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RESEARCH ARTICLE

Gerber-Shiu analysis in the compound Poisson model with constant inter-observation times

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Abstract

In this paper, the classical compound Poisson model under periodic observation is studied. Different from the random observation assumption widely used in the literature, we suppose that the inter-observation time is a constant. In this model, both the finite-time and infinite-time Gerber-Shiu functions are studied via the Laguerre series expansion method. We show that the expansion coefficients can be recursively determined and also analyze the approximation errors in detail. Numerical results for several claim size density functions are given to demonstrate effectiveness of our method, and the effect of some parameters is also studied.

1. Introduction

In this paper, we consider the classical compound Poisson model $U = \{U_t\}_{t\geq 0}$ (also called Cramér–Lundberg model) for an insurance company that is described by

$$U_t = u + ct - S_t, \quad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus level and $c > 0$ is the constant premium rate per unit time. The aggregate claims process $S_t = \sum_{n=1}^{N_t} X_n$ is a compound Poisson process, where the claim number process $\{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the individual claim amounts $X_1, X_2, \ldots$, independent of N_t, form a sequence of positive i.i.d. random variables with common probability density function f_X. Here, we use X to denote a generic version of X_n which has probability density function f_X.

In insurance risk theory, it is usually assumed that the surplus process is continuously observed, and the insurers monitor the business risk by considering the event of ruin, which is defined as the first time when the surplus process drops below the zero level. The commonly used risk measures are the ultimate ruin probability, defined by $P(\inf_{t\geq 0} U_t < 0 \,|\, U_0 = u)$, and the expected discounted penalty function, that is, the Gerber-Shiu function [16]. Note that the latter is an extension of the ruin probability, since it incorporates as special cases more ruin-related quantities, such as the time of ruin, surplus immediately before ruin and the deficit at ruin. For the study of ruin problems under the compound Poisson model (and its various extension, e.g., Sparre Andersen model and Lévy risk model), we refer the interested readers to Asmussen and Albrecher [5].

Instead of continuous observation, Albrecher *et al.* [3,4] first proposed to study ruin problems under periodic observation of the surplus process. In the compound Poisson risk model, they supposed that the insurer only observes the surplus process at a sequence of discrete time points, and makes decisions on paying dividends and declaring ruin based on the observed surplus levels. We remark that the idea in their papers is both theoretically interesting and practically applicable, since the board of the insurance company usually checks the balance on a periodic basis. Recently, the idea of periodic observation of various risk models has been adopted by many authors. For example, Avanzi *et al.* [7], Zhang [31] and Zhang and Cheung [32,33] studied some dividend problems, and Zhang *et al.* [36,37] considered problems on capital injections and taxation, respectively. Note that in the above-mentioned literature, the underlying surplus is modeled by a continuous-time process, and business decisions (e.g., ruin, dividend, capital injection, etc.) are made at the periodic observation time points. This does not change the nature of the surplus being a continuous-time process, and corresponds to a discrete observation of the continuous stochastic process. Similar situations arise in mathematical finance; see, for example, discretely monitored barrier options correspond to checking the barrier breaching only at periodic observation times [23]; in applied probability; see, for example, optimal stopping problems with random intervention times [14], that is, optimal stopping decisions are made only at sequence of Poisson jump times; and also in statistics; see, for example, maximum likelihood estimation based on discretely observed diffusions [2].

In the above-mentioned papers on periodic observation, a common assumption is that the inter-observation times are random variables. Let Δ denote the generic inter-observation time. Albrecher *et al.* [3,4] consider the case when Δ follows either exponential distribution or Erlang distribution. Note that the mathematical treatment under these two distributions is relatively not hard, since the traditional methods (such as integro-differential equations, Laplace transform, and renewal theory) used in the analysis of continuous observation are still applicable. When Δ is an Erlang(n) random variable with density function

$$f_\Delta(t) = \frac{\gamma^n t^{n-1} e^{-\gamma t}}{(n-1)!}, \quad t > 0,$$

we can either (i) fix n and let $\gamma \to \infty$ to approximate the continuous observation; or (ii) fix $E\Delta = h > 0$ and let $n \to \infty$ to approximate the fixed observation time h. The latter setting is also called Erlangization technique, which is proposed by Asmussen *et al.* [6] to approximate the finite-time ruin probabilities. In the numerical sections of Albrecher *et al.* [3,4], it is shown that the Erlangization technique can be used to approximate some dividend and ruin related functions when the model is observed with fixed frequency. However, when the parameter n in the above Erlang density function is very large, some computation obstacles may appear since we usually have to find all the roots of the Lundberg equation and solve some linear equations.

In this paper, we shall directly consider the case when the inter-observation time Δ is a fixed constant, and suppose that U_t is monitored periodically at times $t = 0, \Delta, 2\Delta, \ldots$. The event of ruin is declared as soon as the monitored surplus level is negative, and the ruin time is defined by

$$\tau = \inf\{n\Delta : U_{n\Delta} < 0, n = 1, 2, \ldots\}.$$

The infinite-time expected discounted penalty function (also called Gerber-Shiu function) under above periodic observation setting is defined by

$$\phi(u) = E[e^{-\delta\tau} w(|U_\tau|)\mathbf{1}_{\{\tau<\infty\}}|U_0 = u], \quad u \ge 0,$$

where $\delta \ge 0$ is the force of interest, w is a nonnegative penalty function of the deficit at ruin and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. When $\delta = 0$ and $w \equiv 1$, the Gerber-Shiu function becomes the ultimate ruin probability under periodic observation.

In the practical management of the insurance business operation, the insurer may pay more attention to the risk over some finite-time horizon. In insurance risk theory, the study of finite-time ruin problems dates back to Prahbu [24] and Seal [26], where the finite-time ruin probability (or survival probability) is studied under the compound Poisson model. Over the last few decades, a series of contributions have been made by actuarial researchers; see, for example, Dickson and Willmot [13], Dickson and Li [12], Li and Sendova [20], Kuznetsov and Morales [17], Li and Lu [19] and Li *et al.* [21]. In particular, the finite-time ruin probability and the finite-time Gerber-Shiu function are studied in Lee *et al.* [18]

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and Li *et al.* [22] by the Fourier-cosine series expansion method, respectively. However, the scope of research in these papers is still limited to continuous observation of the surplus process. For the study on finite-time ruin problems with periodic observation, we refer the readers to Xie and Zhang [29,30].

In this paper, we shall also pay attention to the following finite-time Gerber-Shiu function defined by

$$\phi(u;T) = E[e^{-\delta\tau}w(|U_\tau|)\mathbf{1}_{\{\tau\le T\}}|U_0 = u], \quad u \ge 0, \tag{1.2}$$

where $T > 0$ is a deterministic time. To study $\phi(u;T)$, we can introduce the following auxiliary functions, for $n = 1, 2, \ldots$,

$$\phi_n(u) = E[e^{-\delta\tau}w(|U_\tau|)\mathbf{1}_{\{\tau\le n\Delta\}}|U_0 = u], \quad u \ge 0, \tag{1.3}$$

which are finite-time Gerber-Shiu functions when ruin occurs during the first n monitoring times. If we define $n_T = \max\{n : n\Delta \le T\}$, then we have $\phi(u;T) = \phi_{n_T}(u)$. Note that n_T is not a random variable. Hence, it suffices to study the finite-time Gerber-Shiu functions $\phi_n(u)$.

Throughout this paper, the following two conditions on the individual claim size density function f_X and the penalty function w will be used.

Condition 1. The claim size density function is square integrable.

Condition 2. The penalty function w is nonnegative, and $w(x) \le C_w[1 \vee x^\kappa]$ for some $C_w > 0$ and positive integer $\kappa \ge 1$.

Here, $x \vee y = \max(x, y)$ for real numbers x, y, and C_w is a generic constant which is used to express the error bounds in our error analysis. For many widely used penalty functions, Condition 2 is satisfied with $C_w = 1$. Note that Condition 1 is not very restrictive, since almost all claim size density functions, such as exponential, Erlang, Pareto and their linear combinations, satisfy this condition. Condition 2 means that the penalty function has at most a polynomial growth rate, which is satisfied by almost all the penalty functions used in risk theory.

The main technique adopted in this paper is the Laguerre series expansion, which shows that any square integrable function on the positive real line can be expanded on the Laguerre basis. Note that both Conditions 1 and 2 will provide some sufficient conditions on the square integrability of some functions considered in this paper, so that the Laguerre series expansion method is applicable. We remark that the Laguerre series expansion method has been applied to solve some ruin problems in the literature. For example, Zhang and Su [34,35] use Laguerre series expansion to estimate the Gerber-Shiu function, and Cheung and Zhang [9] study the approximation of ruin probability in a class of renewal risk models under interest force. However, these papers mainly consider the infinite-time ruin problems under continuous observation. For the finite-time ruin problems under periodic observation, it is worth mentioning the contributions made by Xie and Zhang [29,30]. In these two papers, it is assumed that an additional Brownian motion exists in their models, so that some Fourier transform methods can be well utilized to approximate the density function of the increments between successive observation times. In our paper, since the classical risk model U_t does not have the Brownian motion term, on the one hand, the Fourier transform methods can not approximate the density function of the aggregate claims process at high accuracy; on the other hand, the approximation formulas are usually very complex. Furthermore, we note that the Fourier transform methods fail to solve our problem when the individual claim size density function does not have a closed-form Fourier transform (e.g., Pareto distribution). The Laguerre series expansion method is totally different from the Fourier transform methods in Xie and Zhang [29,30]. Moreover, different from Xie and Zhang [29,30], we shall show that the Laguerre series expansion method can not only solve the finite-time ruin problems, but also approximate the infinite-time Gerber-Shiu function under periodic observation.

The reminder of this article is organized as follows. In Section 2, we give some preliminaries on Laguerre functions. Analysis of the density function of S_Δ is presented in Section 3, and in particular,

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we show how to approximate this density function by the Laguerre series expansion method. In Section 4, we derive integral equations for the finite-time Gerber-Shiu functions $\phi_n(u)$, and we also show that $\phi_n(u)$ can be approximated by some auxiliary functions $\tilde{\phi}_n(u)$. The Laguerre series expansions for $\tilde{\phi}_n(u)$ are derived in Section 5, and the approximation error is also analyzed. In Section 6, we study the Laguerre series expansion of the infinite-time Gerber-Shiu function $\phi(u)$. Some numerical results are given in Section 7 to show effectiveness of our method. Finally, a conclusion is given in Section 8. Some proofs of propositions are given in Appendix A.

2. Preliminaries on Laguerre functions

In this paper, we shall use Laguerre series expansion to study the finite-time Gerber-Shiu functions $\phi_n(u)$. Throughout this paper, we let $L^1(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+)$ denote the classes of real-valued absolutely integrable and square integrable functions on the positive half-line $\mathbb{R}_+$, respectively. Define the scalar product and L^2-norm on $L^2(\mathbb{R}_+)$ as

$$\langle f, g\rangle := \int_0^\infty f(x)g(x)\,dx, \quad \|f\|_2 := \sqrt{\langle f, f\rangle}, \quad \forall f, g \in L^2(\mathbb{R}_+),$$

respectively. For any $f \in L^1(\mathbb{R}_+)$, define its Fourier transform by

$$\mathcal{F}f(s) = \int_0^\infty e^{isx} f(x)\,dx, \quad s \in \mathbb{R},$$

and define its j-fold ($j \geq 2$) convolution recursively by

$$f^{*j}(x) = \int_0^x f(x-y)f^{*(j-1)}(y)\,dy, \quad x \geq 0,$$

with $f^{*1}(x) = f(x)$.

The Laguerre functions are defined by

$$\varphi_k(x) = \sqrt{2}L_k(2x)e^{-x}, \quad x \geq 0;\ k = 0, 1, \ldots,$$

where $L_k(x)$ are the Laguerre polynomials given by

$$L_k(x) = \sum_{j=0}^{k}(-1)^j \binom{k}{j}\frac{x^j}{j!}, \quad x \geq 0;\ k = 0, 1, \ldots.$$

It is known that $\{\varphi_k\}_{k=0,1,\ldots}$ is a complete orthonormal basis of $L^2(\mathbb{R}_+)$, which satisfies (i) $\|\varphi_k\|_2 = 1$ for each $k \geq 0$; and (ii) $\langle \varphi_k, \varphi_j\rangle = 0$ for $k \neq j$. Hence, we can expand any $f \in L^2(\mathbb{R}_+)$ on the Laguerre basis as follows,

$$f(x) = \sum_{k=0}^{\infty} A_{f,k}\varphi_k(x), \quad x \geq 0, \tag{2.1}$$

where the Laguerre coefficients $A_{f,k}$ are given by

$$A_{f,k} = \langle f, \varphi_k\rangle = \int_0^\infty f(x)\varphi_k(x)\,dx, \quad k = 0, 1, \ldots.$$

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Furthermore, we have $\sup_k |A_{f,k}| \le \|f\|_2$, since by Cauchy–Schwarz inequality

$$|A_{f,k}| \le \left(\int_0^\infty f^2(x)dx\right)^{1/2} \left(\int_0^\infty \varphi_k^2(x)dx\right)^{1/2} = \|f\|_2.$$

The Laguerre functions have many nice properties. The following three properties will be used in the remainder of this paper.

Property 1. The collection $\{\varphi_k\}$ is uniformly bounded, that is, for each $k = 0, 1, 2, \ldots$,

$$\sup_{x\ge 0} |\varphi_k(x)| \le \sqrt{2}. \tag{2.2}$$

Property 2. The convolution of two Laguerre functions is a linear combination of two Laguerre functions, that is, for any $k, l = 0, 1, 2 \ldots$,

$$\int_0^x \varphi_k(x-y)\varphi_l(y)\,dy = \frac{1}{\sqrt{2}}[\varphi_{k+l}(x) - \varphi_{k+l+1}(x)], \quad x \ge 0. \tag{2.3}$$

Property 3. For integers $j, l = 0, 1, 2 \ldots$,

$$x^l \varphi_j(x) = \sum_{m=0}^{j+l} \Xi_{l,j,m}\varphi_m(x), \tag{2.4}$$

where $\Xi_{0,j,m} = \mathbf{1}_{\{j=m\}}$, and for $l = 0, 1, \ldots$,

$$\Xi_{l+1,j,m} = \begin{cases} \frac{1}{2}\Xi_{l,j,0} - \frac{1}{2}\Xi_{l,j,1}, & m = 0, \\ \left(m + \frac{1}{2}\right)\Xi_{l,j,m} - \frac{m}{2}\Xi_{l,j,m-1} - \frac{m+1}{2}\Xi_{l,j,m+1}, & m = 1, 2, \ldots, l+j-1, \\ \left(l + j + \frac{1}{2}\right)\Xi_{l,j,l+j} - \frac{l+j}{2}\Xi_{l,j,l+j-1}, & m = l+j, \\ -\frac{l+j+1}{2}\Xi_{l,j,l+j}, & m = l+j+1. \end{cases} \tag{2.5}$$

Furthermore, we have

$$|\Xi_{l,j,m}| \le 2^l(j+1+l)^l, \quad m = 0, 1, \ldots, l+j. \tag{2.6}$$

Properties 1 and 2 can be found in Abramowitz and Stegun [1], and Property 3 is proved in Lemma 2.1 of Cheung and Zhang [9].

In practical applications, we need to truncate the infinite series summation in (2.1) to compute the function f, that is, for some positive integer K,

$$f(x) \approx \hat{f}_K(x) := \sum_{k=0}^{K} A_{f,k}\varphi_k(x).$$

The above approximation error can be obtained by introducing the Sobolev–Laguerre space (see [8]). For constant $\theta > 0$, the Sobolev–Laguerre space is defined by

$$W(\mathbb{R}_+, \theta) = \left\{ f : f \in L^2(\mathbb{R}_+), \sum_{k=0}^{\infty} k^\theta A_{f,k}^2 < \infty \right\}.$$

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If $f \in W(\mathbb{R}_+, \theta)$ for some $\theta > 1$, by Cauchy–Schwarz inequality, we can obtain

$$\sup_{x\geq 0} |f(x) - \hat{f}_K(x)| = O(K^{-(\theta-1)/2}). \tag{2.7}$$

See Shimizu and Zhang [27] Prop. 3. Here and after, for two positive sequences $\{x_K\}$ and $\{y_K\}$, we use $x_K = O(y_K)$ to mean $\lim_{K\to\infty} x_K/y_K \leq C$ for some $C > 0$.

Remark 1. In general, it may not be easy to directly verify the condition $f \in W(\mathbb{R}_+, \theta)$. However, if θ is a positive integer, by Comte and Genon-Catalot [10] Sect. 2.2 we know that one sufficient condition for $\sum_{k=0}^{\infty} k^\theta A_{f,k}^2 < \infty$ is that f has derivatives up to order $\theta - 1$ with $f^{(\theta-1)}$ absolutely continuous and for $k = 0, 1, \ldots, \theta - 1$ the functions

$$x^{(k+1)/2}(fe^x)^{(k+1)}e^{-x} := x^{(k+1)/2} \sum_{j=0}^{k+1} \binom{k+1}{j} f^{(j)} \tag{2.8}$$

belong to $L^2(\mathbb{R}_+)$.

3. Analysis of the density function of S_Δ

In this section, we study the distribution of the size of total claims S_Δ during the time interval $[0, \Delta]$, which is a compound Poisson random variable. It is known that S_Δ follows a mixed distribution such that S_Δ has a probability mass at zero with $P(S_\Delta = 0) = e^{-\lambda\Delta}$, and on $(0, \infty)$ it has a probability density function given by

$$g(x) = e^{-\lambda\Delta} \sum_{j=1}^{\infty} \frac{(\lambda\Delta)^j}{j!} f_X^{*j}(x), \quad x > 0. \tag{3.1}$$

For the above density function g, we further truncate the infinite summation in (3.1) to obtain

$$g(x) \approx \tilde{g}(x) := e^{-\lambda\Delta} \sum_{j=1}^{J} \frac{(\lambda\Delta)^j}{j!} f_X^{*j}(x), \quad x > 0, \tag{3.2}$$

where J is a large positive integer.

The approximation error is given in the following proposition, which implies that the truncation error has an exponential decay rate w.r.t. J.

Proposition 1. *Under Condition 1, we have, for any $r > 0$*

$$\sup_{x\geq 0} |g(x) - \tilde{g}(x)| \leq C_g e^{-rJ}, \tag{3.3}$$

where $C_g = \|f_X\|_2^2 e^{\lambda\Delta(e^r-1)} e^{-r}$.

Remark 2. Although the coefficient C_g in the upper bound (3.3) is an increasing function of the parameter r, for each fixed r, we can take large truncation parameter J, so that the approximation error can be arbitrarily small.

When the claim size density function $f_X \in L^2(\mathbb{R}_+)$, we also have $f_X^{*j} \in L^2(\mathbb{R}_+)$ for each $j = 2, 3 \ldots$, since by Parseval's theorem we have, for each $j \geq 2$,

$$\|f_X^{*j}\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F} f_X(s)|^{2j} ds \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F} f_X(s)|^2 ds = \|f_X\|_2^2 < \infty,$$

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where we have used the fact $|\mathcal{F} f_X(s)| \leq 1$, $\mathcal{F} f_X^{*j}(s) = [\mathcal{F} f_X(s)]^j$. Furthermore, since $\tilde{g}$ is a finite linear combination of f_X^{*j}, we have $\tilde{g} \in L^2(\mathbb{R}_+)$. Hence, we can expand f_X^{*j} and $\tilde{g}$ on the Laguerre basis as follows,

$$f_X^{*j}(x) = \sum_{k=0}^{\infty} A_{f_X^{*j},k}\varphi_k(x), \quad \tilde{g}(x) = \sum_{k=0}^{\infty} A_{\tilde{g},k}\varphi_k(x), \quad x > 0, \tag{3.4}$$

where the Laguerre coefficients $A_{\tilde{g},k}$ and $A_{f_X^{*j},k}$ satisfy the following relation,

$$A_{\tilde{g},k} = \int_0^{\infty} \tilde{g}(x)\varphi_k(x)\,dx = e^{-\lambda\Delta}\sum_{j=1}^{J}\frac{(\lambda\Delta)^j}{j!}\int_0^{\infty} f_X^{*j}(x)\varphi_k(x)\,dx = e^{-\lambda\Delta}\sum_{j=1}^{J}\frac{(\lambda\Delta)^j}{j!}A_{f_X^{*j},k}. \tag{3.5}$$

In order to obtain the Laguerre coefficients $A_{\tilde{g},k}$, it suffices to determine the Laguerre coefficients $A_{f_X^{*j},k}$. First, we require the following condition.

Condition 3. For each $j = 1, 2, \ldots, J$,

$$\sum_{k=0}^{\infty} |A_{f_X^{*j},k}| < \infty. \tag{3.6}$$

Remark 3. Note that if $f_X^{*j} \in W(\mathbb{R}_+, \theta)$ for some $\theta > 1$, we can use Cauchy–Schwarz inequality to get

$$\sum_{k=0}^{\infty} |A_{f,k}| = |A_{f,0}| + \sum_{k=1}^{\infty} k^{\theta/2}|A_{f,k}|k^{-\theta/2} \leq |A_{f,0}| + \left(\sum_{k=1}^{\infty} k^{\theta}A_{f,k}^2\right)^{1/2}\left(\sum_{k=1}^{\infty} k^{-\theta}\right)^{1/2} < \infty.$$

Hence, Condition 3 is satisfied as long as $f_X^{*j} \in W(\mathbb{R}_+, \theta)$ for some $\theta > 1$.

Next, we show that the Laguerre coefficients $A_{f_X^{*j},k}$ can be recursively determined. When $j = 1$, we have $A_{f_X^{*1},k} = A_{f_X,k}$. For the case $j \geq 2$, by (2.2) we have

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sup_{0\leq y\leq x}|A_{f_X,k}\varphi_k(x-y)A_{f_X^{*(j-1)},l}\varphi_l(y)| \leq 2\sum_{k=0}^{\infty}|A_{f_X,k}|\sum_{l=0}^{\infty}|A_{f_X^{*(j-1)},l}| < \infty,$$

under Condition 3. Then, we can use Fubini theorem and the convolution formula (2.3) to obtain

$$\begin{aligned} f_X^{*j}(x) &= \int_0^x f_X(x-y)f_X^{*(j-1)}(y)\,dy \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{f_X,k}A_{f_X^{*(j-1)},l}\int_0^x \varphi_k(x-y)\varphi_l(y)\,dy \\ &= \frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A_{f_X,k}A_{f_X^{*(j-1)},l}[\varphi_{k+l}(x) - \varphi_{k+l+1}(x)] \\ &= \frac{1}{\sqrt{2}}\sum_{k=0}^{\infty}\sum_{m=0}^{k} A_{f_X,m}[A_{f_X^{*(j-1)},k-m} - A_{f_X^{*(j-1)},k-m-1}]\varphi_k(x) \end{aligned} \tag{3.7}$$

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with $A_{f_X^{*(j-1)},-1} = 0$, from which we know that the Laguerre coefficients $A_{f_X^{*j},k}$ can be recursively determined according to the following formulas

$$A_{f_X^{*j},k} = \frac{1}{\sqrt{2}} \sum_{m=0}^{k} A_{f_X,m}[A_{f_X^{*(j-1)},k-m} - A_{f_X^{*(j-1)},k-m-1}], \quad j = 2, 3, \ldots, J.$$

4. Integral equations

Using the density function g, we can derive some integral equations satisfied by the finite-time Gerber-Shiu functions. First, for $\phi_1(u)$, by conditioning on the size of total claims during $[0, \Delta]$ we have

$$\begin{aligned}\phi_1(u) &= e^{-\delta\Delta} \int_{u+c\Delta}^{\infty} w(x-u-c\Delta)g(x)\,dx \\ &= e^{-\delta\Delta} \int_0^{\infty} w(x)g(x+u+c\Delta)\,dx,\end{aligned} \tag{4.1}$$

and for $\phi_n(u)$ with $n \geq 2$,

$$\begin{aligned}\phi_n(u) &= e^{-\delta\Delta} \int_{u+c\Delta}^{\infty} w(x-u-c\Delta)g(x)\,dx + e^{-\delta\Delta} \int_0^{u+c\Delta} \phi_{n-1}(u+c\Delta-x)g(x)\,dx \\ &\quad + e^{-\delta\Delta} \times P(S_\Delta = 0) \times \phi_{n-1}(u+c\Delta) \\ &= \phi_1(u) + e^{-\delta\Delta} \int_0^{u+c\Delta} \phi_{n-1}(u+c\Delta-x)g(x)\,dx + e^{-(\delta+\lambda)\Delta}\phi_{n-1}(u+c\Delta).\end{aligned} \tag{4.2}$$

Next, using the approximation density function $\tilde{g}$, we obtain from (4.1) and (4.2) that

$$\phi_1(u) \approx \tilde{\phi}_1(u) := e^{-\delta\Delta} \int_0^{\infty} w(x)\tilde{g}(x+u+c\Delta)\,dx, \tag{4.3}$$

and for $n \geq 2$,

$$\phi_n(u) \approx \tilde{\phi}_n(u) := \tilde{\phi}_1(u) + e^{-\delta\Delta} \int_0^{u+c\Delta} \tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)\,dx + e^{-(\delta+\lambda)\Delta}\tilde{\phi}_{n-1}(u+c\Delta). \tag{4.4}$$

Using the fact that $g(x) \geq \tilde{g}(x)$, we find that $\tilde{\phi}_1(u) \leq \phi_1(u)$, which together with (4.2) and (4.4) yields that for each n,

$$\tilde{\phi}_n(u) \leq \phi_n(u) \leq \phi(u). \tag{4.5}$$

The error of the above approximation is derived in the following proposition.

Proposition 2. *Under Conditions 1 and 2, we have, for any $r > 0$*

$$\sup_{u \geq 0} |\phi_n(u) - \tilde{\phi}_n(u)| \leq C_n e^{-rJ}, \tag{4.6}$$

where $C_1 = C_w \cdot E[e^{rN_\Delta}(1 + N_\Delta^\kappa EX^\kappa)]e^{-r}$, and for $n \geq 2$

$$C_n = [2^{n-1} - 1]\left(C_1 + C_g \int_0^{\infty} \phi(u)\,du\right) + 2^{n-1}C_1.$$

In order to apply Laguerre series expansion to approximate the finite-time Gerber-Shiu functions, we need to suppose that the functions $\tilde{\phi}_n(u)$ are all square integrable. By the integral equation (4.4) and mathematical induction, we can obtain the following proposition, which proves the square integrability of $\tilde{\phi}_n$ under Condition 1 together with an additional condition $\tilde{\phi}_1 \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$.

Proposition 3. *Under Condition 1 and the condition $\tilde{\phi}_1 \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, we have, for all $n \geq 2$, $\tilde{\phi}_n \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$.*

Remark 4. We claim that the condition $\tilde{\phi}_1 \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is not very restrictive if Condition 2 is satisfied. First, we have

$$\begin{aligned}
\tilde{\phi}_1(u) &\leq C_w e^{-\delta\Delta} \int_0^\infty [1 \vee x^\kappa] \tilde{g}(x + u + c\Delta)\, dx \\
&\leq C_w e^{-\delta\Delta} \int_0^\infty [1 + x^\kappa] \tilde{g}(x + u + c\Delta)\, dx \\
&\leq C_w e^{-\delta\Delta} \int_0^\infty [1 + x^\kappa] \tilde{g}(x)\, dx \\
&\leq C_w e^{-\delta\Delta} \int_0^\infty [1 + x^\kappa] g(x)\, dx \\
&\leq C_w e^{-\delta\Delta} [1 + E[S_\Delta^\kappa]].
\end{aligned}$$

Next, we have

$$\begin{aligned}
\int_0^\infty \tilde{\phi}_1(u)\, du &\leq C_w e^{-\delta\Delta} \int_0^\infty \int_0^\infty [1 \vee x^\kappa] \tilde{g}(x + u + c\Delta)\, dx\, du \\
&= C_w e^{-\delta\Delta} \int_0^\infty \int_{u+c\Delta}^\infty [1 \vee (y - u - c\Delta)^\kappa] \tilde{g}(y)\, dy\, du \\
&\leq C_w e^{-\delta\Delta} \int_0^\infty \int_u^\infty [1 \vee y^\kappa] g(y)\, dy\, du \\
&= C_w e^{-\delta\Delta} \int_0^\infty \int_0^y [1 \vee y^\kappa] g(y)\, du\, dy \\
&\leq C_w e^{-\delta\Delta} \int_0^\infty [y + y^{\kappa+1}] g(y)\, dy \\
&\leq C_w e^{-\delta\Delta} [E[S_\Delta] + E[S_\Delta^{\kappa+1}]].
\end{aligned}$$

Hence, if $E[S_\Delta^{\kappa+1}] < \infty$, we have $\tilde{\phi}_1 \in L^1(\mathbb{R}_+)$, and

$$\int_0^\infty [\tilde{\phi}_1(u)]^2\, du \leq C_w e^{-\delta\Delta}[1 + E[S_\Delta^\kappa]] \int_0^\infty \tilde{\phi}_1(u)\, du \leq C_w^2 e^{-2\delta\Delta}[1 + E[S_\Delta^\kappa]] \cdot [E[S_\Delta] + E[S_\Delta^{\kappa+1}]],$$

that is, $\tilde{\phi}_1 \in L^2(\mathbb{R}_+)$. Note that the condition $E[S_\Delta^{\kappa+1}] < \infty$ is satisfied when the individual claim sizes have finite $(\kappa + 1)$th moment, that is, $EX^{\kappa+1} < \infty$. When the penalty function w is bounded, we can take $\kappa = 0$, so that $\tilde{\phi}_n \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ as long as $EX < \infty$.

5. Laguerre series expansion for $\tilde{\phi}_n(u)$

In this section, we show how to use the Laguerre series expansion to approximate the finite-time Gerber-Shiu functions. In the remainder of this section, we suppose that the conditions in Proposition 3 hold true, so that $\tilde{\phi}_n \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for each $n = 1, 2, \ldots$. For each $\tilde{\phi}_n$, we have the following unique Laguerre series expansion

$$\tilde{\phi}_n(u) = \sum_{k=0}^\infty A_{\tilde{\phi}_n,k} \varphi_k(u), \quad u \geq 0. \tag{5.1}$$

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In the remainder of this section, we are devoted to computing the Laguerre coefficients $A_{\tilde{\phi}_n,k}$. Throughout the following two subsections, we shall also need the following condition.

Condition 4. There exist positive numbers $\theta^* > 1$ and $B_\phi > 0$ such that for all $n = 1, 2, \ldots$, $\tilde{\phi}_n \in W(\mathbb{R}_+, \theta^*)$ and $\sum_{k=0}^{\infty} k^{\theta^*} A^2_{\tilde{\phi}_n,k} \le B_\phi < \infty$.

5.1. Computing the Laguerre coefficients $A_{\tilde{\phi}_1,k}$

In this subsection, we study how to compute the Laguerre coefficients $A_{\tilde{\phi}_1,k}$. First, we need the following proposition.

Proposition 4. *Suppose that Condition 2 holds true and $\tilde{g} \in W(\mathbb{R}_+, \theta)$ for some $\theta > 2\kappa+5$, then we have*

$$\sum_{k=0}^{\infty} |A_{\tilde{g},k}| \int_0^\infty w(x)|\varphi_k(x + u + c\Delta)|\, dx < \infty. \tag{5.2}$$

Replacing $\tilde{g}$ in (4.3) by its Laguerre expansion, we know from Proposition 4 that the term-by-term integration is permitted due to the Fubini theorem, so that we have

$$\tilde{\phi}_1(u) = e^{-\delta\Delta} \sum_{k=0}^{\infty} A_{\tilde{g},k} \int_0^\infty w(x)\varphi_k(x + u + c\Delta)\, dx. \tag{5.3}$$

Furthermore, for each $z \ge 0$, since the function $u \mapsto \varphi_k(u + z)$, $u \ge 0$, is square integrable, then it has Laguerre series expansion given by

$$\varphi_k(u + z) = \sum_{j=0}^{k} \zeta_{k,j}(z)\varphi_j(u), \tag{5.4}$$

where the Laguerre coefficients are given by

$$\begin{aligned} \zeta_{k,j}(z) &= \int_0^\infty \varphi_k(u + z)\varphi_j(u)\, du \\ &= e^{-z} \sum_{m=0}^{k} \sum_{n=0}^{j} \sum_{l=0}^{m} (-1)^{m+n} (z)^{m-l} \binom{k}{m} \binom{j}{n} \binom{m}{l} \frac{(2)^{m-l}(l+n)!}{m!n!}. \end{aligned} \tag{5.5}$$

By (5.4), we can rewrite (5.3) as

$$\begin{aligned} \tilde{\phi}_1(u) &= e^{-\delta\Delta} \sum_{k=0}^{\infty} A_{\tilde{g},k} \sum_{j=0}^{k} \int_0^\infty w(x)\zeta_{k,j}(x + c\Delta)\, dx \varphi_j(u) \\ &= e^{-\delta\Delta} \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} A_{\tilde{g},k} \int_0^\infty w(x)\zeta_{k,j}(x + c\Delta)\, dx \varphi_j(u) \\ &= \sum_{k=0}^{\infty} A_{\tilde{\phi}_1,k}\varphi_k(u), \end{aligned}$$

where the Laguerre coefficients $A_{\tilde{\phi}_1,k}$ are given by

$$A_{\tilde{\phi}_1,k} = e^{-\delta\Delta} \sum_{j=k}^{\infty} A_{\tilde{g},j} \int_0^\infty w(x)\zeta_{j,k}(x + c\Delta)\, dx. \tag{5.6}$$

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In most applications, the integral in (5.6) can be explicitly computed. Some examples are given in Appendix B.

From the point of view of numerical computation, we should truncate the infinite summation in (5.6), that is, for some large integer $K > k$ we have

$$A_{\tilde{\phi}_1,k} \approx \hat{A}_{\tilde{\phi}_1,k} := e^{-\delta\Delta} \sum_{j=k}^{K} A_{\tilde{g},j} \int_0^\infty w(x)\zeta_{j,k}(x + c\Delta)\, dx. \tag{5.7}$$

Furthermore, for the Laguerre series expansion of $\tilde{\phi}_1(u)$, by series truncation we obtain

$$\tilde{\phi}_1(u) \approx \hat{\phi}_1(u) := \sum_{k=0}^{K} \hat{A}_{\tilde{\phi}_1,k}\varphi_k(u), \quad u \geq 0.$$

The approximation errors are given in the following propositions.

Proposition 5. *Suppose that Condition 2 holds true, and for some $\rho > \frac{1}{2}$, $\tilde{g} \in W(\mathbb{R}_+, \theta)$ for some $\theta \geq 2(\kappa + 3 + \rho)$, then we have*

$$|A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| = k^2 \cdot O(K^{-\rho+1/2}), \quad k < K. \tag{5.8}$$

Proposition 6. *Suppose that Conditions 2 and 4 hold true, and for some $\rho > \frac{7}{2}$, $\tilde{g} \in W(\mathbb{R}_+, \theta)$ for some $\theta \geq 2(\kappa + 3 + \rho)$, then we have*

$$\sup_{u\geq 0} |\tilde{\phi}_1(u) - \hat{\phi}_1(u)| = O(\max(K^{-(\theta^*-1)/2}, K^{-\rho+7/2})). \tag{5.9}$$

Proof. First, we have

$$\tilde{\phi}_1(u) - \hat{\phi}_1(u) = \sum_{k=K+1}^{\infty} A_{\tilde{\phi}_1,k}\varphi_k(u) + \sum_{k=0}^{K}[A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}]\varphi_k(u),$$

which together with (2.2) yields

$$\sup_{u\geq 0} |\tilde{\phi}_1(u) - \hat{\phi}_1(u)| \leq \sqrt{2} \sum_{k=K+1}^{\infty} |A_{\tilde{\phi}_1,k}| + \sqrt{2} \sum_{k=0}^{K} |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}|. \tag{5.10}$$

By Shimizu and Zhang [27] Prop. 3, we have $\sum_{k=K+1}^{\infty} |A_{\tilde{\phi}_1,k}| = O(K^{-(\theta^*-1)/2})$ under Condition 4. By Proposition 5, we obtain

$$\sum_{k=0}^{K} |A_{\tilde{\phi}_1,k} - \tilde{A}_{\tilde{\phi}_1,k}| = O(K^3) \cdot O(K^{-\rho+1/2}) = O(K^{-\rho+7/2}).$$

Hence, the desired convergence rate is obtained. □

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5.2. *Computing the Laguerre coefficients $A_{\tilde{\phi}_n,k}$ for $n \geq 2$*

In this subsection, we derive the Laguerre coefficients $A_{\tilde{\phi}_n,k}$ for $n \geq 2$. First, by formula (5.4), we obtain

$$\begin{aligned}\tilde{\phi}_{n-1}(u+c\Delta) &= \sum_{k=0}^{\infty} A_{\tilde{\phi}_{n-1},k}\varphi_k(u+c\Delta) = \sum_{k=0}^{\infty} A_{\tilde{\phi}_{n-1},k}\sum_{j=0}^{k}\zeta_{k,j}(c\Delta)\varphi_j(u)\\ &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} A_{\tilde{\phi}_{n-1},j}\zeta_{j,k}(c\Delta)\varphi_k(u). \end{aligned}\tag{5.11}$$

Next, by the convolution formula (2.3), we have

$$\begin{aligned}&\int_0^{u+c\Delta}\tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)\,dx\\ &= \int_0^{u+c\Delta}\sum_{j=0}^{\infty}A_{\tilde{\phi}_{n-1},j}\varphi_j(u+c\Delta-x)\sum_{k=0}^{\infty}A_{\tilde{g},k}\varphi_k(x)\,dx\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A_{\tilde{\phi}_{n-1},j}A_{\tilde{g},k}\int_0^{u+c\Delta}\varphi_j(u+c\Delta-x)\varphi_k(x)\,dx\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}A_{\tilde{\phi}_{n-1},j}A_{\tilde{g},k}\frac{1}{\sqrt{2}}[\varphi_{j+k}(u+c\Delta)-\varphi_{j+k+1}(u+c\Delta)]\\ &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\varphi_m(u+c\Delta), \end{aligned}\tag{5.12}$$

where we set $A_{\tilde{g},k} = 0$ for $k < 0$. Note that the above term-by-term integration is permitted by the Fubini theorem and Conditions 3 and 4. Furthermore, using the expansion formula (5.4) and changing the order of summation, we can obtain from (5.12) that

$$\begin{aligned}&\int_0^{u+c\Delta}\tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)\,dx\\ &= \sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\sum_{k=0}^{m}\zeta_{m,k}(c\Delta)\varphi_k(u)\\ &= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta)\varphi_k(u). \end{aligned}\tag{5.13}$$

Finally, substituting (5.11) and (5.13) back into (4.4) gives

$$\begin{aligned}\sum_{k=0}^{\infty}A_{\tilde{\phi}_n,k}\varphi_k(u) &= \sum_{k=0}^{\infty}A_{\tilde{\phi}_1,k}\varphi_k(u) + e^{-(\delta+\lambda)\Delta}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}A_{\tilde{\phi}_{n-1},j}\zeta_{j,k}(c\Delta)\varphi_k(u)\\ &\quad + e^{-\delta\Delta}\sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta)\varphi_k(u). \end{aligned}\tag{5.14}$$

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By comparing the Laguerre coefficients on both sides of (5.14), we obtain, for $k = 0, 1, 2\ldots$,

$$A_{\tilde{\phi}_n,k} = A_{\tilde{\phi}_1,k} + e^{-(\delta+\lambda)\Delta}\sum_{j=k}^{\infty} A_{\tilde{\phi}_{n-1},j}\zeta_{j,k}(c\Delta)$$
$$+ e^{-\delta\Delta}\sum_{m=k}^{\infty}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta), \tag{5.15}$$

which shows a recursive method for computing the Laguerre coefficients $A_{\tilde{\phi}_n,k}$.

For computation of $A_{\tilde{\phi}_n,k}$, we truncate the infinite series summation in (5.15) to get, for a large integer K,

$$A_{\tilde{\phi}_n,k} \approx A_{\tilde{\phi}_1,k} + e^{-(\delta+\lambda)\Delta}\sum_{j=k}^{K} A_{\tilde{\phi}_{n-1},j}\zeta_{j,k}(c\Delta)$$
$$+ e^{-\delta\Delta}\sum_{m=k}^{K}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}A_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta). \tag{5.16}$$

Furthermore, replacing the Laguerre coefficients $A_{\tilde{\phi}_1,k}$ by their approximates $\hat{A}_{\tilde{\phi}_1,k}$ defined in (5.7), we can approximate the Laguerre coefficients $A_{\tilde{\phi}_n,k}$ by $\hat{A}_{\tilde{\phi}_n,k}$, which are recursively defined by

$$\hat{A}_{\tilde{\phi}_n,k} = \hat{A}_{\tilde{\phi}_1,k} + e^{-(\delta+\lambda)\Delta}\sum_{j=k}^{K} \hat{A}_{\tilde{\phi}_{n-1},j}\zeta_{j,k}(c\Delta)$$
$$+ e^{-\delta\Delta}\sum_{m=k}^{K}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}\hat{A}_{\tilde{\phi}_{n-1},j}[A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta). \tag{5.17}$$

Accordingly, we obtain the following approximation for $\tilde{\phi}_n(u)$,

$$\tilde{\phi}_n(u) \approx \hat{\phi}_n(u) := \sum_{k=0}^{K}\hat{A}_{\tilde{\phi}_n,k}\varphi_k(u), \quad u \geq 0. \tag{5.18}$$

For the approximation error of $\hat{A}_{\tilde{\phi}_n,k}$, we have the following result.

Proposition 7. *For each fixed $n = 1, 2, \ldots$, suppose that Conditions 1, 2 and 4 (with $\theta^* > 2(n+1)$), and for some $\rho > n + \frac{3}{2}$, $\tilde{g} \in W(\mathbb{R}, \theta)$ for some $\theta > 2(\kappa + 3 + \rho)$, then we have for each large integer K,*

$$\sup_{k\leq K}|A_{\tilde{\phi}_n,k} - \hat{A}_{\tilde{\phi}_n,k}| = O(\max(K^{-(\theta^*/2-n-1)}, K^{-(\rho-n-3/2)})). \tag{5.19}$$

Furthermore, using the same arguments as in the proof of Proposition 6, we can obtain the approximation error of $\hat{\phi}_n(u)$.

Proposition 8. *For each fixed $n = 1, 2, \ldots$, suppose that Conditions 1, 2 and 4 (with $\theta^* > 2(n+1)$), and for some $\rho > n + \frac{3}{2}$, $\tilde{g} \in W(\mathbb{R}, \theta)$ for some $\theta > 2(\kappa + 3 + \rho)$, then we have for each large integer K,*

$$\sup_{u\geq 0}|\tilde{\phi}_n(u) - \hat{\phi}_n(u)| = O(\max(K^{-(\theta^*/2-n-2)}, K^{-(\rho-n-5/2)})). \tag{5.20}$$

Remark 5. By the proof of Proposition 7, we know that there exists error propagation when using $\hat{\phi}_n(u)$ to approximate the finite-time Gerber-Shiu function, however, the numerical results in Section 7 show that good approximation performance can also be obtained for a large n (for example $n = 20{,}000$).

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6. Laguerre series expansion for $\phi(u)$

In this section, we study how to apply the Laguerre series expansion method to study the infinite-time Gerber-Shiu function $\phi(u)$. First, using monotone convergence theorem, we find from (4.2) that the infinite-time Gerber-Shiu function satisfies the following integral equation

$$\phi(u) = \phi_1(u) + e^{-\delta\Delta} \int_0^{u+c\Delta} \phi(u + c\Delta - x) g(x)\, dx + e^{-(\delta+\lambda)\Delta} \phi(u + c\Delta). \tag{6.1}$$

Furthermore, replacing g and ϕ_1 by their approximations $\tilde{g}$ and $\tilde{\phi}_1$ in (6.1), and let $\tilde{\phi}$ denote the solution of the following integral equation

$$\tilde{\phi}(u) = \tilde{\phi}_1(u) + e^{-\delta\Delta} \int_0^{u+c\Delta} \tilde{\phi}(u + c\Delta - x) \tilde{g}(x)\, dx + e^{-(\delta+\lambda)\Delta} \tilde{\phi}(u + c\Delta). \tag{6.2}$$

Note that monotone convergence theorem yields

$$\lim_{n\to\infty} \tilde{\phi}_n(u) = \tilde{\phi}(u).$$

Remark 6. The approximation error of $\tilde{\phi}(u)$ can be analyzed as in the proof of Proposition 2 as follows. By (6.1) and (6.2), we obtain

$$\begin{aligned}
\sup_{u\geq 0} |\phi(u) - \tilde{\phi}(u)| \leq\ & \sup_{u\geq 0} |\phi_1(u) - \tilde{\phi}_1(u)| + e^{-(\lambda+\delta)\Delta} \sup_{u\geq 0} |\phi(u + c\Delta) - \tilde{\phi}(u + c\Delta)| \\
& + e^{-\delta\Delta} \sup_{u\geq 0} \int_0^{u+c\Delta} |\phi(u + c\Delta - x)[g(x) - \tilde{g}(x)]|\, dx \\
& + e^{-\delta\Delta} \sup_{u\geq 0} \int_0^{u+c\Delta} |\tilde{g}(x)[\phi(u + c\Delta - x) - \tilde{\phi}(u + c\Delta - x)]|\, dx \\
\leq\ & \sup_{u\geq 0} |\phi_1(u) - \tilde{\phi}_1(u)| + e^{-(\lambda+\delta)\Delta} \sup_{u\geq 0} |\phi(u) - \tilde{\phi}(u)| \\
& + e^{-\delta\Delta} \int_0^{\infty} \phi(u)\, du \cdot \sup_{x\geq 0} |g(x) - \tilde{g}(x)| + e^{-\delta\Delta} \int_0^{\infty} \tilde{g}(x)\, dx \\
& \cdot \sup_{u\geq 0} |\phi(u) - \tilde{\phi}(u)|.
\end{aligned} \tag{6.3}$$

When $\delta > 0$, the above inequality, together with Propositions 1 and 2, gives

$$\begin{aligned}
\sup_{u\geq 0} |\phi(u) - \tilde{\phi}(u)| &\leq \frac{\sup_{u\geq 0} |\phi_1(u) - \tilde{\phi}_1(u)| + e^{-\delta\Delta} \int_0^{\infty} \phi(u)\, du \cdot \sup_{x\geq 0} |g(x) - \tilde{g}(x)|}{1 - e^{-(\lambda+\delta)\Delta} - e^{-\delta\Delta} \int_0^{\infty} \tilde{g}(x)\, dx} \\
&\leq \frac{\sup_{u\geq 0} |\phi_1(u) - \tilde{\phi}_1(u)| + e^{-\delta\Delta} \int_0^{\infty} \phi(u)\, du \cdot \sup_{x\geq 0} |g(x) - \tilde{g}(x)|}{1 - e^{-\delta\Delta}} \\
&= O(e^{-rJ}),
\end{aligned} \tag{6.4}$$

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an exponential decay rate w.r.t. the truncation parameter J. When $\delta = 0$, we consider the error $\sup_{0\le u\le C}|\phi(u) - \tilde{\phi}(u)|$ for some fixed number $C > 0$. Similar to (6.3), we can easily obtain

$$\begin{aligned}\sup_{0\le u\le C}|\phi(u)-\tilde{\phi}(u)| \le {} & \sup_{0\le u\le C}|\phi_1(u)-\tilde{\phi}_1(u)| + e^{-\lambda\Delta}\sup_{0\le u\le C}|\phi(u)-\tilde{\phi}(u)| \\ & + \int_0^\infty \phi(u)\,du \cdot \sup_{x\ge 0}|g(x)-\tilde{g}(x)| + \int_0^{C+c\Delta}\tilde{g}(x)\,dx \\ & \cdot \sup_{0\le u\le C}|\phi(u)-\tilde{\phi}(u)|,\end{aligned}$$

from which we can obtain exponential decay rate for the error as follows,

$$\begin{aligned}\sup_{0\le u\le C}|\phi(u)-\tilde{\phi}(u)| &\le \frac{\sup_{0\le u\le C}|\phi_1(u)-\tilde{\phi}_1(u)| + \int_0^\infty \phi(u)\,du\cdot \sup_{x\ge 0}|g(x)-\tilde{g}(x)|}{1-e^{-\lambda\Delta}-\int_0^{C+c\Delta}\tilde{g}(x)\,dx} \\ &\le \frac{\sup_{0\le u\le C}|\phi_1(u)-\tilde{\phi}_1(u)| + \int_0^\infty \phi(u)\,du\cdot \sup_{x\ge 0}|g(x)-\tilde{g}(x)|}{\int_{C+c\Delta}^\infty g(x)\,dx} \\ &= O(e^{-rJ}). \end{aligned} \tag{6.5}$$

Now, we use Laguerre series expansion method to solve the integral equation (6.2). First, we have

$$\tilde{\phi}(u) = \sum_{k=0}^\infty A_{\tilde{\phi},k}\varphi_k(u), \tag{6.6}$$

and by formula (5.4)

$$\tilde{\phi}(u+c\Delta) = \sum_{k=0}^\infty A_{\tilde{\phi},k}\varphi_k(u+c\Delta) = \sum_{k=0}^\infty\sum_{j=k}^\infty A_{\tilde{\phi},j}\zeta_{j,k}(c\Delta)\varphi_k(u). \tag{6.7}$$

By the same arguments leading to (5.13), we obtain

$$\int_0^{u+c\Delta}\tilde{\phi}(u+c\Delta-x)\tilde{g}(x)\,dx = \sum_{k=0}^\infty\sum_{m=k}^\infty\sum_{j=0}^m \frac{1}{\sqrt{2}}A_{\tilde{\phi},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta)\varphi_k(u). \tag{6.8}$$

Plugging the Laguerre series expansions (5.1), (6.6), (6.7) and (6.8) into (6.2), we obtain

$$\begin{aligned}\sum_{k=0}^\infty A_{\tilde{\phi},k}\varphi_k(u) = {} & \sum_{k=0}^\infty A_{\tilde{\phi}_1,k}\varphi_k(u) + e^{-(\lambda+\delta)\Delta}\sum_{k=0}^\infty\sum_{j=k}^\infty A_{\tilde{\phi},j}\zeta_{j,k}(c\Delta)\varphi_k(u) \\ & + e^{-\delta\Delta}\sum_{k=0}^\infty\sum_{m=k}^\infty\sum_{j=0}^m\frac{1}{\sqrt{2}}A_{\tilde{\phi},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta)\varphi_k(u).\end{aligned} \tag{6.9}$$

By comparing the Laguerre coefficients on both sides of (6.9), we find that the coefficients $A_{\tilde{\phi},k}$ satisfy the following linear system of equations, for $k = 0, 1, \ldots,$

$$\begin{aligned}A_{\tilde{\phi},k} = {} & A_{\tilde{\phi}_1,k} + e^{-(\lambda+\delta)\Delta}\sum_{j=k}^\infty A_{\tilde{\phi},j}\zeta_{j,k}(c\Delta) \\ & + e^{-\delta\Delta}\sum_{m=k}^\infty\sum_{j=0}^m\frac{1}{\sqrt{2}}A_{\tilde{\phi},j}[A_{\tilde{g},m-j}-A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta).\end{aligned} \tag{6.10}$$

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Furthermore, we consider the series truncation approximation

$$\hat{\phi}(u) = \sum_{k=0}^{K} \hat{A}_{\tilde{\phi},k}\varphi_k(u), \quad u \geq 0, \tag{6.11}$$

where the coefficients $\hat{A}_{\phi,k}$ satisfy the following linear system of equations, for $k = 0, 1, \ldots, K$,

$$\begin{aligned}\hat{A}_{\tilde{\phi},k} &= \hat{A}_{\tilde{\phi}_1,k} + e^{-(\lambda+\delta)\Delta}\sum_{j=k}^{K}\hat{A}_{\tilde{\phi},j}\zeta_{j,k}(c\Delta)\\ &\quad + e^{-\delta\Delta}\sum_{m=k}^{K}\sum_{j=0}^{m}\frac{1}{\sqrt{2}}\hat{A}_{\tilde{\phi},j}[A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}]\zeta_{m,k}(c\Delta).\end{aligned} \tag{6.12}$$

7. Numerical results

In this section, we present some numerical results to show the effectiveness of our method. All computations are performed in MATLAB on a MacBook, with Intel(R) Core(TM) i5 CPU, at 1.6 GHz and a RAM of 8 GB. Throughout this section, we set $c = 1.2$, $\lambda = 1$ and $J = 15$, and consider the following claim size density functions:

- Exp(1): $f_X(x) = e^{-x}$, $x > 0$;
- Erlang(2,2): $f_X(x) = 4xe^{-2x}$, $x > 0$;
- Pareto(7,6): $f_X(x) = \frac{7\cdot 6^7}{(6+x)^8}$, $x > 0$;
- Combination of exponentials (CoE): $f_X(x) = \frac{1}{3}(\frac{1}{2}e^{-(1/2)x}) + \frac{2}{3}(2e^{-2x})$, $x > 0$.

It is easily seen that the above density functions have a common mean of 1, but with different variances given by 1(Exp(1)), 0.5(Erlang(2,2)), 1.4(Pareto(7,6)) and 2(CoE). For the light-tailed density functions (Exp(1), Erlang(2,2) and CoE), we can easily check that they belong to any Sobolev–Laguerre space $W(\mathbb{R}_+, \theta)$ with $\theta \geq 0$. For the heavy-tailed density function Pareto(7,6), by Remark 1 we can find that it belongs to the Sobolev–Laguerre space $W(\mathbb{R}_+, \theta)$ with $0 \leq \theta \leq 14$.

First, we test the accuracy of the Laguerre series expansion method. It is known that there are no explicit formulas for the Gerber-Shiu functions under fixed periodic observation. Hence, in order to provide benchmark results, we perform Monte Carlo simulation for each risk model. Both the mean values and the 95% confidence intervals will be calculated based on 10^6 sample paths of the surplus processes. In Tables 1–4, we report the numerical results for the finite-time Gerber-Shiu function $\phi(u;T)$ with $\delta = 0.01$, $\Delta = 1$ and (i) $T = 10$, $w \equiv 1$; (ii) $T = 10$, $w(x) = x$; (iii) $T = 50$, $w \equiv 1$ and (iv) $T = 50$, $w(x) = x$. For these two penalty functions, the finite-time Gerber-Shiu functions are respectively the Laplace transform of the ruin time and the expected discounted value of the deficit at ruin when ruin occurs within the first T/Δ observation times.

In each table, we set the initial surplus level $u = 0, 1, 5, 10$, and take the truncation parameter $K = 5, 10, 20, 30$. It follows from each row (i.e., u is fixed) that the approximation performance gets better as K becomes larger, and the values are stable when K is 30. Compared with the Monte Carlo values, we can find that each relative error of Laguerre series expansion is controlled within 1%, and the approximate values are well covered by each of the confidence intervals. On the other hand, fix the truncation parameter K, we observe from each column that the finite-time Gerber-Shiu function is a decreasing function of the initial surplus level u.

Next, we study the impact of the observation parameters T and Δ on the finite-time Gerber-Shiu functions. Set $\delta = 0.01$. Fix the initial surplus level $u = 1, 5, 10$ and the inter-observation time $\Delta = 0.01$. In Figure 1, we plot the curves for the finite-time Gerber-Shiu functions for different claim size density functions and penalty functions. Note that $\phi(u;T) = \phi_n(u)$ with $n = T/\Delta$. The curves are plotted based

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Table 1. *Approximating results of $\phi(u,T)$, $T = 10$, $w \equiv 1$.*

		Laguerre				MC	
Model	u	$K = 5$	$K = 10$	$K = 20$	$K = 30$	Mean	95% CI
Exp	0	0.621307	0.622271	0.622279	0.622279	0.621893	[0.620974, 0.622811]
	1	0.468765	0.467634	0.467629	0.467629	0.467519	[0.466570, 0.468468]
	5	0.128426	0.126283	0.126289	0.126289	0.125802	[0.125172, 0.126433]
	10	0.013308	0.018535	0.018543	0.018543	0.018449	[0.018194, 0.018704]
Erlang	0	0.650735	0.630997	0.631902	0.631904	0.632275	[0.631361, 0.633190]
	1	0.478392	0.446654	0.448070	0.448073	0.448373	[0.447426, 0.449320]
	5	0.126986	0.082589	0.084377	0.084380	0.084540	[0.084011, 0.085068]
	10	0.015895	0.004768	0.006587	0.006590	0.006531	[0.006379, 0.006684]
Pareto	0	0.603039	0.607708	0.607914	0.607913	0.607668	[0.606742, 0.608593]
	1	0.459245	0.461739	0.461850	0.461850	0.461273	[0.460325, 0.462221]
	5	0.143708	0.146193	0.146475	0.146474	0.146286	[0.145613, 0.146959]
	10	0.016917	0.031304	0.031626	0.031626	0.031603	[0.031270, 0.031935]
CoE	0	0.580331	0.588944	0.589204	0.589204	0.590044	[0.589113, 0.590976]
	1	0.456842	0.462360	0.462302	0.462302	0.462888	[0.461941, 0.463835]
	5	0.173537	0.179714	0.179973	0.179973	0.181026	[0.178708, 0.183344]
	10	0.022946	0.049653	0.049991	0.049991	0.050324	[0.049908, 0.050739]

Table 2. *Approximating results of $\phi(u,T)$, $T = 10$, $w(x) = x$.*

		Laguerre				MC	
Model	u	$K = 5$	$K = 10$	$K = 20$	$K = 30$	Mean	95% CI
Exp	0	0.915673	0.917098	0.917108	0.917108	0.915631	[0.913068, 0.918194]
	1	0.675534	0.674046	0.674039	0.674039	0.673963	[0.671660, 0.676266]
	5	0.180356	0.177431	0.177439	0.177439	0.178156	[0.176865, 0.179447]
	10	0.018612	0.025886	0.025897	0.025897	0.025905	[0.025404, 0.026405]
Erlang	0	0.754764	0.725987	0.730681	0.730689	0.731549	[0.729576, 0.733523]
	1	0.471343	0.488401	0.490698	0.490709	0.491374	[0.489679, 0.493068]
	5	−0.063863	0.088862	0.088726	0.088734	0.088767	[0.087981, 0.089553]
	10	−0.043322	0.015818	0.006867	0.006867	0.006886	[0.006664, 0.007108]
Pareto	0	0.991306	1.040164	1.043728	1.043732	1.041380	[1.038161, 1.044599]
	1	0.760795	0.815076	0.819221	0.819227	0.819267	[0.816219, 0.822315]
	5	0.223441	0.276333	0.281398	0.281409	0.282145	[0.280084, 0.284206]
	10	0.023638	0.061061	0.065444	0.065455	0.066521	[0.065420, 0.067623]
CoE	0	1.181434	1.255289	1.256991	1.256990	1.257471	[1.253574, 1.261368]
	1	0.962120	1.035009	1.036179	1.036178	1.038269	[1.034585, 1.041953]
	5	0.348965	0.408014	0.409973	0.409973	0.410100	[0.407602, 0.412599]
	10	0.044757	0.111850	0.113625	0.113626	0.113340	[0.111992, 0.114688]

on the Laguerre series expansion $\hat{\phi}_n(u)$ with truncation parameter $K = 30$. As expected, each finite-time Gerber-Shiu function is an increasing function of T. This fact is very intuitive, since the larger the observation interval $[0, T]$, the more likely we can monitor the ruin event. It is easily seen from Figure 1 that $\phi(u;T) \to \phi(u)$. In fact, we can find that $\phi(u,T) \approx \phi(u)$ when $T \geq 150$. The limit is obvious due

Table 3. *Approximating results of* $\phi(u,T)$, $T = 50$, $w(x) = x$.

Model	u	Laguerre $K = 5$	$K = 10$	$K = 20$	$K = 30$	MC Mean	95% CI
Exp	0	1.014125	1.027627	1.028065	1.028064	1.026087	[1.023502, 1.028672]
	1	0.810240	0.822614	0.822371	0.822370	0.822622	[0.820232, 0.825013]
	5	0.311414	0.343379	0.343855	0.343854	0.343537	[0.341887, 0.345186]
	10	0.040416	0.110282	0.111356	0.111356	0.110851	[0.109902, 0.111801]
Erlang	0	0.761047	0.725987	0.802307	0.802314	0.803074	[0.801092, 0.805055]
	1	0.473222	0.488401	0.591789	0.591799	0.591401	[0.589641, 0.593160]
	5	−0.086828	0.088862	0.190842	0.190832	0.190347	[0.189275, 0.191419]
	10	−0.048892	0.015818	0.044224	0.044171	0.044229	[0.043711, 0.044747]
Pareto	0	1.101868	1.199757	1.208467	1.208467	1.208720	[1.205381, 1.212059]
	1	0.906400	1.025588	1.034928	1.034940	1.037063	[1.033804, 1.040322]
	5	0.356355	0.513370	0.528713	0.528743	0.530022	[0.527437, 0.532607]
	10	0.045481	0.198943	0.215520	0.215575	0.215945	[0.214216, 0.217675]
CoE	0	1.325686	1.474657	1.480344	1.480288	1.481670	[1.477674, 1.485666]
	1	1.140876	1.312129	1.314989	1.314955	1.314761	[1.310895, 1.318627]
	5	0.501385	0.719067	0.728679	0.728650	0.727940	[0.724885, 0.730995]
	10	0.069806	0.317913	0.334086	0.334118	0.333529	[0.331413, 0.335644]

Table 4. *Approximating results of* $\phi(u,T)$, $T = 50$, $w \equiv 1$.

Model	u	Laguerre $K = 5$	$K = 10$	$K = 20$	$K = 30$	MC Mean	95% CI
Exp	0	0.690835	0.700366	0.700678	0.700677	0.699994	[0.699163, 0.700826]
	1	0.563926	0.572643	0.572471	0.572470	0.571586	[0.570686, 0.572485]
	5	0.221089	0.243751	0.244090	0.244090	0.243695	[0.242925, 0.244465]
	10	0.028729	0.078385	0.079150	0.079151	0.079087	[0.078614, 0.079561]
Erlang	0	0.733063	0.696317	0.698975	0.698980	0.698386	[0.697546, 0.699226]
	1	0.599022	0.538742	0.542785	0.542792	0.542028	[0.541115, 0.542940]
	5	0.245579	0.173936	0.180296	0.180306	0.180002	[0.179312, 0.180692]
	10	0.034767	0.035617	0.041752	0.041763	0.041547	[0.041200, 0.041895]
Pareto	0	0.672835	0.693371	0.694417	0.694409	0.694850	[0.694018, 0.695682]
	1	0.551347	0.574591	0.574841	0.574836	0.575524	[0.574630, 0.576418]
	5	0.228221	0.272681	0.274462	0.274457	0.274619	[0.273820, 0.275419]
	10	0.030828	0.104670	0.107899	0.107904	0.108032	[0.107486, 0.108579]
CoE	0	0.651267	0.685956	0.687560	0.687535	0.687422	[0.686590, 0.688254]
	1	0.544808	0.584933	0.585075	0.585060	0.585181	[0.584295, 0.586066]
	5	0.248653	0.317395	0.320300	0.320287	0.319996	[0.319162, 0.320830]
	10	0.035295	0.140957	0.147056	0.147070	0.147042	[0.146417, 0.147666]

to the monotone convergence theorem. On the other hand, we note that the difference between $\phi(u)$ and $\phi(u;T)$ is the following T-deferred Gerber-Shiu function

$$\bar{\phi}(u;T) = E[e^{-\delta\tau} w(|U_\tau|)\mathbf{1}_{\{T<\tau<\infty\}}|U_0 = u], \quad u \geq 0. \tag{7.1}$$

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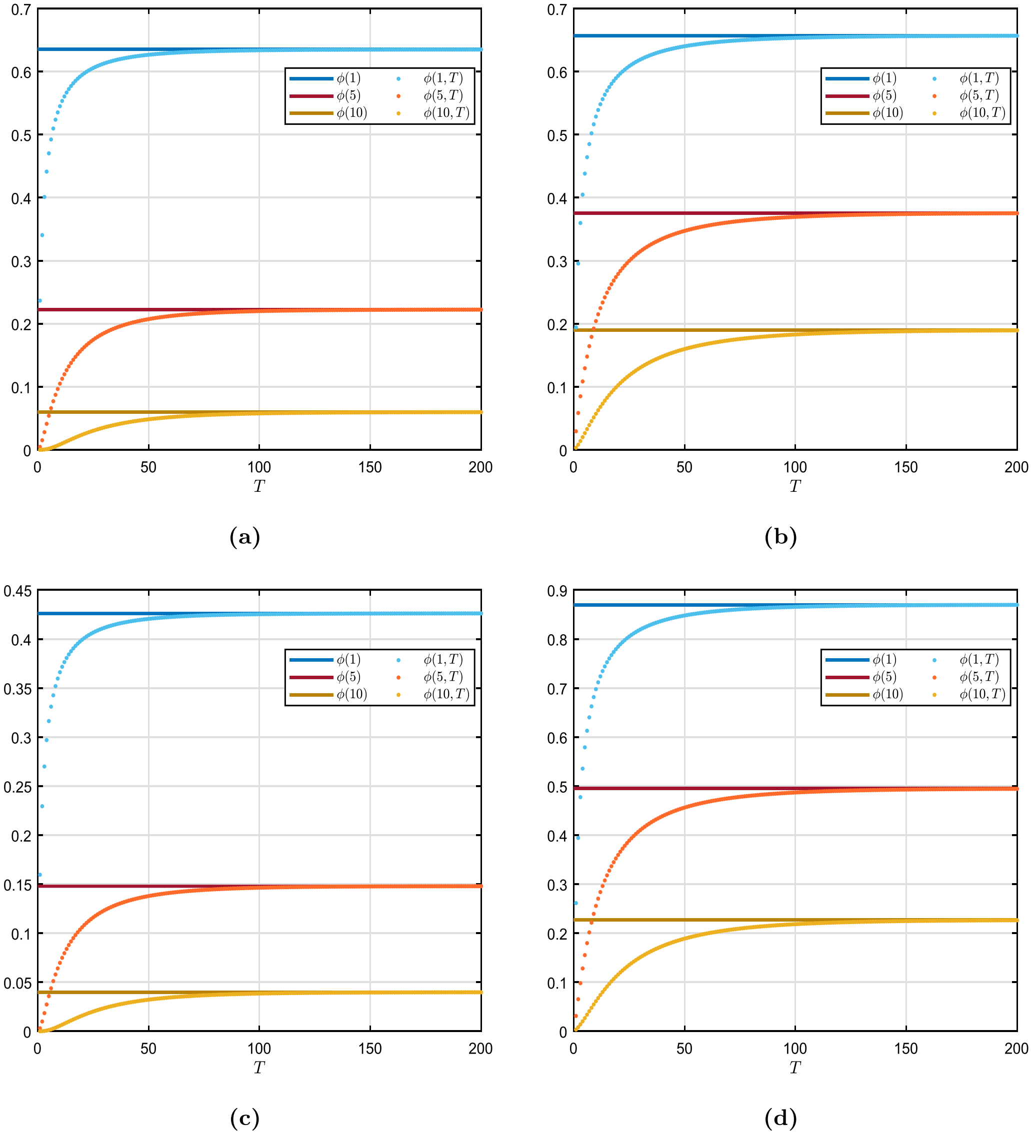

Figure 1. *Convergence of $\phi(u;T)$ for increasing T. $\Delta = 0.01$, $u = 1, 5, 10$. (a) Erlang, $w \equiv 1$; (b) CoE, $w \equiv 1$; (c) Erlang, $w(x) = x$; (d) Pareto, $w(x) = x$.*

Under the additional condition $E[e^{-\delta\tau}\tau^k w(|U_\tau|)\mathbf{1}_{\{\tau<\infty\}}|U_0 = u] < \infty$ for $k > 0$, Markov's inequality yields

$$\bar{\phi}(u;T) \leq \frac{1}{T^k} E[e^{-\delta\tau}\tau^k w(|U_\tau|)\mathbf{1}_{\{\tau<\infty\}}|U_0 = u].$$

Hence, we have $\phi(u;T) - \phi(u) = O(T^{-k})$.

In Figure 2, we plot the finite-time Gerber-Shiu functions (with $\delta = 0.01$ and $w \equiv 1$) on fixed observation interval $[0, 50]$, but with different observation frequency, $\Delta = 1, 0.5, 0.1, 0.01$. Again, we find that under each observation frequency, the finite-time Gerber-Shiu functions are decreasing w.r.t. the initial surplus level u. By Figure 2, we can also find that the corresponding finite-time Gerber-Shiu functions increase as Δ decreases, which may be due to that ruin is more likely to be monitored as the observation frequency increases. On the other hand, when fixing the initial surplus level u, we can

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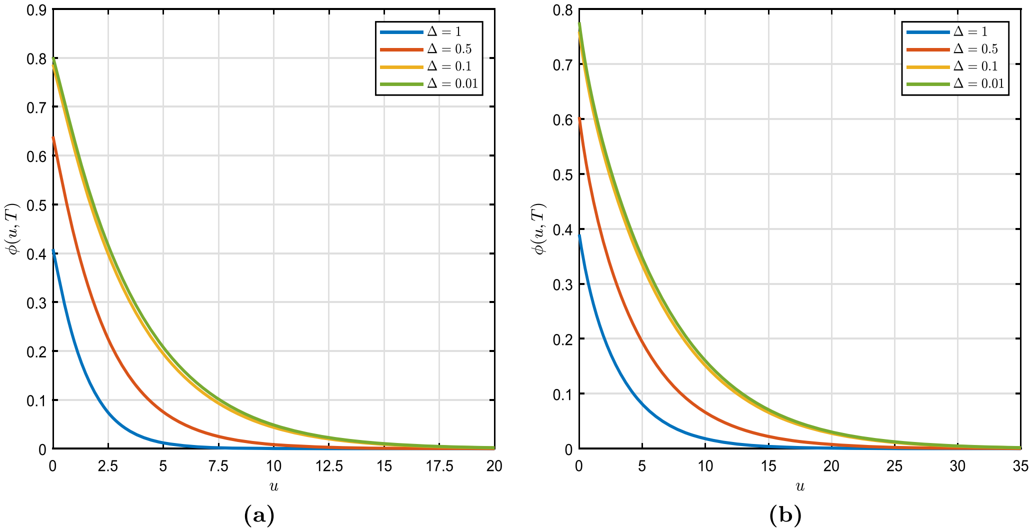

Figure 2. *Finite-time Gerber-Shiu functions vs. initial surplus u. $T = 50$, $\Delta = 1, 0.5, 0.1, 0.01$. (a) Erlang; (b) CoE.*

Table 5. *Approximating results of finite-time ruin probability under different Δ.*

	Exp					Erlang				
Δ	$u = 0$	$u = 1$	$u = 5$	$u = 10$	$u = 20$	$u = 0$	$u = 1$	$u = 5$	$u = 10$	$u = 20$
2	0.560549	0.427137	0.118244	0.017512	0.000191	0.559920	0.403557	0.078205	0.006155	0.000012
1	0.638373	0.483003	0.133141	0.019841	0.000217	0.647294	0.462761	0.089364	0.007096	0.000013
0.5	0.689016	0.519669	0.143780	0.021613	0.000238	0.706099	0.504710	0.098145	0.007897	0.000015
0.1	0.735533	0.554792	0.154797	0.023523	0.000263	0.755048	0.546915	0.107884	0.008836	0.000017
0.01	0.746507	0.563441	0.157656	0.024029	0.000270	0.765468	0.557375	0.110475	0.009094	0.000018
0	0.747733	0.564418	0.157983	0.024087	0.000271	0.767110	0.558764	0.111054	0.009188	0.000018

observe from Figure 2 the convergence behavior of the finite-time Gerber-Shiu functions w.r.t. Δ. In fact, as $\Delta \longrightarrow 0$, the finite-time Gerber-Shiu functions will converge to those under continuous observation. To test this convergence behavior, we set $T = 10$ and compute the finite-time ruin probabilities with $\Delta = 2$, 1, 0.5, 0.1, 0.01, 0. Note that the case $\Delta = 0$ corresponds to continuous observation. In Table 5, we list the numerical results for exponential and Erlang(2,2) claim size density functions, where the results in the last row could be computed by formulas in Garcia [15] and Dickson [11]. It follows from each column that the finite-time ruin probabilities under periodic observation converge to the values under continuous observation.

Finally, we study the infinite-time Gerber-Shiu functions under periodic observation. Set $\delta = 0.01$, $w \equiv 1$, and consider the Erlang(2,2) and CoE claim size density functions. In Figure 3, we plot the infinite-time Gerber-Shiu functions for different claim size density functions. For periodic observation, we set $\Delta = 1, 0.5, 0.1, 0.01$, and plot the curves for the infinite-time Gerber-Shiu functions based on formula (6.11) with truncation parameter $K = 30$. We also consider the case $\Delta = 0$ (i.e., continuous observation), and plot the curves for the infinite-time Gerber-Shiu functions based on the following formulas,

- Erlang: $\phi(u) = 0.8287e^{-0.2626u} - 0.0216e^{-2.9390u}$, $u \geq 0$;
- CoE: $\phi(u) = 0.7431e^{-0.1360u} + 0.0444e^{-1.5615u}$, $u \geq 0$.

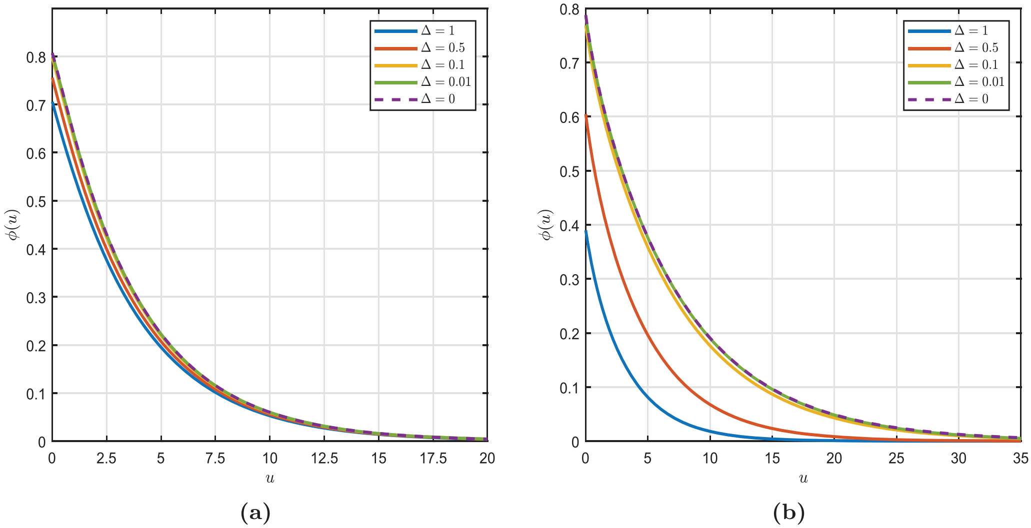

Figure 3. *Infinite-time Gerber-Shiu functions vs. initial surplus u.* $\Delta = 1, 0.5, 0.1, 0.01$. *(a) Erlang; (b) CoE.*

Table 6. *Approximating results of infinite-time ruin probability with Pareto distribution.*

Δ	$u = 0$	$u = 1$	$u = 5$	$u = 10$	$u = 20$	$u = 30$
1	0.760687	0.658533	0.379021	0.193633	0.050970	0.012985
0.5	0.793729	0.682770	0.390919	0.199494	0.052391	0.013242
0.1	0.824759	0.705654	0.402290	0.204979	0.053537	0.013247
0.01	0.832096	0.711122	0.404811	0.205913	0.053597	0.014369
0	0.844195	0.721992	0.412336	0.211048	0.056438	0.015248

The above formulas can be easily obtained by a Laplace transform inversion method. It follows from Figure 3 that all the infinite-time Gerber-Shiu functions are decreasing functions of the initial surplus level u. While for each fixed initial surplus level, $\phi(u)$ increase as Δ decreases, and furthermore, they all converge to those under continuous observation.

In Table 6, we report some approximation results for the infinite-time ruin probability with Pareto claim size density function. Set $\delta = 0$, $w \equiv 1$, $\Delta = 1, 0.5, 0.1, 0.01$, and compute the ruin probabilities based on formula (6.11) with truncation parameter $K = 30$. We also list some numerical results for ruin probability with continuous observation, where the results are computed by formula (20) in Ramsay [25]. From each column in Table 6, we find that the ruin probabilities increase as Δ decreases, and they converge to the corresponding values under continuous observation.

8. Conclusion

In this paper, we have proposed a new efficient method to compute both finite-time and infinite-time Gerber-Shiu functions when the classical risk model is observed periodically with constant inter-observation times. The Laguerre series expansion provides a theoretical basis for our approximation algorithm. For the finite-time Gerber-Shiu functions, we show that the Laguerre coefficients can be easily determined by some recursive equations, and we also study the approximation error. For the approximation of the infinite-time Gerber-Shiu functions, we show that the Laguerre coefficients can be determined by solving some linear equations. However, the approximation error is still very challenging to derive, and we leave it as an open problem for the future study.

As in Albrecher *et al.* [3], we can also consider the risk model with periodic dividend payments, where we can try to apply the Laguerre series expansion method to compute the finite-time expected present value of dividend payments before ruin and the finite-time Gerber-Shiu function. On the other hand, we can also use the Laguerre series expansion method to analyze some capital injection problems, which has been studied by Zhang *et al.* randomized observation periods. Note that when consider the periodic dividend payment (or capital injection) decisions, we can assume that ruin event is continuously or periodically monitored. We shall consider these open problems in the future.

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Appendix A. Proofs of Some Propositions

A.1. Proof of Proposition 1

First, by Fourier inversion formula, we obtain for each $j \geq 2$,

$$\begin{aligned} |f_X^{*j}(x)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \mathcal{F} f_X^{*j}(s)\, ds \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F} f_X(s)|^j \, ds \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{F} f_X(s)|^2 \, ds \\ &= \int_0^{\infty} (f_X(x))^2 \, dx = \|f_X\|_2^2 < \infty, \end{aligned} \tag{A.1}$$

where the second equality follows from Parseval's theorem. Next, using the upper bound (A.1) and Markov inequality, we obtain for $r > 0$,

$$\begin{aligned} \sup_{x>0} |g(x) - \tilde{g}(x)| &\leq \sum_{j=J+1}^{\infty} \frac{(\lambda\Delta)^j}{j!} e^{-\lambda\Delta} \sup_{x\geq 0} f_X^{*j}(x) \leq \|f_X\|_2^2 \sum_{j=J+1}^{\infty} \frac{(\lambda\Delta)^j}{j!} e^{-\lambda\Delta} \\ &= \|f_X\|_2^2 \cdot P(N_\Delta \geq J+1) \leq \|f_X\|_2^2 \cdot \frac{E[e^{rN_\Delta}]}{e^{r(J+1)}} \\ &\leq \|f_X\|_2^2 e^{\lambda\Delta(e^r - 1)} e^{-r(J+1)} = C_g e^{-rJ}, \end{aligned} \tag{A.2}$$

which completes the proof.

A.2. Proof of Proposition 2

First, for the approximation error of $\tilde{\phi}_1(u)$, we have

$$\begin{aligned}\phi_1(u)-\tilde{\phi}_1(u) &= \int_0^\infty w(x)[g(x+u+c\Delta)-\tilde{g}(x+u+c\Delta)]\,dx\\ &= e^{-\lambda\Delta}\sum_{j=J+1}^{\infty}\frac{(\lambda\Delta)^j}{j!}\int_0^\infty w(x)f_X^{*j}(x+u+c\Delta)\,dx\\ &= e^{-\lambda\Delta}\sum_{j=J+1}^{\infty}\frac{(\lambda\Delta)^j}{j!}\int_{u+c\Delta}^\infty w(x-u-c\Delta)f_X^{*j}(x)\,dx. \end{aligned} \tag{A.3}$$

Under Condition 2, we have

$$\begin{aligned}&\int_{u+c\Delta}^\infty w(x-u-c\Delta)f_X^{*j}(x)\,dx\\ &\le C_w\int_{u+c\Delta}^\infty [1\vee(x-u-c\Delta)^\kappa]f_X^{*j}(x)\,dx \le C_w\int_0^\infty[1\vee x^\kappa]f_X^{*j}(x)\,dx\\ &\le C_w + C_w\int_0^\infty x^\kappa f_X^{*j}(x)\,dx = C_w(1+E(X_1+\cdots X_j)^\kappa)\\ &\le C_w(1+j^\kappa\cdot EX^\kappa).\end{aligned}$$

Hence, by (A.3) and Markov inequality, we obtain

$$\begin{aligned}\sup_{u\ge 0}|\phi_1(u)-\tilde{\phi}_1(u)| &\le e^{-\lambda\Delta}\sum_{j=J+1}^{\infty}\frac{(\lambda\Delta)^j}{j!}C_w(1+j^\kappa\cdot EX^\kappa)\\ &= C_w\sum_{j=J+1}^{\infty}P(N_\Delta=j)\cdot(1+j^\kappa\cdot EX^\kappa)\\ &= C_w\cdot E[1+N_\Delta^\kappa EX^\kappa; N_\Delta\ge J+1]\\ &\le C_w\cdot E[e^{rN_\Delta}(1+N_\Delta^\kappa EX^\kappa)]e^{-r(J+1)}\\ &= C_1e^{-rJ}.\end{aligned} \tag{A.4}$$

Next, for $n\ge 2$, we have

$$\begin{aligned}\phi_n(u)-\tilde{\phi}_n(u) &= [\phi_1(u)-\tilde{\phi}_1(u)] + e^{-(\delta+\lambda)\Delta}[\phi_{n-1}(u+c\Delta)-\tilde{\phi}_{n-1}(u+c\Delta)]\\ &\quad + e^{-\delta\Delta}\int_0^{u+c\Delta}[\phi_{n-1}(u+c\Delta-x)g(x)-\tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)]\,dx,\end{aligned}$$

from which, we obtain

$$\begin{aligned}\sup_{u\ge 0}|\phi_n(u)-\tilde{\phi}_n(u)| &\le \sup_{u\ge 0}[\phi_1(u)-\tilde{\phi}_1(u)] + \sup_{u\ge 0}|\phi_{n-1}(u+c\Delta)-\tilde{\phi}_{n-1}(u+c\Delta)|\\ &\quad + \sup_{u\ge 0}\int_0^{u+c\Delta}|\phi_{n-1}(u+c\Delta-x)g(x)-\tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)|\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{u\geq 0}[\phi_1(u)-\tilde{\phi}_1(u)] + \sup_{u\geq 0}|\phi_{n-1}(u)-\tilde{\phi}_{n-1}(u)| \\
&\quad + \sup_{u\geq 0}\int_0^{u+c\Delta} |\phi_{n-1}(u+c\Delta-x)[g(x)-\tilde{g}(x)]|\,dx \\
&\quad + \sup_{u\geq 0}\int_0^{u+c\Delta} |[\phi_{n-1}(u+c\Delta-x)-\tilde{\phi}_{n-1}(u+c\Delta-x)]\tilde{g}(x)|\,dx \\
&\leq C_1 e^{-rJ} + \sup_{u\geq 0}|\phi_{n-1}(u)-\tilde{\phi}_{n-1}(u)| + \int_0^\infty \phi_{n-1}(u)\,du\cdot\sup_{x\geq 0}|g(x)-\tilde{g}(x)| \\
&\quad + \int_0^\infty \tilde{g}(x)\,dx\cdot\sup_{u\geq 0}|\phi_{n-1}(u)-\tilde{\phi}_{n-1}(u)| \\
&\leq \left(C_1 + C_g\int_0^\infty \phi(u)\,du\right)e^{-rJ} + 2\sup_{u\geq 0}|\phi_{n-1}(u)-\tilde{\phi}_{n-1}(u)|, \qquad \text{(A.5)}
\end{aligned}
$$

where the last step follows from $\int_0^\infty \phi_{n-1}(u)\,du \leq \int_0^\infty \phi(u)\,du$ and $\int_0^\infty \tilde{g}(x)\,dx \leq \int_0^\infty g(x)\,dx < 1$.

Finally, applying a recursive argument to (A.5), we obtain

$$
\begin{aligned}
\sup_{u\geq 0}|\phi_n(u)-\tilde{\phi}_n(u)| &\leq [1+2+\cdots+2^{n-2}]\left(C_1 + C_g\int_0^\infty \phi(u)\,du\right)e^{-rJ} + 2^{n-1}\sup_{u\geq 0}|\phi_1(u)-\tilde{\phi}_1(u)| \\
&\leq [2^{n-1}-1]\left(C_1 + C_g\int_0^\infty \phi(u)\,du\right)e^{-rJ} + 2^{n-1}C_1e^{-rJ} \\
&= C_n e^{-rJ}.
\end{aligned}
$$

This completes the proof.

A.3. Proof of Proposition 3

We can use mathematical induction to prove this proposition. First, suppose that $\tilde{\phi}_1,\ldots,\tilde{\phi}_{n-1} \in L^1(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$. The condition $f_X \in L^2(\mathbb{R}_+)$ implies that $\tilde{g} \in L^2(\mathbb{R}_+)$, which together with $\tilde{\phi}_{n-1} \in L^1(\mathbb{R}_+)$ and Theorem 1.4.5 in Stenger [28] implies that the convolution function

$$
\int_0^{u+c\Delta} \tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)\,dx, \quad u\geq 0,
$$

is square integrable. Hence, by (4.4), we know that $\tilde{\phi}_n \in L^2(\mathbb{R}_+)$. Finally, by (4.4), we have

$$
\begin{aligned}
\int_0^\infty \tilde{\phi}_n(u)\,du &= \int_0^\infty \tilde{\phi}_1(u)\,du + e^{-\delta\Delta}\int_0^\infty\int_0^{u+c\Delta} \tilde{\phi}_{n-1}(u+c\Delta-x)\tilde{g}(x)\,dx\,du \\
&\quad + e^{-(\delta+\lambda)\Delta}\int_0^\infty \tilde{\phi}_{n-1}(u+c\Delta)\,du \\
&\leq \int_0^\infty \tilde{\phi}_1(u)\,du + e^{-\delta\Delta}\int_0^\infty \tilde{\phi}_{n-1}(u)\,du\int_0^\infty \tilde{g}(x)\,dx \\
&\quad + e^{-(\delta+\lambda)\Delta}\int_0^\infty \tilde{\phi}_{n-1}(u)\,du \\
&< \int_0^\infty \tilde{\phi}_1(u)\,du + \int_0^\infty \tilde{\phi}_{n-1}(u)\,du < \infty,
\end{aligned}
$$

which yields $\tilde{\phi}_n \in L^1(\mathbb{R}_+)$.

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A.4. Proof of Proposition 4

First, under Condition 2, we have

$$
\begin{aligned}
&\int_0^\infty w(x)|\varphi_k(x+u+c\Delta)|\,dx \\
&\quad\le \int_0^\infty C_w[1\vee x^\kappa]|\varphi_k(x+u+c\Delta)|\,dx \\
&\quad\le C_w\int_0^\infty |\varphi_k(x+u+c\Delta)|\,dx + C_w\int_0^\infty x^\kappa|\varphi_k(x+u+c\Delta)|\,dx \\
&\quad\le C_w\int_0^\infty |\varphi_k(x+u+c\Delta)|\,dx + C_w\int_0^\infty (x+u+c\Delta)^\kappa|\varphi_k(x+u+c\Delta)|\,dx \\
&\quad\le C_w\int_0^\infty [1+x^\kappa]|\varphi_k(x)|\,dx \\
&\quad\le 2C_w\int_0^1 |\varphi_k(x)|\,dx + 2C_w\int_1^\infty x^\kappa|\varphi_k(x)|\,dx. \qquad\qquad (A.6)
\end{aligned}
$$

Next, we derive some upper bounds for the integrals on the right side of (A.6). Using the uniform upper bound (2.2) for Laguerre functions, we have

$$
\sup_{k\ge 0}\int_0^1 |\varphi_k(x)|dx \le \int_0^1 \sup_{k\ge 0}|\varphi_k(x)|\,dx \le \sqrt{2}. \qquad (A.7)
$$

For the second integral in the last line of (A.6), using formula (2.4), we have

$$
\begin{aligned}
\int_1^\infty x^\kappa|\varphi_k(x)|\,dx &= \int_1^\infty \frac{x^{\kappa+1}}{x}|\varphi_k(x)|\,dx = \int_1^\infty \left|\sum_{m=0}^{k+\kappa+1}\Xi_{\kappa+1,k,m}\varphi_m(x)\right|\cdot\frac{1}{x}\,dx \\
&\le \sum_{m=0}^{k+\kappa+1}|\Xi_{\kappa+1,k,m}|\int_1^\infty |\varphi_m(x)|\cdot\frac{1}{x}\,dx.
\end{aligned}
$$

By Cauchy–Schwarz inequality,

$$
\begin{aligned}
\int_1^\infty |\varphi_m(x)|\cdot\frac{1}{x}\,dx &\le \left(\int_1^\infty |\varphi_m(x)|^2\,dx\right)^{1/2}\cdot\left(\int_1^\infty \frac{1}{x^2}\,dx\right)^{1/2} \\
&= \left(\int_1^\infty |\varphi_m(x)|^2\,dx\right)^{1/2} \le \left(\int_0^\infty |\varphi_m(x)|^2\,dx\right)^{1/2} = 1.
\end{aligned}
$$

Hence, using the inequality (2.6), we have

$$
\int_1^\infty x^\kappa|\varphi_k(x)|dx \le \sum_{m=0}^{k+\kappa+1}|\Xi_{\kappa+1,k,m}| \le \sum_{m=0}^{k+\kappa+1}2^{\kappa+1}(k+\kappa+2)^{\kappa+1} = 2^{\kappa+1}(k+\kappa+2)^{\kappa+2}. \qquad (A.8)
$$

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By the inequalities (A.6), (A.7) and (A.8), we obtain

$$\int_0^\infty w(x)|\varphi_k(x+u+c\Delta)|\,dx \le 2\sqrt{2}C_w + 2^{\kappa+2}(k+\kappa+2)^{\kappa+2}C_w, \tag{A.9}$$

which yields

$$\begin{aligned}\sum_{k=0}^{\infty}|A_{\tilde{g},k}|\int_0^\infty w(x)|\varphi_k(x+u+c\Delta)|\,dx \\ < 2\sqrt{2}C_w\sum_{k=0}^{\infty}|A_{\tilde{g},k}| + 2^{\kappa+2}C_w\sum_{k=0}^{\infty}|A_{\tilde{g},k}|(k+\kappa+2)^{\kappa+2}.\end{aligned} \tag{A.10}$$

By Remark 3, we know that $\sum_{k=0}^{\infty}|A_{\tilde{g},k}| < \infty$ as $\tilde{g}$ belongs to the Sobolev–Laguerre space $W(\mathbb{R}_+,\theta)$ for some $\theta > 1$. For the second summation on the right side of (A.10), note that there exists some $C_\kappa > 0$ such that $(k+\kappa+2)^{\kappa+2} \le C_\kappa k^{\kappa+2}$ uniformly for $k = 1, 2, \ldots$, then using Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{k=0}^{\infty}|A_{\tilde{g},k}|(k+\kappa+2)^{\kappa+2} &\le A_{\tilde{g},0}(\kappa+2)^{\kappa+2} + C_\kappa\sum_{k=1}^{\infty}|A_{\tilde{g},k}|k^{\kappa+2} \\ &= A_{\tilde{g},0}(\kappa+2)^{\kappa+2} + C_\kappa\sum_{k=1}^{\infty}|A_{\tilde{g},k}|k^{\kappa+2+\theta}k^{-\theta} \\ &\le A_{\tilde{g},0}(\kappa+2)^{\kappa+2} + C_\kappa\left(\sum_{k=1}^{\infty}|A_{\tilde{g},k}|^2k^{2(\kappa+2+\theta)}\right)^{1/2}\left(\sum_{k=1}^{\infty}k^{-2\theta}\right)^{1/2}.\end{aligned}$$

Hence, we conclude that $\sum_{k=0}^{\infty}|A_{\tilde{g},k}|(k+\kappa+2)^{\kappa+2} < \infty$ as $\tilde{g}$ belongs to the Sobolev–Laguerre space $W(\mathbb{R}_+,\theta)$ for some $\theta > 2\kappa+5$.

Finally, using the fact $W(\mathbb{R}_+,\theta_2) \subset W(\mathbb{R}_+,\theta_1)$ as $0 < \theta_1 < \theta_2$, we conclude that $\sum_{k=0}^{\infty}|A_{\tilde{g},k}| < \infty$ is satisfied as $\tilde{g}$ belongs to the Sobolev–Laguerre space $W(\mathbb{R}_+,\theta)$ for some $\theta > 2\kappa+5$. This completes the proof.

A.5. Proof of Proposition 5

First, we have

$$\begin{aligned}|A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| &\le e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\left|\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx\right| \\ &= e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\left|\int_0^\infty w(x)\int_0^\infty \varphi_j(u+x+c\Delta)\varphi_k(u)\,du\,dx\right| \\ &= e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\left|\int_0^\infty\int_0^\infty w(x)\varphi_j(u+x+c\Delta)\,dx\varphi_k(u)\,du\right| \\ &= e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\left|\sum_{l=0}^{j}\int_0^\infty \zeta_{j,l}(u+c\Delta)\int_0^\infty w(x)\varphi_l(x)\,dx\varphi_k(u)\,du\right| \\ &\le e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\sum_{l=0}^{j}\int_0^\infty|\zeta_{j,l}(u+c\Delta)\varphi_k(u)|\,du\cdot\int_0^\infty|w(x)\varphi_l(x)|\,dx.\end{aligned} \tag{A.11}$$

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Using Schwarz's inequality, we find that for each j, l,

$$|\zeta_{j,l}(u+c\Delta)| \leq \int_0^\infty |\varphi_j(x+u+c\Delta)\varphi_l(x)|\,dx$$
$$\leq \left(\int_0^\infty |\varphi_j(x+u+c\Delta)|^2\,dx\right)^{1/2}\left(\int_0^\infty |\varphi_l(x)|^2\,dx\right)^{1/2} \leq \|\varphi_j\|_2\cdot\|\varphi_l\|_2 = 1, \quad \text{(A.12)}$$

that is, $\zeta_{j,l}$ is uniformly bounded by one. Hence, the inequality (A.11) can be further bounded as

$$|A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| \leq e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\sum_{l=0}^{j}\int_0^\infty|\varphi_k(u)|\,du\cdot\int_0^\infty|w(x)\varphi_l(x)|\,dx. \quad \text{(A.13)}$$

Next, using the same arguments as in the proof of Proposition 4, we have

$$\int_0^\infty |\varphi_k(u)|\,du \leq \sqrt{2}+2(k+2)^2$$

and

$$\int_0^\infty |w(x)\varphi_l(x)|\,dx \leq C_w\int_0^\infty[1+x^\kappa]|\varphi_l(x)|\,dx \leq 2\sqrt{2}C_w + 2^{\kappa+2}(l+\kappa+2)^{\kappa+2}C_w.$$

Then, the inequality (A.13) yields

$$|A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| \leq e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|\cdot\sum_{l=0}^{j}[\sqrt{2}+2(k+2)^2]\cdot[2\sqrt{2}C_w + 2^{\kappa+2}(l+\kappa+2)^{\kappa+2}C_w]$$
$$\leq 3(k+2)^2e^{-\delta\Delta}\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|[2\sqrt{2}C_w(j+1) + 2^{\kappa+2}(j+1)(j+\kappa+2)^{\kappa+2}C_w].$$

It is easily seen that there exists some $C' > 0$ such that

$$2\sqrt{2}C_w(j+1) + 2^{\kappa+2}(j+1)(j+\kappa+2)^{\kappa+2}C_w \leq C'j^{\kappa+3}$$

uniformly in j. Then, using Cauchy–Schwarz inequality, we have

$$|A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| \leq 3(k+2)^2e^{-\delta\Delta}C'\sum_{j=K+1}^{\infty}|A_{\tilde{g},j}|j^{\kappa+3+\rho}j^{-\rho}$$
$$\leq 3(k+2)^2e^{-\delta\Delta}C'\left(\sum_{j=K+1}^{\infty}A_{\tilde{g},j}^2 j^{2(\kappa+3+\rho)}\right)^{1/2}\left(\sum_{j=K+1}^{\infty}j^{-2\rho}\right)^{1/2}$$
$$\leq 3(k+2)^2e^{-\delta\Delta}C'\left(\sum_{j=0}^{\infty}A_{\tilde{g},j}^2 j^{2(\kappa+3+\rho)}\right)^{1/2}\left(\frac{K^{1-2\rho}}{2\rho-1}\right)^{1/2}$$
$$= k^2\cdot O(K^{-\rho+1/2}) \quad \text{(A.14)}$$

under the condition $\tilde{g} \in W(\mathbb{R}_+,\theta)$ for some $\theta \geq 2(\kappa+3+\rho)$. This completes the proof.

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A.6. Proof of Proposition 7

First, by formulas (5.15) and (5.17), we easily obtain

$$
\begin{aligned}
|A_{\tilde{\phi}_n,k} - \hat{A}_{\tilde{\phi}_n,k}| \le |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| &+ e^{-(\delta+\lambda)\Delta} \sum_{j=K+1}^{\infty} |A_{\tilde{\phi}_{n-1},j}| \cdot |\zeta_{j,k}(c\Delta)| \\
&+ e^{-(\delta+\lambda)\Delta} \sum_{j=k}^{K} |A_{\tilde{\phi}_{n-1},j} - \hat{A}_{\tilde{\phi}_{n-1},j}| \cdot |\zeta_{j,k}(c\Delta)| \\
&+ e^{-\delta\Delta} \sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}| \cdot |\zeta_{m,k}(c\Delta)| \\
&+ e^{-\delta\Delta} \sum_{m=k}^{K} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j} - \hat{A}_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}| \cdot |\zeta_{m,k}(c\Delta)|,
\end{aligned}
$$

from which, together with $|\sup_{m,k} \zeta_{m,k}(c\Delta)| < 1$, we obtain

$$
\begin{aligned}
\sup_{k\le K} |A_{\tilde{\phi}_n,k} - \hat{A}_{\tilde{\phi}_n,k}| \le \sup_{k\le K} |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| &+ K \cdot \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| + \sum_{j=K+1}^{\infty} |A_{\tilde{\phi}_{n-1},j}| \\
&+ \sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}| \\
&+ \sum_{m=k}^{K} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j} - \hat{A}_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}| \\
&:= \sup_{k\le K} |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| + K \cdot \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3. \qquad (A.15)
\end{aligned}
$$

For $\mathcal{E}_1$, by (2.7), we have

$$
\mathcal{E}_1 = O(K^{-(\theta^*-1)/2}). \qquad (A.16)
$$

under Condition 4.

For $\mathcal{E}_2$, we have

$$
\mathcal{E}_2 \le \sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| + \sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j-1}|. \qquad (A.17)
$$

For the first summation in (A.17), we rewrite it as

$$
\begin{aligned}
\sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| &= \sum_{m=K+1}^{\infty} \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| \\
&+ \sum_{m=K+1}^{\infty} \sum_{j=\lfloor m/2 \rfloor+1}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}|. \qquad (A.18)
\end{aligned}
$$

Using the fact $\tilde{\phi}_n(u) \le \phi(u)$ and $\tilde{g}(x) \le g(x)$, we have

$$
\sup_j |A_{\tilde{\phi}_n,j}| \le \|\tilde{\phi}_n\|_2 \le \|\phi\|_2, \quad |A_{\tilde{g},j}| \le \|\tilde{g}\|_2 \le \|g\|_2.
$$

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Furthermore, under Condition 4, we have for each $j \geq 1$

$$j^{\theta^*} A^2_{\tilde{\phi}_n,j} \leq \sum_{k=0}^{\infty} k^{\theta^*} A^2_{\tilde{\phi}_n,k} \leq B_\phi < \infty,$$

yielding $|A_{\tilde{\phi}_n,j}| \leq \sqrt{B_\phi} j^{-\theta^*/2}$. Similarly, $\tilde{g} \in W(\mathbb{R}_+, \theta)$ yields that, for some $B_g > 0$, we have $|A_{\tilde{g},j}| \leq \sqrt{B_g} j^{-\theta/2}$, $j \geq 1$. Hence, we have

$$\begin{aligned}
&\sum_{m=K+1}^{\infty} \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| \\
&\leq \frac{\|\phi\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sum_{j=0}^{\lfloor m/2 \rfloor} |A_{\tilde{g},m-j}| \leq \frac{\|\phi\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sum_{j=0}^{\lfloor m/2 \rfloor} \sqrt{B_g} (m-j)^{-\theta/2} \\
&\leq \frac{\|\phi\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sqrt{B_g} (m/2)^{-\theta/2+1} \\
&= O(K^{-(\theta/2-2)})
\end{aligned} \tag{A.19}$$

and

$$\begin{aligned}
&\sum_{m=K+1}^{\infty} \sum_{j=\lfloor m/2 \rfloor+1}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| \\
&\leq \frac{\|g\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sum_{j=\lfloor m/2 \rfloor+1}^{m} |A_{\tilde{g},m-j}| \leq \frac{\|g\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sum_{j=\lfloor m/2 \rfloor+1}^{m} \sqrt{B_\phi} j^{-\theta^*/2} \\
&\leq \frac{\|\phi\|_2}{\sqrt{2}} \sum_{m=K+1}^{\infty} \sqrt{B_\phi} (m/2)^{-\theta^*/2+1} \\
&= O(K^{-(\theta^*/2-2)}),
\end{aligned} \tag{A.20}$$

which together with (A.18) give

$$\sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j}| = O(\max(K^{-(\theta^*/2-2)}, K^{-(\theta/2-2)})).$$

Similarly, we have

$$\sum_{m=K+1}^{\infty} \sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{\phi}_{n-1},j}| \cdot |A_{\tilde{g},m-j-1}| = O(\max(K^{-(\theta^*/2-2)}, K^{-(\theta/2-2)})).$$

Hence, we have for $\theta^*, \theta > 4$,

$$\mathcal{E}_2 = O(\max(K^{-(\theta^*/2-2)}, K^{-(\theta/2-2)})). \tag{A.21}$$

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For $\mathcal{E}_3$, we have

$$\begin{aligned}\mathcal{E}_3 &\le \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| \cdot \sum_{m=k}^{K}\sum_{j=0}^{m} \frac{1}{\sqrt{2}} |A_{\tilde{g},m-j} - A_{\tilde{g},m-j-1}| \\ &\le \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| \cdot \sqrt{2}K \sum_{j=0}^{\infty} |A_{\tilde{g},j}| \\ &\le \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| \cdot C_g K \end{aligned} \tag{A.22}$$

for some $C_g > 0$.

By (A.15), (A.16), (A.21) and (A.22), we obtain

$$\begin{aligned}\sup_{k\le K} |A_{\tilde{\phi}_n,k} - \hat{A}_{\tilde{\phi}_n,k}| &\le \sup_{k\le K} |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| + (1+C_g)K \cdot \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| \\ &\quad + O(\max(K^{-(\theta^*/2-2)}, K^{-(\theta/2-2)})) \\ &\le (1+C_g)K \cdot \sup_{k\le K} |A_{\tilde{\phi}_{n-1},k} - \hat{A}_{\tilde{\phi}_{n-1},k}| + O(\max(K^{-(\theta^*/2-2)}, K^{-\rho+5/2})), \end{aligned} \tag{A.23}$$

where the second inequality follows from Proposition 5. Since $\theta/2-2 > \rho-5/2$, then by (5.8), we obtain

$$\sup_{k\le K} |A_{\tilde{\phi}_1,k} - \hat{A}_{\tilde{\phi}_1,k}| = O(K^{-\rho+5/2}).$$

Finally, applying a recursive argument based on (A.23), we easily obtain

$$\sup_{k\le K} |A_{\tilde{\phi}_n,k} - \hat{A}_{\tilde{\phi}_n,k}| = O(\max(K^{-(\theta^*/2-n-1)}, K^{-(\rho-n-3/2)})).$$

This completes the proof.

Appendix B. Some explicit results for the integral in (5.6)

The following is a list of some widely considered examples for the penalty function w, for which we show that the integral in (5.6) can be explicitly computed.

1. Set the penalty function $w \equiv 1$, then the finite-time Gerber-Shiu function becomes the Laplace transform of ruin time as $\delta > 0$ and the finite-time ruin probability as $\delta = 0$. In this case, the integral in (5.6) can be computed as follows,

$$\begin{aligned}\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx &= \int_0^\infty\int_0^\infty \varphi_j(u+x+c\Delta)\varphi_k(u)\,du\,dx \\ &= \int_0^\infty\int_0^\infty \varphi_j(u+x+c\Delta)\,dx\varphi_k(u)\,du. \end{aligned} \tag{B.1}$$

Furthermore, using the known result

$$\int_u^\infty \varphi_j(x)\,dx = \sum_{i=0}^{j-1} 2(-1)^{i+j}\varphi_i(u) + \varphi_j(u) = \sum_{i=0}^{j} C_{j,i}\varphi_i(u) \tag{B.2}$$

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with $C_{j,i} = 2(-1)^{i+j}$ for $i \neq j$ and $C_{j,j} = 1$, we have

$$\begin{aligned}\int_0^\infty \varphi_j(u+x+c\Delta)\,dx &= \int_{u+c\Delta}^\infty \varphi_j(x)\,dx = \sum_{i=0}^{j} C_{j,i}\varphi_i(u+c\Delta)\\ &= \sum_{i=0}^{j} C_{j,i}\sum_{l=0}^{i}\zeta_{i,l}(c\Delta)\varphi_l(u)\\ &= \sum_{l=0}^{j}\sum_{i=l}^{j} C_{j,i}\zeta_{i,l}(c\Delta)\varphi_l(u),\end{aligned}$$

where the third step follows from formula (5.4). Finally, due to that $\{\varphi_k\}_{k=0,1,\ldots}$ is a complete orthonormal basis, we have

$$\begin{aligned}\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx &= \sum_{l=0}^{j}\sum_{i=l}^{j} C_{j,i}\zeta_{i,l}(c\Delta)\int_0^\infty \varphi_l(u)\varphi_k(u)\,du\\ &= \begin{cases} 0, & \text{for } j<k,\\ \sum_{i=k}^{j} C_{j,i}\zeta_{i,l}(c\Delta), & \text{for } j\geq k.\end{cases}\end{aligned} \tag{B.3}$$

2. Set the penalty function $w(x) = \mathbf{1}_{\{x\leq y\}}$ for some $y>0$, then the Gerber-Shiu function becomes the (discounted) distribution of deficit at ruin. In this case, we have

$$\begin{aligned}\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx &= \int_0^y \zeta_{j,k}(x+c\Delta)\,dx\\ &= \int_0^\infty \zeta_{j,k}(x+c\Delta)\,dx - \int_0^\infty \zeta_{j,k}(x+y+c\Delta)\,dx\\ &= \begin{cases} 0, & \text{for } j<k,\\ \sum_{i=k}^{j} C_{j,i}[\zeta_{i,l}(c\Delta)-\zeta_{i,l}(y+c\Delta)], & \text{for } j\geq k.\end{cases}\end{aligned} \tag{B.4}$$

3. Set the penalty function to be the Dirac Delta function $w(x) = \delta_y(x)$ for some $y>0$, then the Gerber-Shiu function becomes the (discounted) density function of the deficit at ruin. In this case, we have

$$\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx = \int_0^\infty \delta_y(x)\zeta_{j,k}(x+c\Delta)\,dx = \zeta_{j,k}(y+c\Delta). \tag{B.5}$$

4. Set the penalty function $w(x) = x^n$, then the Gerber-Shiu function becomes the (discounted) moment of deficit at ruin. In this case, we have

$$\begin{aligned}&\int_0^\infty w(x)\zeta_{j,k}(x+c\Delta)\,dx\\ &= \int_0^\infty x^n \int_0^\infty \varphi_j(u+x+c\Delta)\varphi_k(u)\,du\,dx\end{aligned}$$

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$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty (u + x + c\Delta - u - c\Delta)^n \varphi_j(u + x + c\Delta)\, dx \varphi_k(u)\, du \\
&= \sum_{l=0}^{n} \binom{n}{l} (-u - c\Delta)^{n-l} \int_0^\infty \int_0^\infty (u + x + c\Delta)^l \varphi_j(u + x + c\Delta)\, dx \varphi_k(u)\, du \\
&= \sum_{l=0}^{n} \binom{n}{l} (-u - c\Delta)^{n-l} \int_0^\infty \int_{u+c\Delta}^\infty x^l \varphi_j(x)\, dx \varphi_k(u)\, du. \qquad \text{(B.6)}
\end{aligned}
$$

For the integral in the last line of (B.6), using formula (2.4), we obtain

$$
\begin{aligned}
&\int_0^\infty \int_{u+c\Delta}^\infty x^l \varphi_j(x)\, dx \varphi_k(u)\, du \\
&= \sum_{m=0}^{j+l} \Xi_{l,j,m} \int_0^\infty \int_{u+c\Delta}^\infty \varphi_m(x)\, dx \varphi_k(u)\, du \\
&= \sum_{m=0}^{j+l} \Xi_{l,j,m} \sum_{\kappa=0}^{m} C_{m,\kappa} \sum_{\iota=0}^{\kappa} \zeta_{\kappa,\iota}(c\Delta) \int_0^\infty \varphi_\iota(u) \varphi_k(u)\, du \\
&= \sum_{m=0}^{j+l} \sum_{\kappa=0}^{m} \sum_{\iota=0}^{\kappa} \Xi_{l,j,m} C_{m,\kappa} \zeta_{\kappa,\iota}(c\Delta) \mathbf{1}_{\{\iota=\kappa\}} \\
&= \sum_{\iota=0}^{j+l} \sum_{m=\iota}^{j+l} \sum_{\kappa=\iota}^{m} \Xi_{l,j,m} C_{m,\kappa} \zeta_{\kappa,\iota}(c\Delta) \mathbf{1}_{\{\iota=\kappa\}} \\
&= \begin{cases} 0, & \text{if } j + l < k, \\ \displaystyle\sum_{m=k}^{j+l} \sum_{\kappa=k}^{m} \Xi_{l,j,m} C_{m,\kappa} \zeta_{\kappa,k}(c\Delta), & \text{if } j + l \geq k. \end{cases} \qquad \text{(B.7)}
\end{aligned}
$$

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