoint in $V(G^+)$; see Equation (3.6) below for a precise statement. To the best of our knowledge, Equation (3.6) is new.

Proof of Proposition 3.1. We write $\tau_\lambda = \tau_\lambda^\Lambda$ and $\eta = \eta_\Lambda$. Given $x_1, x_2 \in \Lambda$, we can partition

$$
\tau_\lambda(x_1, x_2) = \sum_{n \geq 0} \mathbb{P}_\lambda \left(x_1 \longleftrightarrow x_2 \text{ in } \xi_\Lambda^{x_1, x_2}, \left| \mathscr{C}(x_1, \xi_\Lambda^{x_1, x_2}) \right| = n + 2 \right)
$$

$$
= \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \mathbb{P}\left(\Gamma_\varphi(\vec{x}_{[n+2]}) \in \mathcal{C}(\vec{x}_{[n+2]}) \right)
$$

$$
\times \exp\left\{ -\lambda \int_\Lambda \left(1 - \prod_{i=1}^{n+2} (1 - \varphi(x_i - y)) \right) dy \right\} d\vec{x}_{[3, n+2]} \qquad (3.2)
$$

The second identity can be found, for example, in [15, Proposition 3.1]. Set

$$
f(\vec{x}_{[n+2]}, \vec{y}_{[m]}) = \mathbb{P}\left(\Gamma_\varphi(\vec{x}_{[n+2]}) \in \mathcal{C}(\vec{x}_{[n+2]}) \right) \prod_{j=1}^{m} \left(\prod_{i=1}^{n+2} (1 - \varphi(x_i - y_j)) - 1 \right).
$$

Expanding the exponential in (3.2), we find

$$
\tau_\lambda(x_1, x_2) = \sum_{n, m \geq 0} \frac{\lambda^{n+m}}{m! n!} \int_{\Lambda^n} \int_{\Lambda^m} f(\vec{x}_{[n+2]}, \vec{y}_{[m]}) \, d\vec{y}_{[m]} \, d\vec{x}_{[3, n+2]}, \qquad (3.3)
$$

with

$$
\sum_{n, m \geq 0} \frac{\lambda^{n+m}}{m! n!} \int_{\Lambda^n} \int_{\Lambda^m} \left| f(\vec{x}_{[n+2]}, \vec{y}_{[m]}) \right| d\vec{y}_{[m]} \, d\vec{x}_{[3, n+2]}
$$

$$
= \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \mathbb{P}\left(\Gamma_\varphi(\vec{x}_{[n+2]}) \in \mathcal{C}(\vec{x}_{[n+2]}) \right) \exp\left\{ \lambda \int_\Lambda \left(1 - \prod_{i=1}^{n+2} (1 - \varphi(x_i - y)) \right) dy \right\} d\vec{x}_{[3, n+2]}
$$

$$
\leq \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} e^{\lambda(n+2)} \, d\vec{x}_{[3, n+2]}
$$

$$
= \exp\{ 2\lambda + \lambda |\Lambda| e^\lambda \} < \infty. \qquad (3.4)
$$

In the third line, we have used the inequality

$$
\int_\Lambda \left(1 - \prod_{i=1}^{n+2} (1 - \varphi(x_i - y)) \right) dy \leq \int_\Lambda \sum_{i=1}^{n+2} \varphi(x_i - y) \, dy \leq n + 2, \qquad (3.5)
$$

which can be shown as follows. Let $n \in \mathbb{N}$ and let $0 \leq a_1, \ldots, a_n \leq 1$. Notice that the identity $1 - \prod_{i=1}^{n} (1 - a_i) = (1 - a_n)\left(1 - \prod_{i=1}^{n-1} (1 - a_i) \right) + a_n$ and the estimate $(1 - a_n) \leq 1$ hold for all $n \in \mathbb{N}$. The inequality between the integrands in (3.5) now follows by induction with the choice $a_i = \varphi(x_i - y)$. The rescaling introduced in Section 2.3 ensures that $\int_\Lambda \varphi(x_i - y) \, dy \leq 1$, $i \in [n + 2]$, yielding the second inequality.

Next we turn to a combinatorial representation of f as a sum over $(\pm)$-graphs. Recall that $\mathcal{C}^+$ denotes sets of $(\pm)$-graphs that are $(+)$-connected. The definition of f and Observation 3.1 yield

$$f\big(\vec{x}_{[n+2]}, \vec{y}_{[m]}\big) = \left(\sum_{G \in \mathcal{C}^+\big(\vec{x}_{[n+2]}\big)} \mathbf{w}^{\pm}(G)\right) \left(\sum_{\substack{H \in \mathcal{G}\big(\vec{x}_{[n+2]}, \vec{y}_{[m]}\big): \\ y_i \sim \vec{x}_{[n+2]}\ \forall i \in [m]}} \mathbf{w}(H)\right) = \sum_{G \oplus H} \mathbf{w}^{\pm}(G \oplus H),$$

where the last sum is over all $(\pm)$-graphs $G' = G \oplus H$ in $\mathcal{C}^{\pm}\big(\vec{x}_{[n+2]} \cup \vec{y}_{[m]}\big)$ such that, first, there are no edges between points of $\vec{y}$; second, $(G \oplus H)^+$ is connected; and third, the vertices of $(G \oplus H)^+$ are precisely $\vec{x}_{[n+2]}$.

We rearrange the double sum (3.3) over m, n into one sum, indexed by the value of $m + n$, and obtain

$$\tau_{\lambda}(x_1, x_2) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{G \in \mathcal{C}^{\pm}\big(\vec{x}_{[n+2]}\big): \\ \{x_1, x_2\} \subseteq V\big(G^+\big),\, G^+ \text{ connected}, \\ E(G^{|\pm|}[V \setminus V^+]) = \varnothing}} \mathbf{w}^{\pm}(G)\, \mathrm{d}\vec{x}_{[3,n+2]} \tag{3.6}$$

$$= \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{G \in \mathcal{C}^{\pm}\big(\vec{x}_{[n+2]}\big): \\ x_1 \overset{+}{\longleftrightarrow} x_2}} \mathbf{w}^{\pm}(G)\, \mathrm{d}\vec{x}_{[3,n+2]}. \tag{3.7}$$

In the second identity, we have added some graphs to the sum, namely those in which G^+ is not connected or where there exist edges between vertices of $V \setminus V^+$.

We claim that the weights of these added graphs sum up to zero. To see this, first identify $[n + 2]$ with the vertices $\vec{x}_{[n+2]}$ and fix a graph $G \in \mathcal{C}([n + 2])$. Now, let $C \subseteq [n + 2]$ with $\{1, 2\} \subseteq C$ and consider the set $\mathcal{G}_G(C)$ of all $(\pm)$-connected graphs $G^{\pm}$ on $[n + 2]$ such that $G^{|\pm|} = G$ and C is the vertex set of the $(+)$-component of 1 in $G^{\pm}$. If there is at least one edge e in G that has both endpoints outside of C, we partition $\mathcal{G}_G(C)$ into those graphs where e is in E^+ and those where e is in E^-. This induces a pairing between the graphs of $\mathcal{G}_G(C)$, and they cancel out. What remain are precisely the graphs in (3.6). $\qquad\square$

4. The direct-connectedness function

4.1. Motivation and rough outline

The expansion of the direct-connectedness function in powers of the activity given by [6], without proofs and convergence bounds, is

$$g_{\lambda}^{\Lambda}(x_1, x_2) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{G \in \mathcal{D}_{x_1, x_2}^{\pm}\big(\vec{x}_{[n+2]}\big): \\ x_1 \overset{+}{\longleftrightarrow} x_2}} \mathbf{w}^{\pm}(G)\, \mathrm{d}\vec{x}_{[3,n+2]}. \tag{4.1}$$

It is obtained from the expansion of the pair-connectedness function in Proposition 3.1 by discarding graphs that have pivotal points (i.e., graphs G where $\mathsf{Piv}^{\pm}(G)$ is nonempty). Before we

pass to the thermodynamic limit, we perform a re-summation and find another representation of g_λ^Λ which has the conjectured advantage of increasing the domain of convergence.

Let $G = (V, E^+, E^-) \in \mathcal{C}^\pm(\vec{x}_{[n+2]})$ be a $(\pm)$-graph appearing in the expansion (3.6). Thus $V = \{x_i : 1 \le i \le n + 2\}$, the graph G^+ is connected, x_1 and x_2 belong to $V^+ = V(G^+)$, every vertex $y \in V(G) \setminus V(G^+)$ is linked by at least one $(-)$-edge to V^+, and there are no edges between two vertices in $V \setminus V^+$. We impose the additional constraint that $G^{|\pm|} = (\vec{x}_{[n+2]}, E^+ \cup E^-)$ has no pivotal points for paths from x_1 to x_2.

Since x_1 and x_2 are connected by a path of $(+)$-edges, G admits a $(+)$-pivot decomposition $\vec{W} = (u_0, V_0, \ldots, u_k, V_k, u_{k+1})$ (with $u_0 = x_1$ and $u_{k+1} = x_2$), where $k \in \mathbb{N}_0$ is the number of pivotal points in $\mathsf{Piv}^+(x_1, x_2; G)$. Then, G decomposes into a *core* graph $G_{\text{core}} = (V(G^+), E^+, E_{\text{core}}^-)$, with E_{core}^- the set of $(-)$-edges of G with both endpoints in $V_i \cup \{u_i, u_{i+1}\}$ for some $i \in [k]_0$, and a *shell* graph $H = (V, \varnothing, E^- \setminus E_{\text{core}}^-)$. By our choice of E_{core}^-, we have $\mathsf{PD}^\pm(G_{\text{core}}) = \mathsf{PD}^+(G_{\text{core}}) = \vec{W}$. Clearly

$$\mathbf{w}^\pm(G) = \mathbf{w}^\pm(G_{\text{core}})\mathbf{w}^\pm(H).$$

In the right-hand side of (4.1), we restrict to graphs that also appear in (3.6) and rewrite the resulting sum as a double sum over core graphs and shell graphs. This gives rise to the series

$$\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \int_{\Lambda^r} \sum_{\vec{W}} \sum_{G_{\text{core}}} \mathbf{w}^\pm(G_{\text{core}}) \left(\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \int_{\Lambda^m} \sum_H \mathbf{w}^\pm(H)\, d\vec{y}_{[m]} \right) d\vec{x}_{[3,r+2]}.$$

The outer sum is over potential pivot decompositions $\vec{W}$ of core vertices $\vec{x}_{[r+2]}$, the second sum over $(\pm)$-graphs $G_{\text{core}} = (\vec{x}_{[r+2]}, E^+, E_{\text{core}}^-)$ that are $(+)$-connected and for which $\vec{W}$ is both the $(\pm)$-pivot decomposition and the $(+)$-pivot decomposition (in other words, the simple graph $(\vec{x}_{[r+2]}, E^+)$ is connected and $\mathsf{PD}^\pm(x_1, x_2, G) = \mathsf{PD}^+(x_1, x_2, G) = \vec{W}$). The inner sum is over $(\pm)$-graphs $H = (V(H), \varnothing, E^-(H))$ with vertex set $\vec{x}_{[r+2]} \cup \vec{y}_{[m]}$ and $(-)$-edges $\{y_i, x_j\}$ such that every vertex y_i is linked to at least one vertex x_j, under the additional constraint that $(\vec{x}_{[r+2]} \cup \vec{y}_{[m]}, E^+, E_{\text{core}}^- \cup E^-(H))$ has no $(\pm)$-pivotal points for paths from x_1 to x_2. Let us denote the series associated to such graphs H by $h_\lambda^\Lambda(G_{\text{core}})$:

$$h_\lambda^\Lambda(G_{\text{core}}) = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \int_{\Lambda^m} \sum_H \mathbf{w}^\pm(H)\, d\vec{y}_{[m]}. \tag{4.2}$$

The right-hand side of (4.2) depends on G_{core} only through the pivot decomposition $\vec{W}$. We obtain the representation

$$g_\lambda^\Lambda(x_1, x_2) = \sum_{r=1}^{\infty} \frac{\lambda^r}{r!} \int_{\Lambda^r} \sum_{\vec{W}} \sum_{G_{\text{core}}} \mathbf{w}^\pm(G_{\text{core}}) h_\lambda^\Lambda(G_{\text{core}})\, d\vec{x}_{[3,r+3]}. \tag{4.3}$$

This expression, written in a slightly different form (see Definition 4.2), forms the starting point of this section. The main results of this section are the following:

1. Let G_{core} be a $(\pm)$-graph as above. Then the corresponding power series $h_\lambda^\Lambda(G_{\text{core}})$ is absolutely convergent for all intensities $\lambda \ge 0$ (Proposition 4.1). In addition, $h_\lambda^\Lambda(G_{\text{core}})$ can be expressed in terms of probabilities involving the random connection model on

the fixed vertex set $V(G_{\text{core}})$ and of Poisson processes in Λ. This alternative expression is used to show that the (pointwise) limit

$$h_\lambda(G_{\text{core}}) = \lim_{\Lambda \nearrow \mathbb{R}^d} h_\lambda^\Lambda(G_{\text{core}})$$

exists for all $\lambda > 0$ (Lemma 4.5).

2. Then we show in Theorem 4.1 that

$$\sum_{r=0}^\infty \frac{\lambda^r}{r!} \int_{(\mathbb{R}^d)^r} \sum_{\vec{W}} \sum_{G_{\text{core}}} \mathbf{w}^\pm(G_{\text{core}}) \left| h_\lambda(G_{\text{core}}) \right| \, d\vec{x}_{[3,r+3]} < \infty.$$

This allows us to define

$$g_\lambda(x_1, x_2) := \sum_{r=0}^\infty \frac{\lambda^r}{r!} \int_{(\mathbb{R}^d)^r} \sum_{G_{\text{core}}} \mathbf{w}^\pm(G_{\text{core}}) h_\lambda(G_{\text{core}}) \, d\vec{x}_{[3,r+3]}$$

and to pass to the limit in (4.3), showing that

$$\lim_{\Lambda \nearrow \mathbb{R}^d} g_\lambda^\Lambda(x_1, x_2) = g_\lambda(x_1, x_2)$$

as part of Theorem 4.1.

4.2. Definition

Here we introduce the precise definitions of core graphs and shell graphs as well as of the functions h_λ^Λ and g_λ^Λ. We follow the ideas outlined in the previous section but make two small changes. First, shell graphs H are defined not as $(\pm)$-graphs with minus edges only but right away as standard graphs. Second, a close look reveals that the shell function $h_\lambda^\Lambda(G_{\text{core}})$ defined in (4.2) depends on the core graph only via $\vec{W}$; accordingly we view h_λ^Λ as a function of a sequence of sets. In addition we drop the index from the core graph; thus the graph G in Definition 4.1 below corresponds to G_{core} in the previous section (see Figure 2).

Definition 4.1. (*Core graphs and shell graphs.*)

1. Let $x_1, x_2 \in \mathbb{R}^d$ and let $\{x_1, x_2\} \subset W \subset \mathbb{R}^d$ be a finite set of vertices. We call a graph $G \in \mathcal{C}^+(W)$ with $\mathsf{PD}^\pm(x_1, x_2, G) = \mathsf{PD}^+(x_1, x_2, G) = \vec{W}$ a *core* graph with pivot decomposition $\vec{W}$ and denote the set of such graphs by $\mathcal{G}_{\text{core}}^{\vec{W}}$.

2. Let $G \in \mathcal{C}^+(W)$ be a core graph with pivot decomposition $\vec{W} = (u_0, V_0, \ldots, V_k, u_{k+1})$, $k \in \mathbb{N}_0$, where we set $u_0 := x_1$ and $u_{k+1} := x_2$. Moreover, let $\overline{V}_i := V_i \cup \{u_i, u_{i+1}\}$ and let Y be a finite subset of $\mathbb{R}^d$. A *shell* graph on $W \cup Y$ associated to $\vec{W}$ is a $(k+1)$-partite graph $H \in \mathcal{G}(\overline{V}_1, \ldots, \overline{V}_k, Y)$ such that $G \oplus H \in \mathcal{D}_{x_1, x_2}^\pm(W \cup Y)$. We call the vertices $Y \subset V(H)$ satellite vertices and write $S(H) = Y$. Notice that the set of all shell graphs on $W \cup Y$ associated to $\vec{W}$ does not depend on the choice of the core graph G. We denote it by $\mathcal{G}_{\text{shell}}^{Y, \vec{W}}$.

We define h_λ^Λ and g_λ^Λ by expansions similar to (4.2) and (4.3) and postpone the proof of convergence to Proposition 4.3 and Theorem 4.4. By some abuse of language, we refer to the series (4.6) as the direct-connectedness function, and we use the same letter g_λ as in (1.2). This
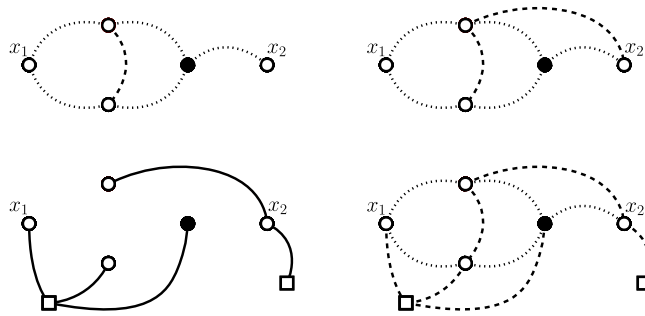
FIGURE 2. In the first line, we see an example of two $(\pm)$-graphs; the $(+)$-edges are depicted by dotted lines and the (minus;)-edges by dashed lines. Notice that both graphs are $(+)$-connected. However, the solid black vertex—which is $(+)$-pivotal for the x_1–x_2 connection in both graphs—is $(\pm)$-pivotal for the x_1–x_2 connection in the graph on the left but not in the graph on the right. Hence, the graph on the left is a core graph according to Definition 4.2 but the graph on the right is not. In the second line, the simple graph on the left is a shell graph for the core graph above, since the $(\pm)$-graph given by their sum (depicted on the right) is $(\pm)$-doubly connected; in particular there are no $(\pm)$-pivotal points for the x_1–x_2 connection.

is justified a posteriori by the proof of Theorem 1.1, where we show that the series is indeed the expansion for the direct-connectedness function g_λ defined as the unique solution of the OZE (1.2).

Definition 4.2. (*Shell functions and direct-connectedness function.*)

1. Let $W \subset \mathbb{R}^d$ be finite and let $\vec{W}$ be given as in Definition 4.2. For $m \in \mathbb{N}_0$, define the *m-shell function $h^{(m)}$* by

$$h^{(m)}(\vec{W}, Y) := \sum_{H \in \mathcal{G}_{\text{shell}}^{Y, \vec{W}}} \mathbf{w}(H), \qquad Y = \{y_1, \ldots, y_m\} \subset \mathbb{R}^d, \qquad (4.4)$$

and the *shell function h_λ^Λ* in finite volume $\Lambda \subset \mathbb{R}^d$ by

$$h_\lambda^\Lambda(\vec{W}) := \sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} h^{(m)}(\vec{W}, \vec{y}_{[m]}) \, d\vec{y}_{[m]}. \qquad (4.5)$$

2. Let $\lambda < \lambda_*$. We define the *direct-connectedness function* as $g_\lambda : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$g_\lambda^\Lambda(x_1, x_2) := \sum_{r \geq 0} \frac{\lambda^r}{r!} \int_{\Lambda^r} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\text{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) h_\lambda^\Lambda(\vec{W}) \, d\vec{x}_{[3, r+2]}, \qquad (4.6)$$

where $W := \{x_1, \ldots, x_{r+2}\}$ and we sum over decompositions $\vec{W}$ of W given as in Definition 4.1. In the pathological case $x_1 = x_2$, (4.6) is to be read as $g_\lambda^\Lambda(x_1, x_2) := 1$. Let $g_\lambda^\Lambda : \mathbb{R}^d \to \mathbb{R}$ be defined by $g_\lambda^\Lambda(x) = g_\lambda^\Lambda(\mathbf{0}, x)$.

The 0-shell function $h^{(0)}$ is understood to be given in terms of shell graphs without satellite vertices, i.e.,

$$h^{(0)}(\vec{W}) = \sum_{H \in \mathcal{G}^{\varnothing, \vec{W}}_{\text{shell}}} \mathbf{w}(H).$$

Note that because of translation-invariance, $g_\lambda^\Lambda(x_1, x_2) = g_\lambda^\Lambda(\mathbf{0}, x_2 - x_1) = g_\lambda^\Lambda(x_2 - x_1)$.

4.3. Analysis of the shell functions: laces

If we take a look at the graphs that are summed over in the shell function, we note that the associated minimal structures have a form which is very reminiscent of graphs that are known as *laces* and famously appear in the analysis of, for example, self-avoiding walks [3, 22]. They are also the namesake of the lace-expansion technique.

Proposition 4.1 is the central result of this section. It allows us to bound the shell function by the probability that the points in a PPP η are not connected to the core vertices W. Moreover, we introduce laces and partition the shell graphs with respect to them. For every lace, we obtain a precise expression for its contribution to the shell function.

To prove Proposition 4.1, we will need quite a few definitions (see Definitions 4.3, 4.4, and 4.5) and some intermediate results thereon.

Proposition 4.1. (Bounds on the shell functions) *Let $\lambda \geq 0$ and let $\Lambda \subset \mathbb{R}^d$ be bounded. Let $u_0, \ldots, u_{k+1} \in \Lambda$ for $k \in \mathbb{N}_0$, let $V_0, \ldots, V_k \subset \Lambda$ be finite sets, and set $\vec{W} = (u_0, V_0, \ldots, V_k, u_{k+1})$. Then*

$$\left| h_\lambda^\Lambda(\vec{W}) \right| \leq \mathbb{P}_\lambda(\eta_\Lambda \longleftrightarrow\!\!\!\!\!| \; W \text{ in } \xi^W). \tag{4.7}$$

Moreover,

$$\sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} \left| h^{(m)}(\vec{W}, \vec{y}_{[m]}) \right| \mathrm{d}\vec{y}_{[m]} \leq \frac{1}{\sqrt{5}} \mathrm{e}^{3\lambda|W|} (3 + \sqrt{5})^{|W|}. \tag{4.8}$$

Proposition 4.1 consists of two parts, and it is (4.8) that guarantees the well-definedness of the shell function h_λ^Λ of Definition 4.2.

Proposition 4.1 is easy to prove for $k = 0$, and we mostly focus on $k \geq 1$. Throughout the remainder of this section, we fix a pivot decomposition $\vec{W} = (u_0, V_0, \ldots, V_k, u_{k+1})$ and recall that $\overline{V}_i = V_i \cup \{u_i, u_{i+1}\}$.

We now work towards a deeper understanding of the shell graphs H summed over in (4.4).

Definition 4.3. (Skeletons) Let $W \subset \mathbb{R}^d$ and let $\vec{W} = (u_0, V_0, \ldots, u_{k+1})$ be a pivot decomposition of some core graph on W. Furthermore, let $Y \subset \mathbb{R}^d$ be finite and let H be a shell graph associated to $\vec{W}$ with satellite vertices $S(H) = Y$. Then we define the skeleton $\hat{H}$ of H as the following graph: its vertex set is $V(\hat{H}) = \{0, \ldots, k + 1\}$. A bond $\alpha\beta$ is in $E(\hat{H})$ if and only if $|\alpha - \beta| \geq 2$ and there exist $s \in \{u_\alpha\} \cup V_\alpha$, $t \in V_{\beta-1} \cup \{u_\beta\}$ such that

- $st \in E(H)$, or

- $sy, yt \in E(H)$ for some $y \in S(H)$.

In the first case we call $\{s, t\}$ a *direct stitch*, and in the second case we call it an *indirect stitch*. We call an edge $\alpha\beta$ in $E(\hat{H})$ a *bond* to distinguish it from the edge of the underlying graph H.

Thus, the graph $\hat{H}$ has no nearest-neighbor bonds, and $\alpha\beta$ with $|\alpha - \beta| \geq 2$ is a bond in $E(\hat{H})$ if and only if $\{u_\alpha\} \cup V_\alpha$ and $V_{\beta-1} \cup \{u_\beta\}$ are connected by a direct or indirect stitch. See

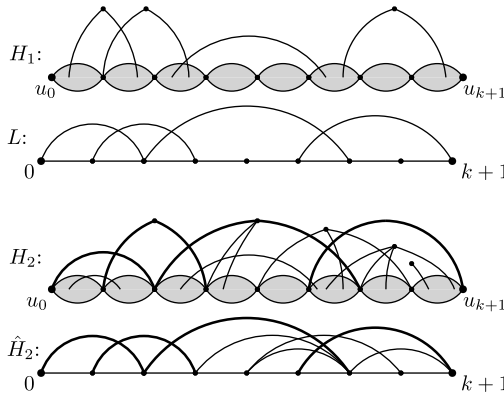FIGURE 3. In the first line, we see a schematic shell graph H_1. Its skeleton $\hat{H}_1$ is already a lace, namely L. The skeleton of the graph H_2 in the second line is not a lace, but $H_2 \in \langle\langle L \rangle\rangle$. The structure of L is indicated in H_2 and in $\hat{H}_2$ by the thicker edges.

Figure 3 for an illustration. We may now apply the standard vocabulary of lace expansion (for self-avoiding walks) to the graph $\hat{H}$ [22, Section 3.3].

Definition 4.4. (*Laces*)

- The graph $\hat{H}$ with vertex set $\{0, \ldots, k+1\}$ is *irreducible* if 0 and $k+1$ are endpoints of edges in $E(\hat{H})$ and for every $i \in [k]$ there exists $\alpha\beta \in E(\hat{H})$ with $\alpha < i < \beta$.

- The graph $\hat{H}$ is a *lace* if it is irreducible and, for every bond $\alpha\beta \in E(\hat{H})$, removal of the bond destroys the irreducibility.

- We denote by $\mathcal{L}_k$ the set of all laces on $\{0, \ldots, k+1\}$.

In the context of lace expansions, usually the word 'connected' is used instead of 'irreducible', but 'connected' is clearly misleading in our setup; Brydges and Spencer originally called those graphs 'primitive' [3]. We observe that the skeleton graphs $\hat{H}$ arising from our shell graphs H are precisely the irreducible graphs (and so $G \oplus H$ being 2-connected corresponds to the skeleton $\hat{H}$ being irreducible).

We map irreducible graphs to laces by following a standard procedure [22, Section 3.3], performed backwards. That is, we define bonds $\alpha'_j\beta'_j$ with $\beta'_1 > \beta'_2 > \cdots$ inductively as follows: we set

$$\beta'_1 := k+1, \qquad \alpha'_1 := \min\{\alpha : \alpha\beta'_1 \in E(\hat{H})\},$$

and

$$\alpha'_{j+1} = \min\{\alpha : \exists \beta > \alpha'_j \text{ with } \alpha\beta \in E(\hat{H})\}, \qquad \beta'_{j+1} = \max\{\beta : \alpha'_{j+1}\beta \in E(\hat{H})\}.$$

The procedure terminates when $\alpha'_j = 0$. At the end, we let $\alpha_j\beta_j$ be a relabeling of the bonds $\alpha'_j\beta'_j$ from left to right.

It is well known that the algorithm maps irreducible graphs to laces; moreover, the set of irreducible graphs that are mapped to a given lace L can be characterized as follows.

Definition 4.5. (*Compatible bonds and the span of a lace.*)

1. Let L be a lace with vertex set $\{0, \ldots, k+1\}$. A bond is *compatible* with a lace L if the algorithm described above maps the graph $(V(L), E(L) \cup \{\alpha\beta\})$ to the lace L.

2. Let $W \subset \mathbb{R}^d$ and let $\vec{W} = (u_0, V_0, \ldots, u_{k+1})$ be a pivot decomposition of some core graph on W. Further let $Y \subset \mathbb{R}^d$ be finite and let H be a shell graph associated to $\vec{W}$ with $S(H) = Y$. Then we say that H belongs to the span of the lace L, and write $H \in \langle\!\langle L \rangle\!\rangle$, if $E(L) \subseteq E(\hat{H})$ and every bond $\alpha\beta \in E(\hat{H}) \setminus E(L)$ is compatible with L.

In other words, H is in the span of L if the above algorithm maps $\hat{H}$ to L. See Figure 3. Given $\vec{W}$ and a lace L, we define

$$h_\lambda^\Lambda(\vec{W}; L) := \sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} \sum_{H \in \langle\!\langle L \rangle\!\rangle \,:\, S(H) = \vec{y}_{[m]}} \mathbf{w}(H) \, d\vec{y}_{[m]}. \tag{4.9}$$

The series $h_\lambda^\Lambda(\vec{W}; L)$ converges absolutely for every fixed λ. This is shown as part of the proof of (4.8) in Proposition 4.1. Now,

$$h_\lambda^\Lambda(\vec{W}) = \sum_{L \in \mathcal{L}_k} h_\lambda^\Lambda(\vec{W}; L).$$

The following characterization of compatible bonds will be useful. We recall that the bonds of a lace with m bonds can be labeled as $\alpha_j \beta_j$ with

$$0 = \alpha_1 < \alpha_2 < \beta_1 \leq \alpha_3 < \beta_2 \leq \cdots \leq \alpha_m < \beta_{m-1} < \beta_m = k+1;$$

see [22, Equations (3.15) and (3.16)].

Lemma 4.1. (Characterization of compatible bonds.) *Let L be a lace with vertex set $V(L) = \{0, \ldots, k+1\}$ and bonds $\alpha_j \beta_j$, $j = 1, \ldots, m$, labeled from left to right (i.e., $\alpha_j < \alpha_{j+1}$). Then a bond $\alpha\beta \notin E(L)$ with $\alpha < \beta - 1$ is compatible with L if and only if either*

(a) $\alpha_i \leq \alpha < \beta \leq \beta_i$ *for $i \in [m]$ or*

(b) $\alpha_i < \alpha < \beta \leq \alpha_{i+2}$ *for $i \in [m-1]$ (where we set $\alpha_{m+1} := k$).*

Proof. Let $\alpha\beta \notin E(L)$ be compatible with L; that is, the algorithm below Definition 4.4 maps $E(L) \cup \{\alpha\beta\}$ to $E(L)$, which in turn means that $\alpha\beta$ is not selected to be part of the output lace. We show that then either (a) or (b) is satisfied. Assume the algorithm has already constructed the partial lace up to some $j < m$, producing the bonds $(\alpha_i', \beta_i')_{i=1}^j$ (note that they are in reverse order and make up the last j bonds of the lace). Assume moreover that $\alpha_j' < \beta \leq \alpha_{j-1}'$; that is, $\alpha\beta$ is a potential candidate to be chosen as the next bond of the lace. Since it is *not* chosen, there is $\alpha_{j+1}' \beta_{j+1}'$ with $\beta_{j+1}' \in (\alpha_j', \alpha_{j-1}']$ such that either

- $\alpha_{j+1}' < \alpha$, or
- $\alpha_{j+1}' = \alpha$ and $\beta_{j+1}' > \beta$.

Both the second case and the first case under the additional assumption $\beta_{j+1}' \geq \beta$ imply that $\alpha\beta$ satisfies (a). Let us thus focus on the case where $\alpha_{j+1}' < \alpha$ and $\beta_{j+1}' < \beta$. Remembering the stage of the algorithm, we have $\beta \leq \alpha_{j-1}'$, implying (b).
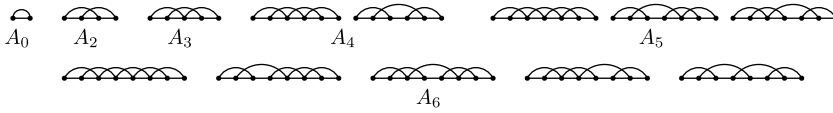
FIGURE 4. Schematic illustration of A_i from the proof of Lemma 4.2 for $i = 0, 2, 3, 4, 5, 6$.

Now let $\alpha\beta \notin E(L)$ be a bond that satisfies (a) or (b). We claim that $\alpha\beta$ is compatible with L. Let i be the index such that $\alpha_i\beta_i$ satisfies (a) or (b). Note that in the execution of the algorithm below Definition 4.3, $\alpha\beta$ does not appear as a candidate to be added to the constructed lace up until the point where $\alpha_m\beta_m, \alpha_{m-1}\beta_{m-1}, \ldots, \alpha_{i+1}\beta_{i+1}$ have already been added to the partial lace. At this stage of the algorithm, if $\alpha\beta$ satisfies (b), then it is not picked, because the left endpoint of the bond $\alpha_i\beta_i$ has a smaller value (i.e., $\alpha_i < \alpha$). If $\alpha\beta$ satisfies (a), however, then either also $\alpha_i < \alpha$, or $\alpha_i = \alpha$, but $\alpha_i\beta_i$ has its right endpoint further to the right (i.e., $\beta < \beta_i$, since the two bonds cannot be equal), and so again, $\alpha_i\beta_i$ is picked by the algorithm. $\square$

To prove the second result of Proposition 4.1, we need the following counting lemma, which may be of independent interest.

Lemma 4.2. (On the number of laces.) *Let f_i be the ith Fibonacci number with $f_1 = 0$, $f_2 = 1$. Then*

$$|\mathcal{L}_k| = 1 + \sum_{i=1}^{k} \binom{k}{i} f_i \qquad and, \ as \ k \to \infty, \qquad |\mathcal{L}_k| \sim \frac{1}{\sqrt{5}} \left(\frac{3 + \sqrt{5}}{2} \right)^k.$$

Proof. We first choose i vertices in $\{1, \ldots, k\}$ and then count the laces that use exactly those vertices. To this end, let A_i be the set of laces L with $V(L) = \{0, \ldots, i+1\}$ so that every vertex is the endpoint of at least one stitch. We claim that $|A_i| = f_i$ for $i \geq 1$. Clearly, $|A_0| = 1$, $|A_1| = 0$, $|A_2| = 1$. See Figure 4 for an illustration.

Let $i \geq 3$. We now establish the Fibonacci recursion. First, note that the bond incident to 0 (the 'first' bond) must always have 2 as the second endpoint. Now, depending on whether or not the third bond is incident to 2, the remaining lace lives on $\{1, 2, \ldots, i+1\}$ or on $\{1, 3, 4, \ldots, i+1\}$, and so $|A_i| = |A_{i-1}| + |A_{i-2}|$.

The asymptotic behavior follows from the fact that $f_n \sim \Phi^n/\sqrt{5}$, where $\Phi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. $\square$

We can now work towards finding an explicit expression for $h_\lambda^\Lambda(\vec{W}; L)$ for a fixed lace. The next lemma is in the spirit of Observation 3.1 and will help us find probabilistic factors in the shell function.

Lemma 4.3. (Bipartite graphs and probabilities.) *Let $Y, A, B, C \subset \mathbb{R}^d$ be finite, disjoint sets.*

1. *Then*

$$\sum_{\substack{H \in \mathcal{G}(A \cup C, Y): \\ \forall y \in Y : y \sim A}} \mathbf{w}(H) = \prod_{y \in Y} \left(-\mathbb{P}(A \sim y \nsim C) \right) = (-1)^{|Y|} \mathbb{P}(\forall y \in Y : A \sim y \nsim C).$$

2. *Moreover,*

$$\sum_{\substack{H \in \mathcal{G}(A \cup B \cup C, Y): \\ \forall y \in Y : A \sim y \sim B}} \mathbf{w}(H) = \prod_{y \in Y} \mathbb{P}(A \sim y \sim B, y \nsim C).$$

3. Lastly,

$$\sum_{\substack{H \in \mathcal{G}(A,Y): \\ E(H) \neq \varnothing}} \mathbf{w}(H) = -\mathbb{P}(A \sim Y).$$

Proof. The first part of the statement is rather straightforward. If $Y = \{y\}$, then $\mathcal{G}(A \cup C, \{y\})$ is the set of star graphs (with center y). Observe first that

$$\sum_{H \in \mathcal{G}(A \cup C, \{y\}): y \sim A} \mathbf{w}(H) = \left(\sum_{H' \in \mathcal{G}(A, \{y\}): y \sim A} \mathbf{w}(H') \right) \left(\sum_{H'' \in \mathcal{G}(C, \{y\})} \mathbf{w}(H'') \right).$$

The first sum is over all star graphs in $\mathcal{G}(A, \{y\})$ except the empty one, the second is over all star graphs in $\mathcal{G}(C, \{y\})$, and so

$$\sum_{H \in \mathcal{G}(A \cup C, \{y\}): y \sim A} \mathbf{w}(H) = -\left(1 - \prod_{x \in A}(1 - \varphi(y, x)) \right) \prod_{x \in C}(1 - \varphi(y, x)) = -\mathbb{P}(A \sim y \approx C).$$

It is an easy induction to prove that for general Y, the sum factors into a product over sums over star graphs. For the second statement, assume again that $Y = \{y\}$ and observe that

$$\sum_{\substack{H \in \mathcal{G}(A \cup B \cup C, \{y\}): \\ A \sim y \sim B}} \mathbf{w}(H) = \left(\sum_{H \in \mathcal{G}(A \cup C, \{y\}): y \sim A} \mathbf{w}(H) \right) \left(\sum_{H \in \mathcal{G}(B, \{y\}): y \sim B} \mathbf{w}(H) \right)$$

$$= \mathbb{P}(A \sim y \sim B, y \approx C),$$

where the last identity is due to independence. The statement easily extends to general Y (again, the sum factors).

For the third statement, note that we sum over every graph except the empty one. $\qquad\square$

Since the explicit expression for $h_\lambda^\Lambda(\vec{W}; L)$ is a lengthy product of probabilities, we first introduce some notation to represent the factors of this product compactly. Let A, B be two subsets of $[k+1]_0$. We define the set of all possible direct stitches in H leading to bonds $\alpha\beta \in E(\hat{H})$ with $\alpha \in A$, $\beta \in B$ as

$$\Upsilon(A, B) := \{xy \subset W : \exists \alpha \in A, \beta \in B \text{ with } \alpha < \beta - 1 \text{ and } x \in \{u_\alpha\} \cup V_\alpha, \ y \in V_{\beta-1} \cup \{u_\beta\}\},$$

and we write $\Upsilon(A) = \Upsilon(A, A)$. We define

$$q_{\alpha,\beta} := \prod_{xy \in \Upsilon([\alpha,\beta)) \cup \Upsilon((\alpha,\beta])} (1 - \varphi(x - y))$$

and, for $\alpha_1 < \alpha_2 < \alpha_3$,

$$q_{\alpha_1,\alpha_2,\alpha_3} := \prod_{xy \in \Upsilon([\alpha_1+1,\alpha_2),[\alpha_2,\alpha_3))} (1 - \varphi(x - y)).$$

Note that these products encode the sum over all $\mathbf{w}$-weighted graphs on the set of edges multiplied over.

To lighten notation, for $0 \leq \alpha \leq \beta \leq k+1$, set

$$[\![u_\alpha]\!] := \{u_\alpha\} \cup V_\alpha, \qquad [\![u_\beta]\!] := V_{\beta-1} \cup \{u_\beta\},$$
$$[\![u_\alpha, u_\beta]\!] := \{u_\alpha\} \cup V_\alpha \cup \cdots \cup V_{\beta-1} \cup \{u_\beta\}.$$
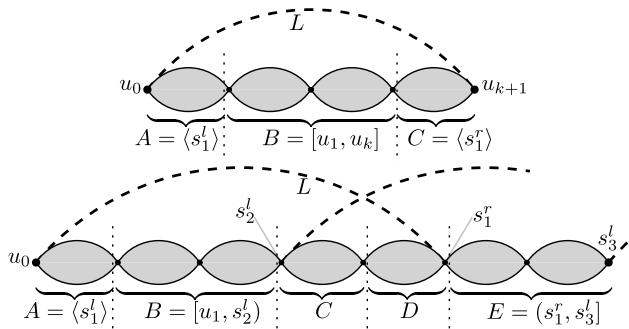
FIGURE 5. Illustration of the induction proof of Lemma 4.4. The lace L is sketched using dashed lines. The left picture shows the base case $m = 1$, where $u_0 = s_1$ and $u_{k+1} = t_1$. To the right, the first three stitches of L are (partially) sketched. The sets C, D are defined as $C = [s_2, t_1)\!)$ and $D = [\![t_1]\!]$.

We extend this notation further: for $a, b \in \{u_0, \ldots, u_{k+1}\}$, let $(a, b) := [a, b] \setminus \{a, b\}$, let $[a, b) := [a, b] \setminus \{b\}$, and let $(a, b] := [a, b] \setminus \{a\}$. We set $(\!(a, b)\!) := [a, b] \setminus ([\![a]\!] \cup [\![b]\!])$ and define sets $(\!(a, b]$ etc. accordingly.

Moreover, define

$$Q_{\alpha, \beta} = \mathbb{P}_\lambda \big(\nexists y \in \eta_\Lambda \text{ s.t. } [u_\alpha]\!] \sim y \sim [\![u_\beta], \, y \nsim [u_{\alpha+1}, u_{\beta-1}] \big)$$

for $\beta \geq \alpha + 2$. We extend this notation by writing

$$Q_{A, B} = \prod_{\alpha \in A} \prod_{\beta \in B} Q_{\alpha, \beta}$$

for sets of pivotal points A, B; we abbreviate $Q_{a, [b, c]} = Q_{\{a\}, [b, c]}$.

We are now ready to state Lemma 4.4, for which we recall the definition of $h_\lambda^\Lambda(\vec{W}; L)$ in (4.9).

Lemma 4.4. (The shell function of a lace.) *Let $\lambda \geq 0$ and let $\Lambda \subset \mathbb{R}^d$ be bounded. Let $W \subset \mathbb{R}^d$ be a core vertex set with pivot decomposition $\vec{W} = (u_0, V_0, \ldots, u_{k+1})$. Let L be a lace with vertex set $[k+1]_0$ and m bonds $\alpha_i \beta_i$, $i \in [m]$. Then, setting $\alpha_{m+1} = k$, we have*

$$h_\lambda^\Lambda(\vec{W}; L) = \mathbb{P}_\lambda(\eta_\Lambda \longleftrightarrow\!\!\!\!\!/\, W) \prod_{i=1}^{m} q_{\alpha_i, \beta_i} \Big[1 - Q_{\alpha_i, \beta_i} - \mathbb{P}([u_{\alpha_i}]\!] \sim [\![u_{\beta_i}]) \Big]$$

$$\times \prod_{i=1}^{m-1} q_{\alpha_i, \alpha_{i+1}, \alpha_{i+2}} Q_{\alpha_i, (\beta_i, k+1]} Q_{(\alpha_i, \alpha_{i+1}), (\alpha_{i+2}, k+1]}. \tag{4.10}$$

Moreover,

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \Big| \sum_{H \in (\!(L)\!) : \mathcal{S}(H) = \vec{y}_{[n]}} \mathbf{w}(H) \Big| \, \mathrm{d}\vec{y}_{[n]} \leq 2^m \mathrm{e}^{3\lambda |W|}. \tag{4.11}$$

Proof. We abbreviate $\eta = \eta_\Lambda$, $h = h_\lambda^\Lambda$, and prove the statement by induction on m.

Base case. Let $m = 1$, which means that $\alpha_1 = 0$ and $\beta_1 = k + 1$. Set $A = [u_0]\!]$, $B = [u_1, u_k]$, and $C = [\![u_{k+1}]$. See Figure 5 for an illustration of A, B, C.

Note first that the edge set $\Upsilon([k+1]_0) \setminus E(A, C)$, that is, the possible direct stitches between points of W except the direct ones between A and C, do not determine membership of H in

$\langle\!\langle L \rangle\!\rangle$. Any such edge xy may or may not be present, resulting in a factor $(1 - \varphi(x - y))$ that can be extracted. In total, this produces the factor $q_{0,k+1}$, and we can restrict to considering graphs $H \in \langle\!\langle L \rangle\!\rangle$ that do not possess any such edge. The remaining graphs H only have edges that are incident to $A \cup C \cup \mathcal{S}(H)$.

We split this set of remaining graphs H into those that have a direct stitch between A and C and those that do not. Among the former, the sum over graphs factors into graphs $H' \in \mathcal{G}(A, C)$ (the direct stitches) and graphs $H'' \in \mathcal{G}(W, \mathcal{S}(H))$. With Lemma 4.3,

$$
h(\vec{W}; L) = q_{0,k+1} \left[\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left(\sum_{\substack{H' \in \mathcal{G}(A,C) : E(H') \neq \varnothing}} \mathbf{w}(H') \right) \left(\sum_{\substack{H'' \in \mathcal{G}(W, \vec{y}_{[n]}) : \\ y_i \sim W \forall i \in [n]}} \mathbf{w}(H'') \right) \mathrm{d}\vec{y}_{[n]} \right.
$$

$$
\left. + \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(W, \vec{y}_{[n]}) : \\ \deg(y_i) \geq 1 \forall i \in [n], \\ \exists i : A \sim y_i \sim C}} \mathbf{w}(H) \, \mathrm{d}\vec{y}_{[n]} \right]
$$

$$
= q_{0,k+1} \left[-\mathbb{P}(A \sim C) \mathbb{P}_\lambda(\eta \not\longleftrightarrow W) + \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(W, \vec{y}_{[n]}) : \\ \deg(y_i) \geq 1 \forall i \in [n], \\ \exists i : A \sim y_i \sim C}} \mathbf{w}(H) \, \mathrm{d}\vec{y}_{[n]} \right].
$$

(4.12)

For now the power series are treated as formal power series; convergence is proven later. To treat the sum in (4.12), we define

$$
\mathcal{S}_1 := \{y : A \sim y \sim C\}, \quad \mathcal{S}_2 := \{y : C \not\sim y \sim (A \cup B)\}, \quad \text{and } \mathcal{S}_3 := \{y : C \sim y \not\sim A\}.
$$

With these definitions, we can partition $\vec{y} = \mathcal{S}(H) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Moreover, we know that $\mathcal{S}_1 \neq \varnothing$. Re-summing and then applying Lemma 4.3, the sum over n in (4.12) becomes

$$
\sum_{n_1, n_2, n_3 \geq 0} \frac{\lambda^{n_1 + n_2 + n_3}}{n_1! n_2! n_3!} \int_{\Lambda^{n_1 + n_2 + n_3}} \sum_{\substack{H \in \langle\!\langle L \rangle\!\rangle : \\ \mathcal{S}_i(H) = \vec{y}_{i,[n_i]} \forall i \in [3]}} \mathbf{w}(H) \, \mathrm{d}(\vec{y}_{1,[n_1]}, \vec{y}_{2,[n_2]}, \vec{y}_{3,[n_3]})
$$

$$
= \left(\sum_{n \geq 1} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(A \cup B \cup C, \vec{y}_{[n]}) : \\ \forall i \in [n] : A \sim y_i \sim C}} \mathbf{w}(H) \, \mathrm{d}\vec{y}_{[n]} \right) \left(\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(A \cup B, \vec{y}_{[n]}) : \\ \forall i \in [n] : y_i \sim (A \cup B)}} \mathbf{w}(H) \, \mathrm{d}\vec{y}_{[n]} \right)
$$

$$
\times \left(\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(B \cup C, \vec{y}_{[n]}) : \\ \forall i \in [n] : y_i \sim C}} \mathbf{w}(H) \, \mathrm{d}\vec{y}_{[n]} \right)
$$

$$
= \left(\sum_{n \geq 1} \frac{\lambda^n}{n!} \left(\int_\Lambda \mathbb{P}(A \sim y \sim C, y \not\sim B) \mathrm{d}y \right)^n \right) \left(\sum_{n \geq 0} \frac{\lambda^n}{n!} \left(-\int_\Lambda \mathbb{P}(y \sim (A \cup B)) \, \mathrm{d}y \right)^n \right)
$$

$$
\times \left(\sum_{n \geq 0} \frac{\lambda^n}{n!} \left(-\int_\Lambda \mathbb{P}(C \sim y \not\sim B) \, \mathrm{d}y \right)^n \right).
$$

(4.13)

Recognizing the exponential series in the expression above, we can rewrite the probabilities with respect to $\mathbb{P}$ (appearing in the exponents) as probabilities with respect to $\mathbb{P}_\lambda$ associated to ξ^y, e.g., $\mathbb{P}(y \sim (A \cup B)) = \mathbb{P}_\lambda(y \sim (A \cup B)$ in $\xi^y)$. Then we can apply the univariate Mecke formula (see (2.1) for $m = 1$) to rewrite (4.13) as

$$\left(e^{\mathbb{E}_\lambda[|\{y \in \eta: A \sim y \sim C, y \nsim B\}|]} - 1\right)e^{-\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B)\}|]}e^{-\mathbb{E}_\lambda[|\{y \in \eta: C \sim y \nsim B\}|]}$$

$$= \left(1 - e^{-\mathbb{E}_\lambda[|\{y \in \eta: A \sim y \sim C, y \nsim B\}|]}\right)e^{-\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B)\}|]}e^{-\mathbb{E}_\lambda[|\{y \in \eta: C \sim y \nsim (A \cup B)\}|]}$$

$$= (1 - Q_{0,k+1})e^{-\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B \cup C)\}|]}.$$

Since $e^{-\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B \cup C)\}|]} = \mathbb{P}_\lambda(\eta \longleftrightarrow\!\!\!\!\!/ \; W)$, we can plug this back into (4.12) and obtain

$$h(\vec{W}; L) = \mathbb{P}_\lambda(\eta \longleftrightarrow\!\!\!\!\!/ \; W)q_{0,k+1}\left(1 - Q_{0,k+1} - \mathbb{P}(A \sim C)\right)$$

on the level of formal power series. Now we prove convergence and check that the previous computational steps are justified not only on the level of formal power series. We first revisit Equation (4.13). On the left-hand side, let us put absolute values inside the integral (but outside the sum over shell graphs H). The resulting expression is bounded by the middle part of (4.13), again with absolute values inside the integral. Each integrand is bounded in absolute value by a probability; hence it is smaller than or equal to 1. The resulting series are exponential series and, in particular, absolutely convergent. As a consequence, Equation (4.13) is justified and the last sum in (4.12) is bounded as

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left| \sum_{\substack{H \in \mathcal{G}(W, \vec{y}_{[n]}): \\ \deg(y_i) \geq 1 \forall i \in [n], \\ \exists i: A \sim y_i \sim C}} \mathbf{w}(H) \right| \, d\vec{y}_{[n]}$$

$$\leq e^{\mathbb{E}_\lambda[|\{y \in \eta: A \sim y \sim C, y \nsim B\}|]}e^{\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B)\}|]}e^{\mathbb{E}_\lambda[|\{y \in \eta: C \sim y \nsim B\}|]}$$

$$\leq e^{\mathbb{E}_\lambda[|\{y \in \eta: y \sim A\}|]}e^{\mathbb{E}_\lambda[|\{y \in \eta: y \sim (A \cup B)\}|]}e^{\mathbb{E}_\lambda[|\{y \in \eta: y \sim C\}|]}$$

$$\leq e^{2\lambda|W|}, \tag{4.14}$$

where for the last inequality we use the fact that the expected number of direct neighbors of any fixed element of W with respect to η is given by $\lambda \int \varphi(x) \, dx$, as well as the rescaling introduced in Section 2.3 ensuring that $\int \varphi(x) \, dx = 1$; compare this bound to the one used in (3.4). For the other contribution to $h(\vec{W}; L)$, we notice that

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left| \left(\sum_{H' \in \mathcal{G}(A,C): E(H) \neq \varnothing} \mathbf{w}(H')\right)\left(\sum_{\substack{H'' \in \mathcal{G}(W, \vec{y}_{[n]}): \\ y_i \sim W \forall i \in [n]}} \mathbf{w}(H'')\right) \right| \, d\vec{y}_{[n]}$$

$$\leq \mathbb{P}(A \sim C)\, e^{\mathbb{E}_\lambda[|\{y \in \eta: y \sim W\}|]} \leq e^{\lambda|W|}, \tag{4.15}$$

by the same argument as in (4.14).

Combining (4.14) and (4.15) with (4.12) and $0 \leq q_{0,k+1} \leq 1$, we deduce

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \left| \sum_{H \in \langle\!\langle L \rangle\!\rangle : \mathcal{S}(H) = \vec{y}_{[n]}} \mathbf{w}(H) \right| \, d\vec{y}_{[n]} \leq e^{\lambda |W|} + e^{2\lambda |W|} \leq 2e^{2\lambda |W|} < \infty.$$

Inductive step. For the inductive step, let $m > 1$. We write the lace L in terms of its vertices (s_i, t_i) in W (that is $s_i = u_{\alpha_i}$ and $t_i = u_{\beta_i}$) and let L' be the lace on $W' := W \setminus [s_1, s_2)$ obtained from L by deleting the first stitch. We note that if $H \in \langle\!\langle L \rangle\!\rangle$, then $H[[s_2, u_{k+1}]] \in \langle\!\langle L' \rangle\!\rangle$. Observe that

$$h(\vec{W}; L) = h(\vec{W}'; L') \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(\overline{V}_0, \ldots, \overline{V}_{\alpha_3 - 1}, \vec{y}_{[n]}) : \\ H \oplus L' \in \langle\!\langle L \rangle\!\rangle}} \mathbf{w}(H) \, d\vec{y}_{[n]}. \tag{4.16}$$

Again we first prove (4.10) and carry out computations on the level of formal power series; we prove convergence (and thus (4.11)) at the end. We can apply the induction hypothesis to $h(\vec{W}'; L)$; it remains to deal with the second factor. We partition the vertices in $[s_1, s_3]$ as $A = [s_1]\!]$, $B = (\!(s_1, s_2)\!)$, $C = [s_2, t_1)\!)$, $D = [\![t_1]\!]$, and $E = (t_1, s_3]$ (see Figure 5). If $m = 2$, we let $E = (t_1, u_k]$.

The graphs summed over in (4.16) must satisfy the following restraints: there must be at least one direct or indirect stitch between A and D, and there cannot be any (direct or indirect) edge between A and E. In particular, the remaining direct stitches may or may not be there, and thus can be extracted as the factor $q_{\alpha_1, \alpha_2, \alpha_3}$.

We partition $\mathcal{S}(H) = \cup_{i=1}^4 \mathcal{S}_i$, where

$$\mathcal{S}_1 = \{y : A \sim y \sim D, N(y) \subseteq [s_1, t_1]\}, \qquad \mathcal{S}_2 = \{y : A \sim y \sim C, N(y) \subseteq [s_1, t_1)\!)\},$$
$$\mathcal{S}_3 = \{y : \varnothing \neq N(y) \subseteq [s_1, s_2)\}, \qquad \mathcal{S}_4 = \{y : B \sim y \sim (C \cup D \cup E), N(y) \subseteq (\!(s_1, s_3]\}.$$

Again, we intend to split the sum over graphs into those that have at least one direct stitch between A and D, and those that do not. We can thus rewrite the second factor in (4.16) as

$$q_{\alpha_1, \alpha_2, \alpha_3} \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}(\overline{V}_0, \ldots, \overline{V}_{\alpha_3 - 1}, \vec{y}_{[n]}) : \\ H \oplus L' \in \langle\!\langle L \rangle\!\rangle, \\ \forall e \in E(H) : e \cap (A \cup D \cup \vec{y}_{[n]}) \neq \varnothing}} \mathbf{w}(H) \, d\vec{y}_{[n]}$$

$$= q_{\alpha_1, \alpha_2, \alpha_3} \prod_{i=2}^4 \left(\sum_{n_i \geq 0} \frac{\lambda^{n_i}}{n_i!} \int_{\Lambda^{n_i}} \sum_{\substack{H \in \mathcal{G}(W, \vec{y}_{i, [n_i]}) : \\ \mathcal{S}(H) = \mathcal{S}_i}} \mathbf{w}(H) \, d\vec{y}_{i, [n_i]} \right)$$

$$\times \left[-\mathbb{P}(A \sim D) \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}([s_1, s_3], \vec{y}_{[n]}) : \\ y_i \sim A \cup B \forall i \in [n]}} \mathbf{w}(H) \, d\vec{y}_{[n]} \right.$$

$$\left. + \sum_{n \geq 1} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\substack{H \in \mathcal{G}([s_1, s_3], \vec{y}_{[n]}) : \\ y_i \sim A \cup B \forall i \in [n]}} \mathbf{w}(H) \, d\vec{y}_{[n]} \right]$$

$$= q_{\alpha_1, \alpha_2, \alpha_3} \left(-\mathbb{P}(A \sim D) e^{\mathbb{E}_\lambda [|\{y \in \eta : y \in \mathcal{S}_1\}|]} + e^{\mathbb{E}_\lambda [|\{y \in \eta : y \in \mathcal{S}_1\}|]} - 1 \right)$$

$$\times \exp\left\{ \mathbb{E}_\lambda [|\{y \in \eta : y \in \mathcal{S}_2\}|] - \mathbb{E}_\lambda [|\{y \in \eta : y \in \mathcal{S}_3\}|] + \mathbb{E}_\lambda [|\{y \in \eta : y \in \mathcal{S}_4\}|] \right\}, \tag{4.17}$$

where the last identity was obtained using Lemma 4.3. Note that the factor $h(\vec{W}'; L')$ contains the factor $\mathbb{P}(\eta \leftrightarrow\!\!\!\!\!/\ [s_2, u_{k+1}]) = \mathrm{e}^{-\mathbb{E}_\lambda[|\{y \in \eta : y \sim [s_2, u_{k+1}]\}|]}$. Together with this factor, (4.17) equals

$$\exp\{\mathbb{E}_\lambda[-|\{y \in \eta : (A \cup B \sim y \sim [s_2, u_{k+1}]\}| + |\{y \in \eta : A \sim y \sim D, y \not\sim (B \cup C)\}|$$
$$+ |\{y \in \eta : A \sim y \sim C, y \not\sim B\}| + |\{y \in \eta : B \sim y \sim (C \cup D \cup E)\}|]\} \qquad (4.18)$$
$$\times \mathbb{P}_\lambda(\eta \leftrightarrow\!\!\!\!\!/\ W)(1 - Q_{A,D} - \mathbb{P}(A \sim D)).$$

It remains to rewrite the argument in the expectation of the exponent in (4.18). Note that

$$- |\{y \in \eta : A \sim y \sim [s_2, u_{k+1}], y \not\sim B\}| - |\{y \in \eta : B \sim y \sim [s_2, u_{k+1}]\}|$$
$$+ |\{y \in \eta : A \sim y \sim (C \cup D), y \not\sim B\}| + |\{y \in \eta : B \sim y \sim (C \cup D \cup E)\}|$$
$$= - |\{y \in \eta : A \sim y \sim (t_1, u_{k+1}], y \not\sim (B \cup C \cup D)\}|$$
$$- |\{y \in \eta : B \sim y \sim (s_3, u_{k+1}], y \not\sim (C \cup D \cup E)\}|.$$

This gives two exponential terms. The first is

$$\exp\Big\{ - \mathbb{E}_\lambda\big[|\{y \in \eta : [u_{\alpha_1}[\!\![\sim y \sim (u_{\beta_1}, u_{k+1}], y \not\sim (\!(u_{\alpha_1}, u_{\beta_1}]\!]\}|\big]\Big\}$$
$$= \prod_{j=\beta_1+1}^{k+1} \exp\Big\{-\mathbb{E}_\lambda\big[|\{y \in \eta : [u_{\alpha_1}[\!\![\sim y \sim [\!\![u_j], y \not\sim (\!(u_{\alpha_1}, u_j)\!)\}|\big]\Big\}$$
$$= Q_{\alpha_1,(\beta_1,k+1]}.$$

Similarly, the second exponential term equals $Q_{(\alpha_1,\alpha_2),(\alpha_3,k+1]}$.

Again, we prove convergence and justify the previous computational steps. Revisiting the left-hand side of (4.17), we insert absolute values inside the integral (and outside the sum over graphs H). As in the base case, this is bounded by the middle part of (4.17) with absolute values in the integrals, and each integrand is a probability. With the Mecke equation, we obtain

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \Bigg| \sum_{\substack{H \in \mathcal{G}(\overline{V}_0, \ldots, \overline{V}_{\alpha_3 - 1}, \vec{y}_{[n]}): \\ H \oplus L' \in \langle\!\langle L \rangle\!\rangle, \\ \forall e \in E(H): e \cap (A \cup D \cup \vec{y}_{[n]}) \neq \varnothing}} \mathbf{w}(H) \Bigg| \, \mathrm{d}\vec{y}_{[n]}$$
$$\leq 2 \exp\Big\{\mathbb{E}_\lambda[|\{y \in \eta : y \in \mathcal{S}_1\}|] + \mathbb{E}_\lambda[|\{y \in \eta : y \in \mathcal{S}_2\}|]$$
$$+ \mathbb{E}_\lambda[|\{y \in \eta : y \in \mathcal{S}_3\}|] + \mathbb{E}_\lambda[|\{y \in \eta : y \in \mathcal{S}_4\}|]\Big\}$$
$$\leq 2\mathrm{e}^{3\lambda|A \cup B|},$$

arguing as in (4.14) for the last inequality.

Note that by the induction hypothesis, the term $h(\vec{W}'; L')$ with absolute values in the respective integrals is bounded by $2^{m-1}\mathrm{e}^{3\lambda|W'|}$. Since $A \cup B$ and W' are disjoint, this proves (4.11). $\qquad \square$

Proof of Proposition 4.1. Again, we abbreviate $\eta = \eta_\Lambda$ and $h = h_\lambda^\Lambda$. First, consider $k = 0$, i.e., pivot decompositions with no pivotal points. Then there are no direct stitches, and we have

$$h^{(m)}(\vec{W}, \vec{y}_{[m]}) = (-1)^m \prod_{i=1}^m \mathbb{P}(y_i \sim W), \qquad h(\vec{W}) = \mathbb{P}_\lambda(\eta \leftrightarrow\!\!\!\!\!/\ W).$$

Moreover,

$$\sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} |h^{(m)}(\vec{W}, \vec{y}_{[m]})| \, d\vec{y}_{[m]} = e^{\mathbb{E}_\lambda[|\{y \in \eta : y \sim W\}|]} \leq e^{\lambda |W|},$$

using the same bound as in (4.15). Since this proves the proposition for $k = 0$, we turn to $k \geq 1$ and we first prove (4.7).

We rewrite $h(\vec{W})$ by explicitly writing out the sum over laces L in terms of the endpoints of their stitches in W (note that any lace can have at most k stitches). We first exhibit this for $k = 2$, where $\vec{W} = (u_0, V_0, u_1, V_1, u_2, V_2, u_3)$ and there are two different laces. With the abbreviation $\tilde{Q}_{i,j} = Q_{i,j} + \mathbb{P}([u_i] \sim [u_j])$,

$$h(\vec{W}) = h(\vec{W}; L_1) + h(\vec{W}; L_2) = \mathbb{P}_\lambda(\eta \nleftrightarrow W)\big(q_{0,3}(1 - \tilde{Q}_{0,3}) + Q_{0,3}(1 - \tilde{Q}_{0,2})(1 - \tilde{Q}_{1,3})\big)$$

$$= \mathbb{P}_\lambda(\eta \nleftrightarrow W) \sum_{\beta_1=2}^{3} q_{0,\beta_1}(1 - \tilde{Q}_{0,\beta_1}) \bigg[\mathbb{1}_{\{\beta_1=3\}} + \mathbb{1}_{\{\beta_1<3\}} \sum_{\alpha_2=1}^{\beta_1-1} Q_{0,3}(1 - \tilde{Q}_{\alpha_2,3}) \bigg].$$

$$(4.19)$$

Clearly, this is unnecessarily complicated for $k = 2$, as the sum over α_2 contains only one term and $q_{0,2} = 1$. However, this turns out to be convenient for general k. We use the convention that $Q_{[a,b],\varnothing} = Q_{\varnothing,[a,b]} = 1$. Carefully rearranging the sum over all laces yields

$$h(\vec{W}) = \sum_{L \in \mathcal{L}(\vec{w})} h(\vec{W}; L) = \mathbb{P}_\lambda(\eta \nleftrightarrow W) \sum_{\beta_1=2}^{k+1} q_{0,\beta_1}(1 - \tilde{Q}_{0,\beta_1}) Q_{0,(\beta_1,k+1]}$$

$$\times \bigg[\mathbb{1}_{\{\beta_1=k+1\}} + \sum_{\alpha_2=1}^{\beta_1-1} \sum_{\beta_2=\beta_1+1}^{k+1} q_{\alpha_2,\beta_2}(1 - \tilde{Q}_{\alpha_2,\beta_2}) Q_{(0,\alpha_2],(\beta_2,k+1]} Q_{(0,\alpha_2),\beta_2}$$

$$\times \bigg[\mathbb{1}_{\{\beta_2=k+1\}} + \sum_{\alpha_3=\beta_1}^{\beta_2-1} \sum_{\beta_3=\beta_2+1}^{k+1} q_{\alpha_3,\beta_3}(1 - \tilde{Q}_{\alpha_3,\beta_3}) q_{\alpha_1,\alpha_2,\alpha_3}$$

$$Q_{(\alpha_2,\alpha_3],(\beta_3,k+1]} Q_{(\alpha_2,\alpha_3),\beta_3} Q_{(0,\alpha_2),(\alpha_3,\beta_2)}$$

$$\times \bigg[\mathbb{1}_{\{\beta_3=k+1\}} + \sum_{\alpha_4=\beta_2}^{\beta_3-1} \sum_{\beta_4=\beta_3+1}^{k+1} q_{\alpha_4,\beta_4}(1 - \tilde{Q}_{\alpha_4,\beta_4}) q_{\alpha_2,\alpha_3,\alpha_4}$$

$$Q_{(\alpha_3,\alpha_4],(\beta_4,k+1]} Q_{(\alpha_3,\alpha_4),\beta_4} Q_{(\alpha_2,\alpha_3),(\alpha_4,\beta_3)} \times \cdots$$

$$\times \bigg[\mathbb{1}_{\{\beta_{k-1}=k+1\}} + \mathbb{1}_{\{\beta_{k-1}<k+1\}} \sum_{\alpha_k=\beta_{k-2}}^{\beta_{k-1}-1} q_{\alpha_k,k+1}(1 - \tilde{Q}_{\alpha_k,k+1}) \prod_{j=k,k+1} q_{\alpha_{j-2},\alpha_{j-1},\alpha_j}$$

$$Q_{(\alpha_{k-1},\alpha_k),k+1} Q_{(\alpha_{k-2},\alpha_{k-1}),(\alpha_k,\beta_{k-1})} \bigg] \cdots \bigg] \bigg] \bigg].$$

Note that if $\beta = k + 1$ for some i, then the double sum following the corresponding indicator breaks down to 0. Also, only the innermost bracketed term contains two factors of $q_{a,b,c}$.

We now show that, starting with the innermost square brackets, the bracketed terms are bounded by 1 in absolute value.

To lighten notation, we write the innermost sum as $\sum_{\alpha=b_1}^{b_2-1} R(\alpha)$. We split the factor

$$1 - \tilde{Q}_{\alpha_k,k+1} = (1 - Q_{\alpha_k,k+1}) - \mathbb{P}([u_{\alpha_k}] \sim [u_{k+1}]).$$

This yields two sums

$$\sum_{\alpha=b_1}^{b_2-1} R(\alpha) = \sum_{\alpha=b_1}^{b_2-1} R'(\alpha) - \sum_{\alpha=b_1}^{b_2-1} R''(\alpha),$$

where R' and R'' are both nonnegative. Now, with the estimate $Q_{(\alpha_{k-1},\alpha),k+1} \le Q_{[\beta_{k-2},\alpha),k+1} = Q_{[b_1,\alpha),k+1}$, we can bound

$$\sum_{\alpha=b_1}^{b_2-1} R'(\alpha) \le \sum_{\alpha=b_1}^{b_2-1} (1 - Q_{\alpha,k+1})Q_{[b_1,\alpha),k+1}$$

$$= (1 - Q_{b_1,k+1}) + Q_{b_1,k+1} \sum_{\alpha=b_1+1}^{b_2-1} (1 - Q_{\alpha,k+1})Q_{[b_1+1,\alpha),k+1}, \qquad (4.20)$$

which is readily proven to be at most 1 by induction. Moreover,

$$\sum_{\alpha=b_1}^{b_2-1} R''(\alpha) \le \sum_{\alpha=b_1}^{b_2-1} q_{\alpha,k+1}\mathbb{P}([u_\alpha]] \sim [\![u_{k+1}]\!]). \qquad (4.21)$$

The above summands can be rewritten as the probability of the event that the direct stitch $(\alpha, k + 1)$ is present, while all direct stitches $(j, k + 1)$ for $j \in (\alpha, k + 1]$ are not. Hence, these are disjoint events for different values of α, and so the sum is at most 1.

In total, we rewrote $\sum_{\alpha=b_1}^{b_2-1} R(\alpha)$ as the difference of two nonnegative values, both at most 1, proving our claim.

To deal with the summands for $2 \le i < k$, we write the double sum as

$$\sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R(\alpha, \beta)$$

and split the term $1 - \tilde{Q}_{\alpha_i,\beta_i} = (1 - Q_{\alpha_i,\beta_i}) - \mathbb{P}([\![u_{\alpha_i}]\!] \sim [\![u_{\beta_i}]\!])$ so that

$$\sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R(\alpha, \beta) = \sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R'(\alpha, \beta) - \sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R''(\alpha, \beta) \qquad (4.22)$$

for nonnegative summands R', R''. We prove a bound on the sum over $R'(\alpha, \beta)$ by induction on $k - b_2$. If $b_2 = k$, then the bound is the same as for the bound (4.20). For $b_2 < k$, we first bound $Q_{(\alpha_i,\alpha],(\beta,k+1]} \le Q_{[b_1,\alpha],(\beta,k+1]}$ and then extract the summand for $\beta = k + 1$, yielding

$$\sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R'(\alpha, \beta) \le \sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} (1 - Q_{\alpha,\beta})Q_{[b_1,\alpha],(\beta,k+1]}Q_{[b_1,\alpha),\beta}$$

$$\le \sum_{\alpha=b_1}^{b_2-1} (1 - Q_{\alpha,k+1})Q_{[b_1,\alpha),k+1}$$

$$+ Q_{[b_1,b_2-1],k+1} \sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k} (1 - Q_{\alpha,\beta})Q_{[b_1,\alpha],(\beta,k]}Q_{[b_1,\alpha),\beta}.$$

$$(4.23)$$

By the induction hypothesis, the double sum in (4.23) is at most 1. Therefore,

$$
\sum_{\alpha=b_1}^{b_2-1} \sum_{\beta=b_2+1}^{k+1} R'(\alpha, \beta) \leq \sum_{\alpha=b_1}^{b_2-2} (1 - Q_{\alpha,k+1}) Q_{[b_1,\alpha),k+1}
$$
$$
+ (1 - Q_{b_2-1,k+1}) Q_{[b_1,b_2-1),k+1} + Q_{b_2-1,k+1} Q_{[b_1,b_2-1),k+1}
$$
$$
= \sum_{\alpha=b_1}^{b_2-2} (1 - Q_{\alpha,k+1}) Q_{[b_1,\alpha),k+1} + Q_{[b_1,b_2-2],k+1}
$$
$$
= 1,
$$

where the last identity is now an easy induction.

Turning to the second summand in (4.22), by the same argument used to treat (4.21), the summands $R''(\alpha, \beta)$ are probabilities of events which are disjoint for different values of (α, β), and so they sum to at most 1.

The observation that the bracket term for $i = 1$ is handled analogously finishes the proof of (4.7).

We proceed to prove (4.8) for $k > 1$. By combining Lemma 4.2 with Lemma 4.4, we obtain

$$
\sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} \left| h^{(m)}(\vec{W}, \vec{y}_{[m]}) \right| \mathrm{d}\vec{y}_{[m]} \leq \sum_{L \in \mathcal{L}_k} \sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} \left| \sum_{H \in \langle\!\langle L \rangle\!\rangle : \mathcal{S}(H) = \vec{y}_{[m]}} \mathbf{w}(H) \right| \mathrm{d}\vec{y}_{[m]}
$$
$$
\leq \frac{1}{\sqrt{5}} \left(\frac{3 + \sqrt{5}}{2} \right)^k 2^k e^{3\lambda|W|}.
$$

Using the bound $k \leq |W|$ finishes the proof. $\qquad\square$

Lemma 4.5. (Thermodynamic limit of the shell function.) *For every $\lambda \geq 0$, the pointwise limit*

$$
\lim_{\Lambda \nearrow \mathbb{R}^d} h_\lambda^\Lambda(\vec{W}) = h_\lambda(\vec{W})
$$

along $\mathbb{R}^d$-exhausting sequences exists.

Proof. Let $(\Lambda_n)_{n \in \mathbb{N}}$ be an $\mathbb{R}^d$-exhausting sequence. For fixed $\vec{W} = (u_0, V_0, \ldots, u_{k+1})$, note that

$$
h_\lambda^{\Lambda_n}(\vec{W}) = \sum_{L \in \mathcal{L}_k} h_\lambda^{\Lambda_n}(\vec{W}; L).
$$

For each lace L, the limit

$$
h_\lambda(\vec{W}; L) = \lim_{n \to \infty} h_\lambda^{\Lambda_n}(\vec{W}; L)
$$

exists and does not depend on the precise choice of $\mathbb{R}^d$-exhausting sequence. This is clear from the representation for $h_\lambda^\Lambda(\vec{W}; L)$ proven in Lemma 4.4. In particular, $h_\lambda^\Lambda(\vec{W}; L)$ is given as the finite product of Λ-independent factors and factors that describe the probability of certain point processes containing no points (namely, $\mathbb{P}_\lambda(\eta_\Lambda \not\longleftrightarrow W)$ and the factors $Q_{i,j}$). As probabilities that are decreasing in the volume, the latter admit a $\Lambda \nearrow \mathbb{R}^d$ limit. It follows that the limit of the shell function exists as well and is given by

$$
h_\lambda(\vec{W}) = \sum_{L \in \mathcal{L}_k} h_\lambda(\vec{W}; L).
$$

$\qquad\square$

4.4. The direct-connectedness function in infinite volume

In this section, we consider the limit $\lim_{\Lambda \nearrow \mathbb{R}^d} g_\lambda^\Lambda$ with g_λ^Λ as in (4.6) and give sufficient conditions under which it exists, thereby proving the two convergence statements from Theorem 1.1.

The candidate limit is given by the analogue of (4.6) with Λ replaced by $\mathbb{R}^d$; the existence of $h_\lambda^{\mathbb{R}^d} \equiv h_\lambda$ has been checked in Lemma 4.5. Thus,

$$g_\lambda(x_1, x_2) = \sum_{r=0}^\infty \frac{\lambda^r}{r!} \int_{(\mathbb{R}^d)^r} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\text{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) h_\lambda(\vec{W}) \, d\vec{x}_{[3,r+2]}, \qquad (4.24)$$

where the inner sum is over core graphs G on $\vec{x}_{[r+2]}$ with pivot decomposition $\vec{W}$, i.e., over $(+)$-connected graphs G on $\vec{x}_{[r+2]}$ with $\mathsf{PD}^+(x_1, x_2, G) = \mathsf{PD}^\pm(x_1, x_2, G) = \vec{W}$. Remember the quantities $0 < \tilde{\lambda}_* \leq \lambda_*$ introduced before Theorem 1.1. We will see in (4.25) that the sum over core graphs for a given pivot decomposition is a probability, hence in particular nonnegative.

Theorem 4.1. (The thermodynamic limit of g_λ^Λ: pointwise convergence.) *If $\lambda < \lambda_*$, then*

$$\sum_{r=0}^\infty \frac{\lambda^r}{r!} \int_{(\mathbb{R}^d)^r} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\text{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) \left| h_\lambda(\vec{W}) \right| d\vec{x}_{[3,r+2]} < \infty$$

for all $x_1, x_2 \in \mathbb{R}^d$. Moreover, for every $\mathbb{R}^d$-exhausting sequence $(\Lambda_n)_{n \in \mathbb{N}}$, we have the pointwise convergence

$$\lim_{n \to \infty} g_\lambda^{\Lambda_n}(x_1, x_2) = g_\lambda(x_1, x_2)$$

with g_λ given in (4.24) (equivalently, Equation (4.6) with Λ replaced by $\mathbb{R}^d$).

Theorem 4.2. (Integrability and convergence in the L^1-norm.) *If $\lambda < \tilde{\lambda}_*$, then for all $x_1 \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} |g_\lambda(x_1, x_2)| \, dx_2 \leq \sum_{r=0}^\infty \frac{\lambda^r}{r!} \int_{\mathbb{R}^d} \left(\int_{(\mathbb{R}^d)^r} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\text{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) \left| h_\lambda(\vec{W}) \right| d\vec{x}_{[3,r+2]} \right) dx_2 < \infty.$$

Proof of Theorem 4.1. We consider a summand in (4.24) for fixed $\vec{W}$ and set $x_1 = u_0$ as well as $x_2 = u_{k+1}$. Let $\vec{W} = (u_0, V_0, \ldots, V_k, u_{k+1})$. Remember $\overline{V}_i = \{u_i\} \cup V_i \cup \{u_{i+1}\}$. A first important observation is the fact that the weight of a core graph with pivot decomposition $\vec{W}$ factors into the product over the $k(\pm)$-subgraphs induced by the vertex sets $\overline{V}_i$. The sum over core graphs thus factors as

$$\sum_{\substack{G \in \mathcal{C}^+(W): \\ \mathsf{PD}^+(G) = \mathsf{PD}^\pm(G) = \vec{W}}} \mathbf{w}^\pm(G) = \prod_{i=0}^k \left(\sum_{H \in \mathcal{D}_{u_i, u_{i+1}}^+(\overline{V}_i)} \mathbf{w}^\pm(H) \right)$$

$$= \prod_{i=0}^k \mathbb{P}\left(\Gamma_\varphi(\overline{V}_i) \in \mathcal{D}_{u_i, u_{i+1}} \right)$$

$$= \mathbb{P}\left(\bigcap_{i=0}^k \{\Gamma_\varphi(\overline{V}_i) \in \mathcal{D}_{u_i, u_{i+1}}\} \right). \qquad (4.25)$$

Hence, the core can be written as a probability. Combining this with Proposition 4.1, we get

$$
\left(\sum_{\substack{G \in \mathcal{C}^+(W): \\ \mathrm{PD}^+(G) = \mathrm{PD}^\pm(G) = \vec{W}}} \mathbf{w}^\pm(G) \right) \left| h_\lambda^\Lambda(\vec{W}) \right| \leq \mathbb{P}_\lambda(\eta_\Lambda \longleftrightarrow\hspace{-1.5em}/\hspace{0.6em} W) \mathbb{P}\left(\bigcap_{i=0}^{k} \{ \Gamma_\varphi(\overline{V}_i) \in \mathcal{D}_{u_i, u_{i+1}} \} \right)
$$

$$
= \mathbb{P}_\lambda\left(\left\{ \mathscr{C}(u_0, \xi_\Lambda^W) = W \right\} \cap \bigcap_{i=0}^{k} \left\{ \xi_\Lambda^W[\overline{V}_i] \in \mathcal{D}_{u_i, u_{i+1}} \right\} \right).
$$

Above, we used independence as well as the fact that for $V \subseteq W$, the two random graphs $\Gamma_\varphi(V)$ and $\xi^W[V]$ are identical in distribution. The inequality holds true for bounded Λ as well as $\Lambda = \mathbb{R}^d$.

We now go back to (4.6) and rearrange the sum by first summing over the number of pivotal points k, giving

$$
\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \int_{\Lambda^r} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\mathrm{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) \left| h_\lambda^\Lambda(\vec{W}) \right| \, \mathrm{d}\vec{x}_{[3, r+2]}
$$

$$
= \sum_{k \geq 0} \lambda^k \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^{k+n}} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\mathrm{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) \left| h_\lambda^\Lambda(\vec{W}) \right| \, \mathrm{d}\vec{v}_{[n]} \, \mathrm{d}\vec{u}_{[k]}. \tag{4.26}
$$

In the second term, the sum is over pivot decompositions $\vec{W} = (u_0, V_0, \ldots, V_k, u_{k+1})$ where $u_0 = x_1$, $u_{k+1} = x_2$, and $\cup_{i=0}^{k} V_i = \{v_1, \ldots, v_n\}$.

When rewriting the integrand of (4.26) as a probability, the event that u_i and u_{i+1} are 2-connected for $i \in [k]_0$ in disjoint vertex sets V_i becomes the event that these connection events occur disjointly within W; see Section 2 and recall the definition (2.2). The inner series can thus be bounded as

$$
\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \sum_{\vec{W}} \left(\sum_{G \in \mathcal{G}_{\mathrm{core}}^{\vec{W}}} \mathbf{w}^\pm(G) \right) |h_\lambda^\Lambda(\vec{W})| \, \mathrm{d}\vec{v}_{[n]}
$$

$$
\leq \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{\Lambda^n} \mathbb{P}_\lambda\left(\left\{ \mathscr{C}\left(u_0, \xi_\Lambda^{\vec{u}_{[k]}, \vec{v}_{[n]}} \right) = \vec{u}_{[k]} \cup \vec{v}_{[n]} \right\} \right.
$$

$$
\cap \left(\left\{ u_0 \Longleftrightarrow u_1 \text{ in } \xi^{u_0, u_1, \vec{v}_{[n]}} \right\} \circ \cdots \circ \left\{ u_k \Longleftrightarrow u_{k+1} \text{ in } \xi^{u_k, u_{k+1}, \vec{v}_{[n]}} \right\} \right) \bigg) \bigg) \mathrm{d}\vec{v}_{[n]}
$$

$$
= \mathbb{P}_\lambda\left(\left\{ u_0 \Longleftrightarrow u_1 \text{ in } \xi^{u_0, u_1} \right\} \circ \cdots \circ \left\{ u_k \Longleftrightarrow u_{k+1} \text{ in } \xi^{u_k, u_{k+1}} \right\} \right), \tag{4.27}
$$

where the identity is due to the Mecke equation and the fact that by summing over $\vec{v}$, we were partitioning over what the joint cluster of $\vec{u}_{[0, k+1]}$ is. We can now use the BK inequality ([10, Theorem 2.1]) to bound (4.27) by

$$
\prod_{i=0}^{k} \mathbb{P}_\lambda\left(u_i \Longleftrightarrow u_{i+1} \text{ in } \xi_\Lambda^{u_i, u_{i+1}} \right) \leq \prod_{i=0}^{k} \sigma_\lambda(u_{i+1} - u_i). \tag{4.28}
$$

Inserting this back into (4.26),

$$
\sum_{k\geq 0}\lambda^k\sum_{n\geq 0}\frac{\lambda^n}{n!}\int_{\Lambda^{k+n}}\sum_{\vec{W}}\left(\sum_{G\in\mathcal{G}_{\mathrm{core}}^{\vec{W}}}\mathbf{w}^{\pm}(G)\right)\left|h_\lambda^\Lambda(\vec{W})\right|\,\mathrm{d}\vec{v}_{[n]}\,\mathrm{d}\vec{u}_{[k]}
$$

$$
\leq\sum_{k\geq 0}\lambda^k\sigma_\lambda^{*(k+1)}(x_2-x_1). \tag{4.29}
$$

The last expression is finite for $\lambda<\lambda_*$, by the definition of λ_*. The pointwise convergence of $g_\lambda^{\Lambda_n}$ to g_λ follows by dominated convergence.   □

Proof of Theorem 4.2. If we integrate over x_2 in (4.29), this yields the upper bound

$$
\lambda^{-1}\sum_{k\geq 1}\left(\lambda\int\sigma_\lambda(x)\,\mathrm{d}x\right)^k,
$$

which is finite for $\lambda<\tilde{\lambda}_*$, by definition of $\tilde{\lambda}_*$. The theorem follows by Fubini–Tonelli and the triangle inequality.   □

5. The Ornstein–Zernike equation

Here we complete the proof of Theorem 1.1. In view of Theorems 4.1 and 4.2, it remains to prove that the expansion (4.24) is indeed equal to the direct-connectedness function given by the OZE (1.2). This is proven by showing first that g_λ^Λ from Definition 4.2 fulfills the OZE in finite volume and then passing to the limit $\Lambda\nearrow\mathbb{R}^d$.

The idea of the proof in finite volume is basically well known; the same proof works for the OZE for the total correlation function.

Proposition 5.1. (The Ornstein–Zernike equation in finite volume.) *Let* $\Lambda\subset\mathbb{R}^d$ *be bounded and let* $x_1, x_2\in\Lambda$. *Then*

$$
\tau_\lambda^\Lambda(x_1,x_2)=g_\lambda^\Lambda(x_1,x_2)+\lambda\int_\Lambda g_\lambda^\Lambda(x_1,x_3)\tau_\lambda^\Lambda(x_3,x_2)\,\mathrm{d}x_3.
$$

Proof. We drop the Λ-dependence in the superscript of τ_λ^Λ and g_λ^Λ. Thanks to Proposition 3.1, we can re-sum the series expansion for τ_λ at will. Given a pivot decomposition $\vec{W}=(u_0,V_0,\ldots,u_{k+1})$ of an arbitrary core graph G with the vertex set W, define

$$
\bar{h}_\lambda^{(m)}\big(\vec{W},\vec{y}_{[m]}\big):=\sum_{\substack{H\in\mathcal{G}(\overline{V}_1,\ldots,\overline{V}_k,\vec{y}_{[m]}):\\ G\oplus H\in\mathcal{C}_{u_0,u_{k+1}}^{\pm}\left(W\cup\vec{y}_{[m]}\right)}}\mathbf{w}(H),\qquad \bar{h}_\lambda\big(\vec{W}\big):=\sum_{m\geq 0}\frac{\lambda^m}{m!}\int_{\Lambda^m}\bar{h}_\lambda^{(m)}\big(\vec{W},\vec{y}_{[m]}\big)\,\mathrm{d}\vec{y}_{[m]},
$$

$$\tag{5.1}$$

in analogy to the shell function h_λ in (4.5) (just like the latter, $\bar{h}_\lambda$ only depends on G through its pivot decomposition $\vec{W}$). To be more precise, the shell function h_λ is recovered from $\bar{h}_\lambda$ by summing over a smaller subset of graphs H in (5.1), adding the restriction that $G\oplus H$ shall not contain ($\pm$)-pivot points for the u_0–u_{k+1} connection. Note that

$$
0\leq\bar{h}_\lambda\big(\vec{W}\big)=\mathrm{e}^{-\mathbb{E}_\lambda[|\{y\in\eta:y\sim W\}|]}\prod_{\substack{x,y\in W:\nexists i\in[k]_0:\{x,y\}\subseteq\overline{V}_i}}(1-\varphi(x-y))
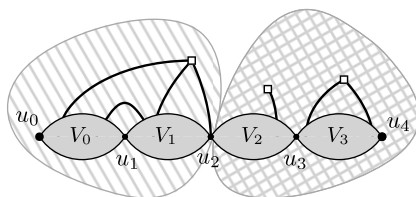$$

FIGURE 6. The schematic representation of a $(\pm)$-graph $G \oplus H$ in $\mathcal{C}^{\pm}_{u_0, u_4}(W \cup \vec{y}_{[3]})$ illustrates the factorization of the graph weight from Equation (5.2): the edges of H explicitly depicted in the picture, while the core graph G is represented by its pivot decomposition $(u_0, V_0, \ldots, u_4)$. The vertices $\vec{y}_{[3]}$ are depicted by squares, ordered from left to right. The first $(\pm)$-pivot point for the u_0–u_4 connection in $G \oplus H$ is u_2. Thus, the weight of the simple graph H factors into the weight of its subgraph induced by $\vec{W}'_2 \cup \vec{y}_{[1]} = \{u_0, u_1, u_2\} \cup V_0 \cup V_1 \cup \{y_1\}$ (hatched on the left) and the weight of the subgraph induced by $\vec{W}''_2 \cup \vec{y}_{[3] \setminus [1]} = \{u_3, u_4\} \cup V_2 \cup V_3 \cup \{y_2, y_3\}$ (crosshatched on the right).

and that when replacing h_λ with $\bar{h}_\lambda$ in the right-hand side of (4.6), we get τ_λ instead of g_λ. We can split the sum $\bar{h}^{(m)}_\lambda(\vec{W}, \vec{y}_{[m]}) = h^{(m)}_\lambda(\vec{W}, \vec{y}_{[m]}) + f^{(m)}_\lambda(\vec{W}, \vec{y}_{[m]})$, where $f^{(m)}_\lambda$ contains the sum over those graphs H such that $G \oplus H$ *does* have $(\pm)$-pivotal points with respect to the u_0–u_{k+1} connection. We set

$$f_\lambda(\vec{W}) := \sum_{m \geq 0} \frac{\lambda^m}{m!} \int_{\Lambda^m} f^{(m)}_\lambda(\vec{W}, \vec{y}_{[m]}) \, \mathrm{d}\vec{y}_{[m]}.$$

Assume now that u_j for $j \in [k]$ is the first pivotal point of $G \oplus H \in \mathcal{C}^{\pm}_{x_1, x_2}(W \cup \vec{y}_{[m]})$. Furthermore, let $\vec{W}'_j := (u_0, V_0, \ldots, u_j)$, let $\vec{W}''_j := (u_j, V_j, \ldots, u_{k+1})$, and let $y_{[s]}$ for $s \leq m$ be the points adjacent to $\vec{W}'_j$ (possibly after reordering the vertices). The weight of such a graph H then factors into the product of the weights of two graphs, namely the subgraphs of H induced by $\vec{W}'_j \cup \vec{y}_{[s]} \subset V(H)$ and by $\vec{W}''_j \cup \vec{y}_{[m] \setminus [s]} \subset V(H)$; see Figure 6. That is,

$$\mathbf{w}(H) = \mathbf{w}\big(H[\vec{W}'_j \cup \vec{y}_{[s]}]\big) \mathbf{w}\big(H[\vec{W}''_j \cup \vec{y}_{[m] \setminus [s]}]\big). \tag{5.2}$$

Moreover, we see that $H[\vec{W}'_j \cup \vec{y}_{[s]}] \oplus G[W'_j]$ does not contain $(\pm)$-pivot points (for the u_0–u_j connection) and $H[\vec{W}''_j \cup \vec{y}_{[m] \setminus [s]}] \oplus G[W''_j]$ is in general just $(\pm)$-connected.

By partitioning over j, we thus obtain the decomposition

$$f_\lambda(\vec{W}) = \sum_{j=1}^{k} h_\lambda(\vec{W}'_j) \bar{h}_\lambda(\vec{W}''_j).$$

Since both h_λ and $\bar{h}_\lambda$ converge absolutely, so does f_λ, justifying all re-summations. Letting $x_1 = u_0$ and $x_2 = u_{k+1}$,

$$(\tau_\lambda - g_\lambda)(x_1, x_2) = \sum_{k \geq 1} \lambda^k \sum_{n_0, \ldots, n_k \geq 0} \frac{\lambda^{\sum_{i=0}^{k} n_i}}{\prod_{i=0}^{k} n_i!} \int_{\Lambda^{k + \sum_{i=0}^{k} n_i}} \left(\prod_{i=0}^{k} \mathbb{P}\big(\Gamma_\varphi(\bar{V}_i) \in \mathcal{D}_{u_i, u_{i+1}}(\bar{V}_i)\big) \right)$$

$$\times \left(\sum_{j=1}^{k} h_\lambda(\vec{W}'_j) \bar{h}_\lambda(\vec{W}''_j) \right) \prod_{i=0}^{k} \mathrm{d}\vec{v}_{i, [n_i]} \, \mathrm{d}\vec{u}_{[k]}$$

$$= \sum_{j\geq 1, k\geq 0} \lambda^{j+k} \int_\Lambda \sum_{n_0,\dots,n_{j+k}\geq 0} \frac{\lambda^{\sum_{i=0}^{j+k} n_i}}{\prod_{i=0}^{j+k} n_i!} \left[\int_{\Lambda^{j-1+\sum_{i=0}^{j-1} n_i}} \prod_{i=0}^{j-1} \mathbb{P}\Big(\Gamma_\varphi(\bar{V}_i) \in \mathcal{D}_{u_i, u_{i+1}}(\bar{V}_i)\Big) \right.$$

$$\times\, h_\lambda\big(\vec{W}'_j\big) \prod_{i=0}^{j-1} \mathrm{d}\vec{v}_{i,[n_i]}\, \mathrm{d}\vec{u}_{[j-1]} \Bigg]$$

$$\times \left[\int_{\Lambda^{k+\sum_{i=j}^{j+k} n_i}} \prod_{i=j}^{j+k} \mathbb{P}\Big(\Gamma_\varphi(\bar{V}_i) \in \mathcal{D}_{u_i, u_{i+1}}(\bar{V}_i)\Big) \right.$$

$$\times\, \bar{h}_\lambda\big(\vec{W}''_j\big) \prod_{i=j}^{j+k} \mathrm{d}\vec{v}_{i,[n_i]}\, \mathrm{d}\vec{u}_{[j+1,j+k]} \Bigg]\, \mathrm{d}u_j$$

$$= \lambda \int_\Lambda g_\lambda(x_1, u)\tau_\lambda(u, x_2)\, \mathrm{d}u.$$

The re-summation with respect to j and k is justified as the resulting series converges for $\lambda < \lambda_*$ even when we put h_λ in absolute values. $\qquad\square$

We can now extend the result of Proposition 5.1 to $\Lambda \nearrow \mathbb{R}^d$ and thus prove that the expansion (4.24) is indeed equal to the direct-connectedness function for $\lambda < \lambda_*$, finalizing the proof of our main result.

Proof of Theorem 1.1. We have

$$\tau_\lambda(x_1, x_2) = \lim_{\Lambda \nearrow \mathbb{R}^d} \tau_\lambda^\Lambda(x_1, x_2)$$

$$= \lim_{\Lambda \nearrow \mathbb{R}^d} g_\lambda^\Lambda(x_1, x_2) + \lambda \lim_{\Lambda \nearrow \mathbb{R}^d} \int_{\mathbb{R}^d} g_\lambda^\Lambda(x_1, x_3) \mathbb{1}_\Lambda(x_3) \tau_\lambda^\Lambda(x_3, x_2)\, \mathrm{d}x_3, \qquad (5.3)$$

where the first equality holds by the continuity of probability measures along sequences of increasing events and the second one by Proposition 5.1.

Note that the integrand in (5.3) is bounded uniformly in Λ by

$$C\tau_\lambda(x_3, x_2),$$

where $C = \sup_{y\in\mathbb{R}^d} \sum_k \lambda^k \sigma_\lambda^{*(k+1)}(y)$ is a constant obtained in (4.29). Since τ_λ is integrable for all $\lambda < \lambda_c$, the theorem follows by dominated convergence. $\qquad\square$

6. Discussion

6.1. Connections to percolation on Gibbs point processes

The Ornstein–Zernike equation gets its name from the seminal paper [20] and has since been a well-known formalism in liquid-state statistical mechanics. It relates the total correlation function to the direct correlation function and it naturally connects to power series expansions of these correlation functions (see [6, 23, 24]; the terminology is not the same in all of these references).

The correlation functions admit graphical expansions that also consist of connected graphs. It was observed [11] that a similar formalism can be formulated for the pair-connectedness function, and a key reference for this is [6]. The pair-connectedness function is deemed part

of the pair-correlation function. The connected graphs appearing in the expansion of the latter are referred to as 'mathematical clusters', and they correspond to our $(\pm)$-connected graphs. Isolating the $(+)$-connected components within these graphs yields the 'physical clusters', and the graphs in which x_1 and x_2 lie in the same physical cluster make up the expansion for $\tau_\lambda(x_1, x_2)$. In the following, we elaborate on this.

The percolation models considered in the physics literature are mostly based not on a PPP (Stell calls the Poisson setup *random percolation* [24]), but on a Gibbs point process (called *correlated percolation* in the language of Stell). (The denomination 'random percolation' for the Poisson setup feels quite misleading for probabilists; but it reflects language commonly adopted across physics, with 'random' understood as 'completely random' in the sense of *completely random measures* [13], a class comprising the PPP.)

To define the latter, consider a nonnegative pair potential $v : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ and some finite volume Λ. Let $\mathbf{N}(\Lambda)$ be the set of finite counting measures on Λ and let $\mu \in \mathbf{N}(\Lambda)$. Then the energy of $\{x_1, \ldots, x_n\}$ under the boundary condition μ is

$$H(\{x_1, \ldots, x_n\} \mid \mu) = \sum_{1 \leq i < j \leq n} v(x_i - x_j) + \sum_{i=1}^{n} \sum_{y \in \mu} v(x_i - y).$$

Let $f : \mathbf{N}(\Lambda) \to \mathbb{R}$ be bounded. We define a probability measure as

$$\mathbb{E}_z[f] := \frac{1}{\Xi(z)} \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\}) \mathrm{e}^{-H(\{x_1, \ldots, x_n\})} \, \mathrm{d}\vec{x}_{[n]},$$

where the partition function $\Xi(z)$ is such that $\mathbb{E}_z[1] = 1$ and $z \in \mathbb{R}_{\geq 0}$ is called the activity. If we denote by η a random variable with law $\mathbb{E}_z$, then η is a point process. Note that we recover the homogeneous PPP with intensity $\lambda = z$ by setting $v \equiv 0$.

We can define the RCM ξ on this general point process, and we denote its probability measure by $\mathbb{P}_{z,\varphi}$. We furthermore define the (one-particle) density as

$$\rho_1(x) = z\mathbb{E}_z\big[\mathrm{e}^{-H(\{x\}|\eta)}\big] = \rho,$$

and we define the pair-correlation function as

$$\rho_2(x, y) = z^2 \mathbb{E}_z\big[\mathrm{e}^{-H(\{x,y\}|\eta)}\big].$$

Again, in the case of a homogeneous PPP with intensity $\lambda = z$, we have $\rho = z$ and $\rho_2 = z^2$. Defining the pair-connectedness function as

$$\tau_{z,\varphi}(x, y) := \mathbb{E}_{z,\varphi}\big[\mathrm{e}^{-H(\{x,y\}|\eta)} \mathbb{1}_{\{x \longleftrightarrow y \text{ in } \xi^{x,y}\}}\big],$$

we can decompose

$$\rho_2(x, y) = z^2 \tau_{z,\varphi}(x, y) + z^2 \mathbb{E}_{z,\varphi}\big[\mathrm{e}^{-H(\{x,y\}|\eta)} \mathbb{1}_{\{x \longleftrightarrow\!\!\!\!\!/\ y \text{ in } \xi^{x,y}\}}\big].$$

In [6], Coniglio *et al.* define the pair-connectedness function as $\tilde{\tau}_{z,\varphi} = \big(z^2/\rho^2\big)\tau_{z,\varphi}$.

The function $\tilde{\tau}_{z,\varphi}$ has a density expansion (note that $\tau_{z,\varphi}$ is better suited for activity expansions) that can be found in [6, Equation (12)], which can be obtained from the density expansion of the pair-correlation function: the latter is obtained by expanding the Mayer f-functions $f(x, y) = \mathrm{e}^{-v(x,y)} - 1$ in the partition function, which is the starting point of a cluster

expansion. Splitting the Mayer f-function as $f = f^+ + f^*$ with $f^+ = e^{-v(x,y)}\varphi(x-y)$ and executing the same expansion for the correlation function 'doubles' every edge into a $(+)$-edge and a $(*)$-edge. Only summing over graphs in which x and y are connected by $(+)$-edges yields the pair-connectedness function.

In general, the graphs appearing in the density expansion are a subset of those in the activity expansion, namely the ones without articulation points (articulation points were defined after Proposition 3.1). In the case of a homogeneous PPP, we have $\lambda = z = \rho$, and so activity and density expansion coincide (and the graphs with articulation points cancel out). Moreover, $f^+(x, y) = -f^*(x, y) = \varphi(x-y)$, and the graphs summed over in the expansion become the $(\pm)$-graphs, yielding the expansion (3.1) for τ_λ.

It is an interesting question which ideas of this paper can be generalized to RCMs based on Gibbs point processes. While some aspects generalize without much effort, the crucial difference lies in the fact that the weight of graphs showing up in expansions for Gibbs point processes also encodes the pair interaction induced by the potential v. To recover probabilistic interpretations for terms after performing re-summations and bounds is therefore much more delicate.

6.2. Connections to Last and Ziesche

In [15], Last and Ziesche use a Margulis–Russo-type formula to prove analyticity of τ_λ in presumably the whole subcritical regime. Moreover, they show the existence of some $\lambda_0 > 0$ (which is not quantified) such that both τ_λ and g_λ have an absolutely convergent graphical expansion in $[0, \lambda_0)$ that seems closely related to the ones discussed here. We want to illustrate how to relate the respective expressions.

The two-point function. Last and Ziesche show that $\tau_\lambda(x_1, x_2)$ is equal to

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int \sum_{J \subset [3, n+2]} (-1)^{n-|J|} \mathbb{P}\big(x_1 \longleftrightarrow x_2 \text{ in } \Gamma_\varphi(\vec{x}_{J \cup \{1,2\}}), \ \Gamma_\varphi(\vec{x}_{[n+2]}) \text{ is connected}\big) \, d\vec{x}_{[3, n+2]}.$$
$$(6.1)$$

We show that the above integrand is the same as the one in (3.1). We can rewrite the one in (6.1) as

$$\mathbb{E}\left[\mathbb{1}_{\left\{ \Gamma_\varphi(\vec{x}_{[n+2]}) \text{ is connected} \right\}} \sum_{J \subset [n+2]} (-1)^{n-|J|} \mathbb{1}_{\left\{ x_1 \longleftrightarrow x_2 \text{ in } \Gamma_\varphi(\vec{x}_J) \right\}} \right]. \qquad (6.2)$$

Note that now, any nonvanishing J needs to contain $\{1, 2\}$. We are now going to observe some cancellations. For a fixed graph $G \in \mathcal{C}(\vec{x}_{[n+2]})$,

$$\sum_{J \subseteq [n+2]} (-1)^{n-|J|} \mathbb{1}_{\{x_1 \longleftrightarrow x_2 \text{ in } G[\vec{x}_J]\}} = \sum_{I, J \subseteq [n+2]} (-1)^{n-|J|} \mathbb{1}_{\{\{1,2\} \subseteq I \subseteq J\}} \mathbb{1}_{\{\mathscr{C}(x_1, G[\vec{x}_J]) = \vec{x}_I\}}$$

$$= \sum_{I, J \subseteq [n+2]} (-1)^{n-|J|} \mathbb{1}_{\{\{1,2\} \subseteq I \subseteq J\}} \mathbb{1}_{\{G[\vec{x}_I] \text{ connected}\}} \mathbb{1}_{\{\forall j \in J \setminus I : x_j \nsim \vec{x}_I\}}$$

$$= \sum_{\{1,2\} \subseteq I \subseteq [n+2]} (-1)^{n-|I|} \mathbb{1}_{\{G[\vec{x}_I] \text{ connected}\}}$$

$$\times \sum_{J \subseteq [n+2] \setminus I} (-1)^{|J|} \mathbb{1}_{\{\forall j \in J : x_j \nsim \vec{x}_I\}}. \qquad (6.3)$$

Note that for given G and I, defining $\mathcal{I}(G, I) = \{j \in [n+2] \setminus I : x_j \nsim \vec{x}_I\}$, we can rewrite

$$\sum_{J \subseteq [n+2] \setminus I} (-1)^{|J|} \mathbb{1}_{\{\forall j \in J : x_j \nsim \vec{x}_I\}} = \sum_{J \subseteq \mathcal{I}(G,I)} (-1)^{|J|}. \tag{6.4}$$

The only case for which (6.4) does not vanish is when $\mathcal{I}(G, I) = \varnothing$. We can therefore rewrite (6.3) as

$$\sum_{J \subseteq [n+2]} (-1)^{n-|J|} \mathbb{1}_{\left\{x_1 \longleftrightarrow x_2 \text{ in } G[\vec{x}_J]\right\}} = \sum_{\{1,2\} \subseteq I \subseteq [n+2]} (-1)^{n-|I|} \mathbb{1}_{\{G[\vec{x}_I] \text{ connected}\}} \mathbb{1}_{\left\{\forall j \in [n+2] \setminus I : x_j \sim \vec{x}_I\right\}},$$

and so (6.2) becomes

$$\sum_{\{1,2\} \subseteq I \subseteq [n+2]} (-1)^{n-|I|} \mathbb{P}\big(\Gamma_\varphi(\vec{x}_I) \text{ is connected, } x_j \sim \vec{x}_I \forall j \in [n+2] \setminus I\big)$$

$$= \sum_{\{1,2\} \subseteq I \subseteq [n+2]} \mathbb{P}\big(\Gamma_\varphi(\vec{x}_I) \text{ is connected}\big) \prod_{j \in [n+2] \setminus I} \left[\prod_{i \in I} \big(1 - \varphi(x_i - x_j)\big) - 1 \right]$$

$$= \sum_{I \subseteq [n+2]} \sum_G \mathbf{w}^{\pm}(G).$$

In the last line, summation is over the same set of graphs as in (3.6), with the additional restriction that $V(G^+) = I$. Resolving the partition over I gives that (6.1) is equal to (3.6).

The direct-connectedness function. In [15, Theorem 5.1], it is shown that there exists λ_0 such that for $\lambda \in [0, \lambda_0)$,

$$g_\lambda(x_1, x_2) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int \sum_{G \in \mathcal{D}_{x_1,x_2}\left(\vec{x}_{[n+2]}\right)} \prod_{e \in E(G)} \varphi(e)$$

$$\times \sum_{\substack{H \in \mathcal{C}\left(\vec{x}_{[n+2]}\right): \\ H \subseteq G}} \sum_{\substack{J \subseteq [n+2]: \\ x_1 \longleftrightarrow x_2 \text{ in } H[\vec{x}_J]}} (-1)^{n-|J|+|E(G) \setminus E(H)|} \, \mathrm{d}\vec{x}_{[3,n+2]}. \tag{6.5}$$

We show that the integrand in (6.5) is equal to the one in (4.1). With the calculations (6.3) and (6.4) performed for the two-point function, letting $I^c = [n+2] \setminus I$, we have

$$\sum_{J \subseteq [n+2]} (-1)^{n-|J|} \mathbb{1}_{\left\{x_1 \longleftrightarrow x_2 \text{ in } H[\vec{x}_J]\right\}} = \sum_{\{1,2\} \subseteq I \subseteq [n+2]} (-1)^{n-|I|} \mathbb{1}_{\{H[\vec{x}_I] \text{ is connected}\}} \mathbb{1}_{\left\{\forall j \in I^c : x_j \sim \vec{x}_I \text{ in } H\right\}}.$$

The two indicators imply that H is connected, and so

$$\sum_{\substack{H \in \mathcal{C}\left(\vec{x}_{[n+2]}\right): \\ H \subseteq G}} \sum_{\substack{J \subseteq [n+2]: \\ x_1 \longleftrightarrow x_2 \text{ in } H[\vec{x}_J]}} (-1)^{n-|J|+|E(G) \setminus E(H)|}$$

$$= \sum_{\{1,2\} \subseteq I \subseteq [n+2]} \sum_{H \subseteq G} (-1)^{n-|I|+|E(G) \setminus E(H)|} \mathbb{1}_{\{H[\vec{x}_I] \text{ is connected}\}} \mathbb{1}_{\left\{\forall j \in I^c : x_j \sim \vec{x}_I \text{ in } H\right\}}$$

$$= \sum_{\{1,2\} \subseteq I \subseteq [n+2]} \sum_{\substack{H' \in \mathcal{C}(\vec{x}_I): \\ H' \subseteq G}} (-1)^{n-|I|+|E(G) \setminus E(H')|} \sum_{\substack{F \subseteq E(G) \cap \left((I \times I^c) \cup \binom{I^c}{2}\right): \\ \forall j \in I^c : F \cap (I \times \{j\}) \neq \varnothing}} (-1)^{|F|}. \tag{6.6}$$

Note that for the second identity in (6.6), we split the edges of H into those contained in H' (the subgraph induced by I) and the remaining ones, called F.

When $E(G) \cap \binom{I^c}{2} \neq \varnothing$, the sum over F vanishes. Hence, the sum over I can be reduced to those I such that $G[I^c]$ contains no edges. For such sets I, we have

$$\sum_{\substack{F \subset E(G) \cap (I \times I^c): \\ \forall j \in I^c: F \cap (I \times \{j\}) \neq \varnothing}} (-1)^{|F|} = \prod_{j \in I^c} \sum_{\varnothing \neq F_j \subseteq E(G) \cap (I \times \{j\})} (-1)^{|F_j|} = \prod_{j \in I^c} (-1) = (-1)^{n-|I|}. \quad (6.7)$$

If we insert (6.7) back into (6.6), the two factors $(-1)^{n-|I|}$ cancel out, and so

$$\sum_{\substack{H \in \mathcal{C}(\vec{x}_{[n+2]}): \\ H \subseteq G}} \sum_{\substack{J \subseteq [n+2]: \\ x_1 \longleftrightarrow x_2 \text{ in } H[\vec{x}_J]}} (-1)^{n-|J|+|E(G)\setminus E(H)|}$$

$$= \sum_{\substack{\{1,2\} \subseteq I \subseteq [n+2]: \\ E(G[I^c])=\varnothing}} \sum_{\substack{H \in \mathcal{C}(\vec{x}_I): \\ H \subseteq G}} (-1)^{|E(G)\setminus E(H)|}$$

$$= \sum_{H \in \mathcal{G}(\vec{x}_{[n+2]}): G \triangleright H} \mathbb{1}_{\{\{1,2\} \subseteq V(H)\}} (-1)^{|E(G)\setminus E(H)|}, \quad (6.8)$$

where $G \triangleright H$ means that $E(H) \subseteq E(G)$, the subgraph of H induced by the vertices incident to at least one edge (call this set $V_{\geq 1}(H)$) is connected, and the subgraph of G induced by $[n+2] \setminus V_{\geq 1}(H)$ contains no edges.

With the identity (6.8), and letting $X = \vec{x}_{[n+2]}$, the integrand of (6.5) is equal to

$$\sum_{\substack{H \in \mathcal{G}(X): \\ \{x_1, x_2\} \subseteq V_{\geq 1}(H), \\ H[V_{\geq 1}(H)] \text{ connected}}} \prod_{e \in E(H)} \varphi(e) \sum_{\substack{F \subseteq \binom{X}{2} \setminus E(H): \\ \forall e \in F: e \cap V_{\geq 1}(H) \neq \varnothing, \\ (X, F \cup E(H)) \in \mathcal{D}_{x_1, x_2}(X)}} (-1)^{|F|} \prod_{e \in F} \varphi(e)$$

$$= \sum_{\substack{H \in \mathcal{G}(X): \\ x_1 \longleftrightarrow x_2}} \prod_{e \in E(H)} \varphi(e) \sum_{\substack{F \subseteq \binom{X}{2} \setminus E(H): \\ (X, F \cup E(H)) \in \mathcal{D}_{x_1, x_2}(X)}} (-1)^{|F|} \prod_{e \in F} \varphi(e)$$

$$= \sum_{\substack{C \in \mathcal{D}_{x_1, x_2}^{\pm}(X): \\ x_1 \overset{+}{\longleftrightarrow} x_2}} \mathbf{w}^{\pm}(G). \quad (6.9)$$

The argument for the first identity in (6.9) is the same as for the identity of (3.6) and (3.7).

6.3. Connections to the lace expansion

Both the graphical power series expansions and the lace expansion provide expressions for the direct-connectedness function. In this section, we show how to get from one to the other. Note that the statements to follow hold for sufficiently small intensities and *cannot* replace the lace expansion, which works all the way up to λ_c. The emphasis of this section is on the qualitative nature of the results.

We first summarize some results of [10], where the lace expansion is applied to the RCM. We keep some of the definitions brief and informal, and we refer to [10] for the detailed definitions in these cases.

On the lace expansion. In [10], among other things, the OZE is proved for τ_λ in *high dimension* (and for certain classes of connection functions φ; see [10, Section 1.2]). In particular, it is shown that

$$g_\lambda(x) = \varphi(x) + \Pi_\lambda(x),$$

with $\Pi_\lambda(x) = \sum_{n \geq 0} (-1)^n \Pi_\lambda^{(n)}(x)$. The functions $\Pi_\lambda^{(n)}$ are called the *lace-expansion coefficients*; they are nonnegative and have a quite involved probabilistic interpretation. To briefly define them, let $\{x \overset{A}{\longleftrightarrow} y \text{ in } \xi^{x,y}\}$ be the event that $x \longleftrightarrow y$ in $\xi^{x,y}$, but x is no longer connected to y in an A-thinning of η^y. Informally, every point $z \in \eta$ survives an A-thinning with probability $\prod_{y \in A} (1 - \varphi(z - y))$. See [10, Definition 3.2] for a formal definition. Letting

$$E\big(x, y; A, \xi^{x,y}\big) = \big\{x \overset{A}{\longleftrightarrow} y \text{ in } \xi^{x,y}\big\} \cap \big\{\nexists w \in \mathsf{Piv}\big(x, y; \xi^{x,y}\big) : x \overset{A}{\longleftrightarrow} w \text{ in } \xi^x\big\},$$

we introduce a sequence $\xi_0, \dots, \xi_n$ of independent RCMs and define

$$\Pi_\lambda^{(0)}(x) := \sigma_\lambda(x) - \varphi(x),$$

$$\Pi_\lambda^{(n)}(u_n) := \lambda^n \int \mathbb{P}_\lambda\left(\big\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\big\} \cap \bigcap_{i=1}^{n} E\big(u_{i-1}, u_i; \mathscr{C}(u_{i-2}, \xi_{i-1}^{u_{i-2}}), \xi_i^{u_{i-1},u_i}\big)\right) d\vec{u}_{[0,n-1]} \tag{6.10}$$

for $n \geq 1$ (with $u_{-1} = \mathbf{0}$). The method of proof is called the *lace expansion*, a perturbative technique in which one first proves via induction that

$$\tau_\lambda(x) = \varphi(x) + \sum_{m=0}^{n} (-1)^m \Pi_\lambda^{(m)}(x) + \lambda\left(\left(\varphi + \sum_{m=0}^{n} (-1)^m \Pi_\lambda^{(m)}\right) * \tau_\lambda\right)(x) + R_{\lambda,n}(x) \tag{6.11}$$

for $n \in \mathbb{N}_0$ and some remainder term $R_{\lambda,n}$ (see [10, Definition 3.7]), and then shows that the partial sum converges to $\Pi_\lambda = g_\lambda - \varphi$ and that $R_{\lambda,n} \to 0$ as $n \to \infty$.

The lace expansion was first devised for self-avoiding walks by Brydges and Spencer [3] and takes some inspiration from cluster expansions. It was later applied to percolation (specifically, bond percolation on $\mathbb{Z}^d$) by Hara and Slade [8]. While the name stems from laces that appear in the pictorial representation in [3], laces are absent in the representation for percolation models.

We show that we can rewrite $\Pi_\lambda^{(n)}$ in terms of graphs that are associated to a lace of size n. More generally, rewriting $\Pi_\lambda^{(n)}$ should serve as a bridge between the graphical expansions for g_λ that are well known in the physics literature, and the expression for g_λ in terms of lace-expansion coefficients.

The big advantage in the lace expansion lies in the probabilistic nature of all the terms that appear, allowing one to bound most of the integrals that appear by the expected cluster size, which is finite for $\lambda < \lambda_c$. The downside is the absence of a direct expression for g_λ and thus a direct proof of the OZE, which is only obtained after performing the $n \to \infty$ limit in (6.11).

We now show how to re-sum the graphical expansion for τ_λ and how to obtain the lace-expansion coefficients by appropriate grouping of terms.

Building the connection. For $x, y \in X$, let $\tilde{\mathcal{C}}_{x,y}^{\pm}(X) \subset \mathcal{C}^{\pm}(X)$ be the set of graphs in $\mathcal{C}^{\pm}(X)$ such that G^+ is connected and contains $\{x, y\}$, and $E(G[V \setminus V^+]) = \varnothing$. Hence, all $(-)$-edges are

incident to at least one vertex in $V(G^+)$. This is exactly the set of graphs summed over in (3.6). Indeed,

$$\tau_\lambda(x_1, x_2) = \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{G \in \tilde{\mathcal{C}}_{x_1, x_2}^{\pm}\left(\vec{x}_{[n+2]}\right)} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec{x}_{[3, n+2]}. \tag{6.12}$$

If we define $\tilde{\mathcal{D}}_{x,y}^{\pm}(X) := \mathcal{D}_{x,y}^{\pm}(X) \cap \tilde{\mathcal{C}}_{x,y}^{\pm}(X)$, we can express $g_\lambda(x_1, x_2)$ by replacing the graphs summed over in (6.12) by $\tilde{\mathcal{D}}_{x,y}^{\pm}\left(\vec{x}_{[n+2]}\right)$.

We are going to recycle some notation from Section 4. We split G into its core G_{core} and its shell H, so that

$$\mathsf{PD}^+(x, y, G_{\mathrm{core}}) = \mathsf{PD}^{\pm}(x, y, G_{\mathrm{core}}) = (u_0, V_0, u_1, \ldots, u_k, V_k, u_{k+1})$$

for some k (where $u_0 = x$ and $u_{k+1} = y$). We also recall that G 'contains' a skeleton (see Definition 4.3), a graph on $[k+1]_0$.

Definition 6.1. (*The minimal lace.*) Let G be a graph with core G_{core} and shell H; let $\vec{W} = (u_0, V_0 \ldots, u_{k+1})$ for $k \in \mathbb{N}$ be its $(+)$-pivot decomposition. We define the minimal lace $L_{\min}(x, y; G)$ as the lace with the following properties:

- L (having bonds $\alpha_i \beta_i$ with $i \in [m]$ for some $m \in \mathbb{N}$) is contained as a subgraph in the skeleton $\hat{H}$;

- for every $i \in [m]$, among all the bonds $\alpha\beta$ in $\hat{H}$ satisfying $\alpha < \beta_{i-1}$, the bond $\alpha_i \beta_i$ maximizes the value of β. For $i = 1$, we take $\beta_0 = 1$.

If $\mathsf{Piv}^+(x, y; G) = \varnothing$, we say that G has a minimal lace of size 0.

In other words, the first stitch $0\beta_1$ maximizes the value of β_1 among all stitches starting at 0, the second stitch has a maximal value of β_2 among the stitches with $1 \leq \alpha_2 < \beta_1$, and so on.

As a side remark, it is worth noting that the minimal laces offer an alternative way of partitioning the set of all shell graphs by mapping every shell graph H onto its minimal lace. This gives a standard procedure used in lace expansion for self-avoiding walks; performing it 'backwards' yields precisely the mapping described below Definition 4.4.

With the notion of minimal laces, we partition

$$g_\lambda(x_1, x_2) = \sum_{m \geq 0} \pi_\lambda^{(m)}(x_1, x_2), \tag{6.13}$$

where

$$\pi_\lambda^{(m)}(x_1, x_2) := \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{\substack{G \in \tilde{\mathcal{D}}_{x_1, x_2}^{\pm}\left(\vec{x}_{[n+2]}\right): \\ \|L_{\min}\| = m}} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec{x}_{[3, n+2]}. \tag{6.14}$$

We also set $\pi_\lambda^{(m)}(x) = \pi_\lambda^{(m)}(\mathbf{0}, x)$.

We strongly expect that the (pointwise) absolute convergence of the power series on the right-hand side of (6.14) holds (at least) in the domain of absolute convergence of the physicists' expansion (4.1) and thus, as already discussed, for sufficiently small intensities $\lambda > 0$.

However, a proof would go beyond the scope of the discussion here; therefore we formulate the absolute convergence of $\pi_\lambda^{(m)}$ (in the above sense) as an assumption for the following result (Lemma 6.1).

Assumption 6.1. *There exists $0 < \lambda_\star \leq \lambda_c$ such that the right-hand side of (6.14) is (point-wise) absolutely convergent for all $m \in \mathbb{N}$ and $\lambda < \lambda_\star$.*

Under Assumption 6.1, we show that the coefficients defined in (6.14) are basically identical to the lace-expansion coefficients introduced in (6.10).

Lemma 6.1. (Identity for the lace-expansion coefficients.) *Let $m \geq 1$ and let $\lambda < \lambda_\star$. Then*

$$\Pi_\lambda^{(0)}(x) = \pi_\lambda^{(0)}(\mathbf{0}, x) - \varphi(x),$$
$$(-1)^m \Pi_\lambda^{(m)}(x) = \pi_\lambda^{(m)}(\mathbf{0}, x).$$

As a side note, since $\Pi_\lambda^{(m)}$ is nonnegative, Lemma 6.1 shows that the sign of $\pi_\lambda^{(m)}$ alternates, which is far from obvious from the definition in (6.14).

Next, we prove an approximate version of the OZE in analogy to [10, Proposition 3.8]. Clearly, Lemma 6.2 follows immediately from the latter via Lemma ; however, we want to present a short independent proof on the level of formal power series, which we consider instructive for the understanding of the underlying combinatorics. We emphasize that the proof presented here treats the claim of Lemma 6.2 as an identity between formal power series; in particular, we do not concern ourselves with absolute convergence of the power series appearing in (6.20) and in (6.21).

Lemma 6.2. (The lace expansion in terms of $(\pm)$-graph coefficients.) *Let $m \in \mathbb{N}_0$, let $\lambda < \lambda_\star$, and set $\pi_{\lambda,m}(x) := \sum_{i=0}^m \pi_\lambda^{(i)}(\mathbf{0}, x)$. Then*

$$\tau_\lambda(x) = \pi_{\lambda,m}(x) + \left(\pi_{\lambda,m} * \tau_\lambda\right)(x) + R_{\lambda,m}(x),$$

where $R_{\lambda,m}$ is defined in [10, Definition 3.7].

Before carrying out the proof of Lemma 6.1, we define

$$\bar{\varphi}(A, B) = \prod_{a \in A} \prod_{b \in B} (1 - \varphi(a - b))$$

and $\bar{\varphi}(a, B) = \bar{\varphi}(\{a\}, B)$. Now, observe that, given a set $A \subset \mathbb{R}^d$ and an RCM event F,

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \bar{\varphi}\left(A, \vec{v}_{[n+2]}\right) \sum_{\substack{G \in \tilde{\mathcal{C}}_{v_1, v_2}^\pm (\vec{v}_{[n+2]}): \\ G^+ \in F}} \mathbf{w}^\pm(G) \, d\vec{v}_{[3,n+2]} = \mathbb{P}_\lambda\left(\xi\left(\eta_{\langle A\rangle}^{v_1, v_2}\right) \in F\right), \quad (6.15)$$

where $\xi(\eta)$ is the RCM on the basis of the point process η and $\eta_{\langle A\rangle}^v$ is an A-thinning of η^v (the usual PPP of intensity λ and added point v). In particular, v may be thinned out as well. We remark that $\eta_{\langle A\rangle}$ has the same distribution as a PPP of intensity $\lambda\bar{\varphi}(A, \cdot)$.
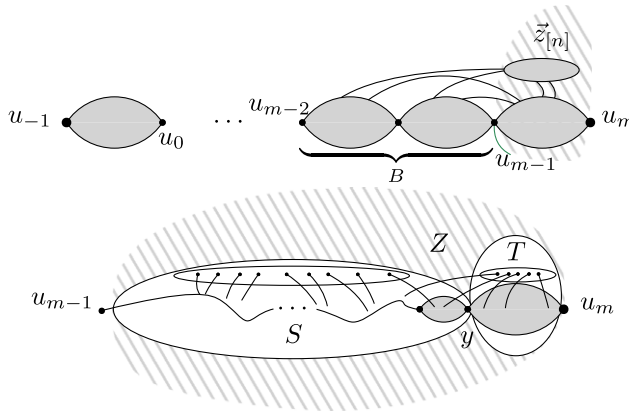
FIGURE 7. Illustration for the proof of Lemma 6.1. On the left, we see an example graph $G \in \mathcal{B}$; the grey bags on the bottom represent $\mathsf{PD}^+(u_{-1}, u_m, G)$ (note that there can be pivotal points within a grey bag). The minimal lace $L_{\min}$ is not depicted; however, note that $p_m \in B$. On the right, we see a schematic zoom into $G[Z]$, where $Z = \vec{z}_{[n]}$, together with the partition $Z = S \cup T$.

Proof of Lemma 6.1. The statement for $m = 0$ is clear. For $m > 0$, we can rewrite $\pi_\lambda^{(m)}$ as

$$\pi_\lambda^{(m)}(u_{-1}, u_m) = \lambda^m \int_{(\mathbb{R}^d)^m} \sum_{k,n \geq 0} \frac{\lambda^{k+n}}{k!n!} \int_{(\mathbb{R}^d)^{k+n}} \sum_{G \in \mathcal{B}} \mathbf{w}^\pm(G) \, \mathrm{d}\big(\vec{u}_{[0,m-1]}, \vec{x}_{[k]}, \vec{z}_{[n]}\big), \qquad (6.16)$$

where $\mathcal{B} \subseteq \tilde{\mathcal{D}}_{u_{-1},u_m}^\pm \big(\vec{u}_{[-1,m]} \cup \vec{x}_{[k]} \cup \vec{z}_{[n]}\big)$ are the graphs such that

- u_0 is the first pivotal point in $\mathsf{Piv}^+(u_{-1}, u_m; G)$ (i.e., $\mathsf{ord}(u_0) = 2$);

- $\vec{u}_{[0,m-1]} \subseteq \mathsf{Piv}^+(u_{-1}, u_m; G)$ and $u_{i-1} \prec u_i$;

- there are points $p_2, \ldots, p_m$ such that $L_{\min} \mathrel{\hat{=}} \{(u_{-1}, u_1), (p_2, u_2), \ldots, (p_m, u_m)\}$;

- $\vec{z}_{[n]}$ are those vertices $z \notin \{u_{m-1}, u_m\}$ in G such that $\{z\} \cup N(z)$ contains at least one vertex y of order $y \succ u_{m-1}$.

Given a graph $G \in \mathcal{B}$, let B denote the set of points x in $V(G^+)$ with $u_{m-2} \preccurlyeq x \prec u_{m-1}$. See Figure 7 for an illustration of such a graph G. We integrate out the points $\vec{z}$ first and claim that their contribution to (6.16) is

$$\lambda \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{H \in \mathcal{B}^\star} \mathbf{w}^\pm(H) \, \mathrm{d}\vec{z}_{[n]} = -\lambda \mathbb{P}_\lambda \big(E\big(u_{m-1}, u_m; B, \xi^{u_{m-1},u_m}\big)\big), \qquad (6.17)$$

where every $H \in \mathcal{B}^\star$ is the subgraph of some $G \in \mathcal{B}$ and has vertex set $B \cup \{u_{m-1}, u_m\} \cup \vec{z}_{[n]}$ and precisely those edges in G that have at least one endpoint in $\{u_m\} \cup \vec{z}_{[n]}$.

We let y be the last pivotal point in $V(G^+)$, that is, $\mathsf{ord}(y) = \mathsf{ord}(u_m) - 2$. We write $Z = \vec{z}_{[n]}$ and split Z once more into those vertices 'in front of' and 'behind' y; that is, $Z = S \cup T$, where T are the points in G^+ of order $\mathsf{ord}(u_m) - 1$ together with the points in $V \setminus V(G^+)$ that are adjacent to the former, and $S = Z \setminus T$. Possibly $y = u_{m-1}$, in which case $S = \varnothing$. See Figure 7 for an illustration of this split of the vertices in Z.

Note that there are no restrictions on the $(-)$-edges between B and $S \cup \{y\}$, whereas there must be at least one $(-)$-edge between B and $T \cup \{u_m\}$. There are no restrictions on the

$(-)$-edges between $\{u_{m-1}\} \cup S \cap V(G^+)$ and $T \cup \{u_m\}$, whereas there cannot be any $(-)$-edges between $S \setminus V(G^+)$ and $T \cup \{u_m\}$. By distinguishing whether or not $S = \varnothing$, we find that the left-hand side of (6.17) is equal to

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \left(\bar\varphi(B, \vec z_{[n]} \cup \{u_m\}) - 1 \right) \sum_{\substack{G \in \tilde{\mathcal{C}}_{u_{m-1}, u_m}^{\pm} \left(\{u_{m-1}, u_m\} \cup \vec z_{[n]} \right) : \\ u_{m-1} \overset{+}{\Longleftrightarrow} u_m}} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec z_{[n]}$$

$$+ \lambda \sum_{k \geq 0} \frac{\lambda^k}{k!} \int_{(\mathbb{R}^d)^{k+1}} \bar\varphi(B, \vec s_{[k]} \cup \{y\}) \sum_{\substack{H \in \tilde{\mathcal{C}}_{u_{m-1}, y}^{\pm} \left(\{u_{m-1}, y\} \cup \vec s_{[k]} \right) : \\ u_{m-1} \overset{+}{\Longleftrightarrow} y}} \mathbf{w}^{\pm}(H)$$

$$\times \left(\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \left(\bar\varphi(B, \vec t_{[n]} \cup \{u_m\}) - 1 \right) \bar\varphi(V^+(H) \setminus \{y\}, \vec t_{[n]} \cup \{u_m\}) \right.$$

$$\left. \sum_{\substack{G \in \tilde{\mathcal{C}}_{y, u_m}^{\pm} (\{y, u_m\} \cup \vec t_{[n]}) : \\ y \overset{+}{\Longleftrightarrow} u_m}} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec t_{[n]} \right) \mathrm{d}\vec s_{[k]} \, \mathrm{d}y$$

$$= \mathbb{P}_\lambda \left(u_{m-1} \Longleftrightarrow u_m \text{ in } \xi\left(\{u_{m-1}\} \cup \eta_{\langle B \rangle}^{u_m} \right) \right) - \mathbb{P}_\lambda \left(u_{m-1} \Longleftrightarrow u_m \text{ in } \xi^{u_{m-1}, u_m} \right)$$

$$+ \lambda \int_{\mathbb{R}^d} \mathbb{E}_\lambda \left[\mathbb{1}_{\left\{ u_{m-1} \longleftrightarrow y \text{ in } \xi\left(\{u_{m-1}\} \cup \eta_{\langle B \rangle}^y \right) \right\}} \right.$$

$$\left. \times \left(\mathbb{P}_\lambda \left(y \Longleftrightarrow u_m \text{ in } \xi\left(\{y\} \cup \eta_{\langle B \cup \mathscr{C} \rangle}^{u_m} \right) \right) - \mathbb{P}_\lambda \left(y \Longleftrightarrow u_m \text{ in } \xi\left(\{y\} \cup \eta_{\langle \mathscr{C} \rangle}^{u_m} \right) \right) \right) \right] \mathrm{d}y, \tag{6.18}$$

where we abbreviate $\mathscr{C} = \mathscr{C}(u_{m-1}, \xi^{u_{m-1}})$. Note that the inner probabilities are conditional on the random variable $\mathscr{C}$. We now resolve the integral over y by use of the Mecke equation and incorporate the first two summands as the case $y = u_{m-1}$. With this, (6.18) becomes

$$\mathbb{E}_\lambda \left[\sum_{y \in \eta^{u_{m-1}}} \mathbb{1}_{\left\{ u_{m-1} \longleftrightarrow y \text{ in } \xi\left(\{u_{m-1}\} \cup \eta_{\langle B \rangle}^y \right) \right\}} \mathbb{1}_{\left\{ y \Longleftrightarrow u_m \text{ in } \xi\left(\{y\} \cup (\eta^{u_m} \setminus \mathscr{C}')_{\langle B \rangle} \right) \right\}} \right]$$

$$- \mathbb{E}_\lambda \left[\sum_{y \in \eta^{u_{m-1}}} \mathbb{1}_{\left\{ u_{m-1} \longleftrightarrow y \text{ in } \xi^{u_{m-1}} \right\}} \mathbb{1}_{\left\{ y \Longleftrightarrow u_m \text{ in } \xi\left(\eta^{u_m} \setminus \mathscr{C}' \right) \right\}} \right], \tag{6.19}$$

where $\mathscr{C}' = \mathscr{C}\left(u_{m-1}, \xi(\eta^{u_{m-1}} \setminus \{y\}) \right)$. But both terms in (6.19) are simply a partition over the last pivotal point for the connection between u_{m-1} and u_m, and so (6.19) equals

$$\mathbb{P}_\lambda \left(u_{m-1} \longleftrightarrow u_m \text{ in } \xi\left(\{u_{m-1}\} \cup \eta_{\langle B \rangle}^{u_m} \right) \right) - \tau_\lambda(u_m - u_{m-1}) = -\mathbb{P}_\lambda \left(E\left(u_{m-1}, u_m; B, \xi^{u_{m-1}, u_m} \right) \right),$$

proving (6.17). Lemma 6.1 can now be proven by iteratively applying (6.17). □

Proof of Lemma 6.2. For $m \in \mathbb{N}_0$, we can write

$$\tau_\lambda(x_1, x_2) = \sum_{l=0}^{m} \pi_\lambda^{(l)}(x_1, x_2) + \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{G \in \mathcal{A}} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec x_{[3, n+2]}, \tag{6.20}$$

where $\mathcal{A}$ is the set of graphs $G \in \tilde{\mathcal{C}}^{\pm}_{x_1,x_2}(\vec{x}_{[n+2]}) \setminus \tilde{\mathcal{D}}^{\pm}_{x_1,x_2}(\vec{x}_{[n+2]})$ together with the graphs $G \in$ $\tilde{\mathcal{D}}^{\pm}_{x_1,x_2}(\vec{x}_{[n+2]})$ where $\|L_{\min}\| > m$. Note that if $G \in \mathcal{A}$, then $\mathsf{Piv}^+(x_1, x_2; G) \neq \varnothing$.

For $G \in \mathcal{A}$ and $u \in \mathsf{Piv}^+(x_1, x_2; G)$, define

$$V^{\preccurlyeq}(u) := \{y \in V(G^+) : y \preccurlyeq u\} \cup \{y \in V(G) \setminus V(G^+) : \exists z \in N(y) \cap V(G^+) \text{ with } z \prec u\},$$

that is, all the core vertices of order at most that of u together with the shell vertices adjacent to at least one vertex of strictly smaller order than u. Next, let $u^{\mathrm{cut}} = u^{\mathrm{cut}}(x_1, x_2; G)$ be the vertex in $\mathsf{Piv}^+(x_1, x_2; G)$ such that

$$E\big(V^{\preccurlyeq}(u^{\mathrm{cut}}) \setminus \{u^{\mathrm{cut}}\}, V \setminus V^{\preccurlyeq}(u^{\mathrm{cut}})\big) = \varnothing \qquad \text{and} \qquad G\big[V^{\preccurlyeq}(u^{\mathrm{cut}})\big] \in \tilde{\mathcal{D}}^{\pm}_{x_1, u^{\mathrm{cut}}}.$$

If such a point exists, it is unique; if no such point exists, set $u^{\mathrm{cut}} = x_2$. We can now partition $\mathcal{A}$ as

$$\mathcal{A} = \left(\bigcup_{i=1}^{m} \mathcal{A}_i\right) \cup \mathcal{A}_{>m},$$

where

$$\mathcal{A}_i := \big\{G \in \mathcal{A} : u^{\mathrm{cut}} \neq x_2 \text{ and } \|L_{\min}(x_1, u^{\mathrm{cut}}; G[V^{\preccurlyeq}(u^{\mathrm{cut}})])\| = i\big\},$$
$$\mathcal{A}_{>m} := \big\{G \in \mathcal{A} : \|L_{\min}(x_1, u^{\mathrm{cut}}; G[V^{\preccurlyeq}(u^{\mathrm{cut}})])\| > m\big\}.$$

Now, if $x_s = u^{\mathrm{cut}}$ and $V' := V^{\preccurlyeq}(u^{\mathrm{cut}})$ as well as $V'' := \{x_s\} \cup (\vec{x}_{[n+2]} \setminus V')$, then

$$\mathbf{w}^{\pm}(G) = \mathbf{w}^{\pm}(G[V'])\mathbf{w}^{\pm}(G[V'']);$$

that is, the weight factors. Therefore, for every $i \in [m]$,

$$\sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{G \in \mathcal{A}_i} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec{x}_{[3,n+2]} = \lambda \int_{\mathbb{R}^d} \pi_{\lambda}^{(i)}(x_1, u)\tau_{\lambda}(u, x_2) \, \mathrm{d}u. \qquad (6.21)$$

Setting

$$\bar{R}_{\lambda,m}(x_2 - x_1) := \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{(\mathbb{R}^d)^n} \sum_{G \in \mathcal{A}_{>m}} \mathbf{w}^{\pm}(G) \, \mathrm{d}\vec{x}_{[3,n+2]},$$

we can rewrite (6.20) as

$$\tau_{\lambda}(x) = \pi_{\lambda,m}(x) + \lambda(\pi_{\lambda,m} * \tau_{\lambda})(x) + \bar{R}_{\lambda,m}(x).$$

One can now prove by hand or by employing Lemma 6.1 that $\bar{R}_{\lambda,m} = R_{\lambda,m}$. $\qquad \square$

6.4. Other percolation models

The results of this paper should apply in quite analogous fashion to all other percolation models that enjoy sufficient independence—in particular, to (long-range) bond and site percolation on $\mathbb{Z}^d$. We take bond percolation on $\mathbb{Z}^d$ with edge parameter p as an example. We can adjust our notation by using $\mathcal{C}(x, y, \mathbb{Z}^d)$ to denote the connected subgraphs of $\mathbb{Z}^d$ containing x

and y, and we define $\mathcal{D}_{x,y}(\mathbb{Z}^d)$ and the notions for $(\pm)$-graphs analogously. Then one can show that, if we restrict to a finite box $\Lambda \subset \mathbb{Z}^d$, the two-point function satisfies

$$\tau_p^\Lambda(x_1, x_2) = \sum_{n \geq 0} p^n \sum_{\substack{G \in \mathcal{C}^\pm(x_1, x_2, \Lambda) : |E(G)| = n, \\ x_1 \overset{+}{\longleftrightarrow} x_2}} (-1)^{|E^-(G)|}. \tag{6.22}$$

One can easily observe that all graphs summed over in (6.22) that contain more than one $(+)$-cluster cancel out, which is also what happens in the RCM. The direct-connectedness function can be defined analogously to Definition 4.2, providing a suitable setup for an analysis analogous to the one in Section 4.

Acknowledgements

Funding information

Competing interests

References

[1] BETSCH, S. AND LAST, G. (2021). On the uniqueness of Gibbs distributions with a non-negative and subcritical pair potential. Preprint. Available at https://arxiv.org/abs/2108.06303.

[2] BRYDGES, D. C. (1986). A short course on cluster expansions. In *Phénomènes Critiques, Systèmes Aléatoires, Théories de Jauge (Les Houches, 1984)*, Parts I–II, North-Holland, Amsterdam, pp. 129–183.

[3] BRYDGES, D. C. AND SPENCER, T. (1985). Self-avoiding walk in 5 or more dimensions. *Commun. Math. Phys.* **97**, 125–148.

[4] CAMPANINO, M. AND IOFFE, D. (2002). Ornstein–Zernike theory for the Bernoulli bond percolation on $\mathbb{Z}^d$. *Ann. Prob.* **30**, 652–682.

[5] CHIEW, Y. C. AND STELL, G. (1989). Connectivity and percolation of randomly centered spheres: correction to the Percus–Yevick approximation. *J. Chemical Phys.* **90**, 4956–4959.

[6] CONIGLIO, A., DEANGELIS, U. AND FORLANI, A. (1977). Pair connectedness and cluster size. *J. Phys. A* **10**, 1123–1139.

[7] HANSEN, J. P. AND MCDONALD, I. R. (2013). *Theory of Simple Liquids: with Applications to Soft Matter*. Academic Press, Amsterdam.

[8] HARA, T. AND SLADE, G. (1990). Mean-field critical behaviour for percolation in high dimensions. *Commun. Math. Phys.* **128**, 333–391.

[9] HEYDENREICH, M. AND VAN DER HOFSTAD, R. (2017). *Progress in High-Dimensional Percolation and Random Graphs*. Springer, Cham.

[10] HEYDENREICH, M., VAN DER HOFSTAD, R., LAST, G. AND MATZKE, K. (2019). Lace expansion and mean-field behavior for the random connection model. Preprint. Available at https://arxiv.org/abs/1908.11356.

[11] HILL, T. L. (1955). Molecular clusters in imperfect gases. *J. Chem. Phys.* **23**, 617–622.

[12] JANSEN, S. (2016). Continuum percolation for Gibbsian point processes with attractive interactions. *Electron. J. Prob.* **21**, article no. 47, 22 pp.

[13] KINGMAN, J. F. C. (1967). Completely random measures. *Pacific J. Math.* **21**, 59–78.

[14] LAST, G. AND PENROSE, M. D. (2018). *Lectures on the Poisson Process*. Cambridge University Press.

[15] LAST, G. AND ZIESCHE, S. (2017). On the Ornstein–Zernike equation for stationary cluster processes and the random connection model. *Adv. Appl. Prob.* **49**, 1260–1287.

[16] MALYSHEV, V. A. AND MINLOS, R. A. (1991). *Gibbs Random Fields: Cluster Expansions* (Mathematics and Its Applications (Soviet Series), Vol. 44). Kluwer Academic Publishers Group, Dordrecht.

[17] MEESTER, R. (1995). Equality of critical densities in continuum percolation. *J. Appl. Prob.* **32**, 90–104.

[18] MEESTER, R. AND ROY, R. (1996). *Continuum Percolation*. Cambridge University Press.

[19] OGATA, Y. AND TANEMURA, M. (1989). Likelihood estimation of soft-core interaction potentials for Gibbsian point patterns. *Ann. Inst. Statist. Math.* **41**, 583–600.

[20] ORNSTEIN, L. S. AND ZERNIKE, F. (1914). Accidental deviations of density and opalescence at the critical point of a single substance. *Proc. Akad. Sci. (Amsterdam)* **17**, 793–806.

[21] PENROSE, M. D. (1993). On the spread-out limit for bond and continuum percolation. *Ann. Appl. Prob.* **3**, 253–276.

[22] SLADE, G. (2006). *The Lace Expansion and Its Applications: École d'été de Probabilités de Saint-Flour XXXIV—2004*. Springer, Berlin.

[23] STELL, G. (1976). Correlation functions and their generating functionals: general relations with applications to the theory of fluids. In *Phase Transitions and Critical Phenomena*, Vol. 5b, Academic Press, London, pp. 205–258.

[24] STELL, G. (1996). Continuum theory of percolation and association. *Physica A* **231**, 1–19.

[25] TORQUATO, S. (2002). *Random Heterogeneous Materials: Microstructure and Macroscopic Properties* (Interdisciplinary *Applied Mathematics*, Vol. 16). Springer, New York.