Adv. Appl. Prob. **46,** 832–845 (2014)
Printed in Northern Ireland

A CELLULAR NETWORK MODEL WITH GINIBRE CONFIGURED BASE STATIONS

NAOTO MIYOSHI,* *Tokyo Institute of Technology*
TOMOYUKI SHIRAI,** *Kyushu University*

Abstract

Stochastic geometry models for wireless communication networks have recently attracted much attention. This is because the performance of such networks critically depends on the spatial configuration of wireless nodes and the irregularity of the node configuration in a real network can be captured by a spatial point process. However, most analysis of such stochastic geometry models for wireless networks assumes, owing to its tractability, that the wireless nodes are deployed according to homogeneous Poisson point processes. This means that the wireless nodes are located independently of each other and their spatial correlation is ignored. In this work we propose a stochastic geometry model of cellular networks such that the wireless base stations are deployed according to the Ginibre point process. The Ginibre point process is one of the determinantal point processes and accounts for the repulsion between the base stations. For the proposed model, we derive a computable representation for the coverage probability—the probability that the signal-to-interference-plus-noise ratio (SINR) for a mobile user achieves a target threshold. To capture its qualitative property, we further investigate the asymptotics of the coverage probability as the SINR threshold becomes large in a special case. We also present the results of some numerical experiments.

Keywords: Stochastic geometry model; wireless communication network; cellular network; determinantal point process; Ginibre point process; signal-to-interference-plus-noise ratio; coverage probability

2010 Mathematics Subject Classification: Primary 60G55
Secondary 90B18; 60D05

1. Introduction

Recently, stochastic geometry models for wireless communication networks have attracted much attention (see, e.g. the introductory articles by Andrews *et al.* [2] and Haenggi *et al.* [12], and the monographs by Baccelli and Błaszczyszyn [3], [4], and Haenggi [11]). This is because the performance of such networks critically depends on the spatial configuration of wireless nodes, and the irregularity of the node configuration in a real network can be captured by a spatial point process. For cellular networks, some works have also proposed and analyzed the stochastic geometry models, where the wireless base stations and mobile users are located randomly on the Euclidean plane, and various performance indices, such as the coverage probability—the probability that the signal-to-interference-plus-noise ratio (SINR)

Received 5 July 2013; revision received 24 September 2013.
* Postal address: Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1-W8-52 Ookayama, Tokyo, 152-8552, Japan. Email address: miyoshi@is.titech.ac.jp
** Postal address: Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan. Email address: shirai@imi.kyushu-u.ac.jp

for a mobile user achieves a target threshold—have been evaluated (see, e.g. [1], [6], [7], [8], [14], [16], and [17], which we briefly review in the next section).

Most analysis of such stochastic geometry models for wireless networks, however, assumes that the wireless nodes are deployed according to homogeneous Poisson point processes though the modeling is possible using general spatial point processes. While this assumption makes the models tractable, it means that the wireless nodes are located independently of each other and their spatial correlation is ignored. Since real networks can be designed such that two wireless nodes are not too close, models which account for repulsion between the nodes are required. Only a few works have so far allowed for non-Poisson configurated wireless nodes, except for some works dealing with the classical grid models for cellular networks. For example, Błaszczyszyn and Yogeshwaran [5] studied the connectivity of sub-Poisson SINR graphs. For general motion-invariant (stationary and isotropic) point process models, Giacomelli *et al.* [10] studied the asymptotics of the coverage probability as the density of interfering nodes goes to 0, and Ganti *et al.* [9] developed a series expansion for functions of interference using the factorial moment expansion.

In this work we propose a stochastic geometry model of cellular networks such that the wireless base stations are deployed according to the Ginibre point process. The Ginibre point process is one of the determinantal point processes, which are used to model fermions in quantum mechanics and account for the repulsion between particles, and has been well studied since it has several desirable features (see, e.g. [13], [18], and [19]). For the proposed model, we derive a computable representation for the coverage probability. Furthermore, to capture its qualitative property, we investigate the asymptotics of coverage probability as the SINR threshold becomes large in the interference-limited (noise-free) case. Though we here focus on the coverage probability in a basic model, it would be possible to extend our results to more practical problems developed in [6], [7], [8], [14], and [17].

The rest of the paper is organized as follows. In the next section we describe our stochastic geometry model of cellular networks by following [1] and present a brief review on some related works with Poisson configured base stations. We also derive a basic formula for the coverage probability, which plays a key role in our analysis. In Section 3 we define the Ginibre point process and give some of its useful properties, as well as defining a scaled version of that process. The computable integral representation for the coverage probability is derived in Section 4. The effect of random frequency reuse is also considered there. In Section 5 we investigate the asymptotic property of the coverage probability as the SINR threshold becomes large in the interference-limited case. The results of numerical experiments are presented in Section 6. Finally, concluding remarks are given in Section 7.

2. Stochastic geometry model of cellular networks

In this section we describe a stochastic geometry model of cellular wireless networks, which mainly follows [1] though some notation is altered for convenience. Let Φ denote a point process on $\mathbb{R}^2$, and let X_i, $i \in \mathbb{N}$, denote the points of Φ, where the order of $X_1, X_2, \dots$ is arbitrary. The point process Φ represents the configuration of wireless base stations and we refer to the base station located at X_i as station i. We assume that Φ is simple and locally finite almost surely (a.s.), and also motion invariant. The transmission power of each base station is constant at $1/\mu$, $\mu > 0$. Each mobile user is associated with the closest base station, that is, the mobile users in the Voronoi cell of a base station are associated with that station. Thus, owing to the motion invariance of the point process and the homogeneity of base stations, we can focus on a typical user located at the origin $o = (0, 0)$. We assume Rayleigh fading for the random effect of fading from each base station to a user (shadowing is ignored), so that the transmission power multiplied by the fading effect from station i to the typical user, denoted by F_i, is an exponential random variable with mean $1/\mu$, where F_i, $i \in \mathbb{N}$, are mutually independent and also independent of the point process Φ. The path-loss function ℓ representing the attenuation of signals with distance is given by $\ell(r) = \alpha r^{-2\beta}$, $r > 0$, for some $\alpha > 0$ and $\beta > 1$.

In the setting described above, the SINR of a typical user from the associated base station is then expressed as

$$\mathsf{SINR}_o = \frac{F_{B_o}\ell(|X_{B_o}|)}{W_o + I_o(B_o)}, \tag{1}$$

where B_o denotes the index of the base station associated with the typical user and W_o denotes a random variable representing the thermal noise at the origin. We assume that W_o is independent of $\Phi_F = \{(X_i, F_i)\}_{i\in\mathbb{N}}$ and its Laplace transform is known to be computable. Also, $I_o(i)$, $i \in \mathbb{N}$, in (1) denotes the cumulative interference from all the base stations except station i received by the typical user and is given by

$$I_o(i) = \sum_{j\in\mathbb{N}\setminus\{i\}} F_j \ell(|X_j|). \tag{2}$$

We consider the coverage probability as the performance index, which is defined as $p(\theta, \beta) = \mathbb{P}(\mathsf{SINR}_o > \theta)$; the probability that the SINR of a typical user achieves a predefined threshold $\theta > 0$ (the coverage probability is, of course, not only a function of θ and β but also of μ, α, and so on, but, as we will see later, the effective parameters are θ and β).

Some works have so far considered similar cellular network models in which the base stations are deployed according to homogeneous Poisson point processes. Andrews *et al.* [1] dealt with more general fading distributions, and evaluated the coverage probability and the mean achievable rate, defined as $\tau(\beta) = \mathbb{E}\ln(1 + \mathsf{SINR}_o)$. Decreusefond *et al.* [6] proposed a model that incorporated time-invariant shadowing and time-variant fading, and evaluated the handover probability under the assumption that the associated base stations are altered when the SINR from the current associated station continues to be lower than the threshold. Most recent works have extended the model to that with multitiers of heterogeneous base stations, which generates the macrocells, picocells, or femtocells (see, e.g. [7], [8], [14], [16], and [17]).

In this paper we adopt the Ginibre point process (or its scaled version) as the point process Φ representing the configuration of base stations. Samples of Poisson and Ginibre point processes with the same intensity are presented in Figure 1, where we can see that the points of the Ginibre process are distributed more evenly. Also, comparing it with Figure 2 of [1], we find that the point configuration of the Ginibre process is relatively closer to a real base station deployment by a major service provider in a relatively flat urban area than that of a Poisson process. Before proceeding to the description of the Ginibre point process, we give a basic formula for the coverage probability, which plays a key role in our analysis.

Lemma 1. *For the cellular network model described above, where the base stations are deployed according to a general stationary point process on $\mathbb{R}^2$, the coverage probability for a typical user satisfies*

$$p(\theta, \beta) = \mathbb{E}\biggl(\mathcal{L}_W\biggl(\frac{\mu\theta|X_{B_0}|^{2\beta}}{\alpha}\biggr)\prod_{j\in\mathbb{N}\setminus\{B_0\}}\biggl(1 + \theta\biggl|\frac{X_{B_0}}{X_j}\biggr|^{2\beta}\biggr)^{-1}\biggr), \tag{3}$$

where $\mathcal{L}_W$ denotes the Laplace transform of W_o.

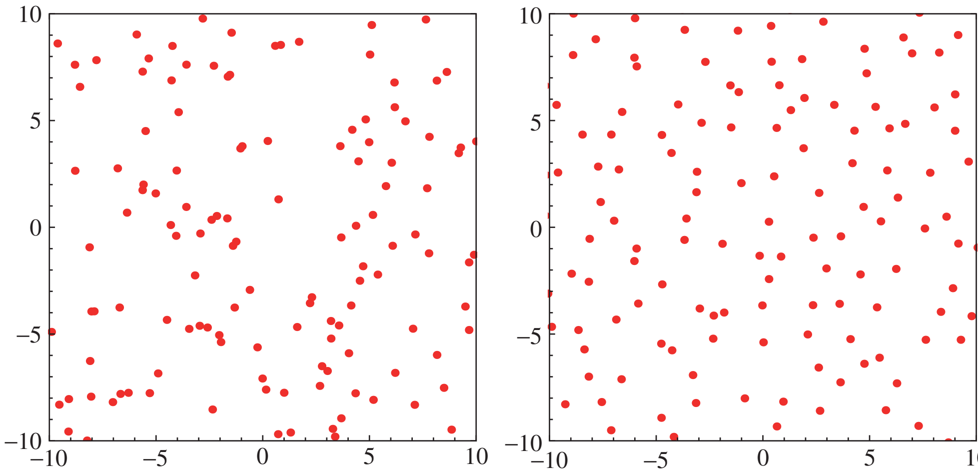

FIGURE 1: Samples of the Poisson point process (*left*) and Ginibre point process (*right*).

Proof. We have, from (1),

$$\begin{aligned}
\mathbb{P}(\mathsf{SINR}_o > \theta) &= \sum_{i=1}^{\infty} \mathbb{P}(\mathsf{SINR}_o > \theta,\ B_o = i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}\biggl(F_i > \frac{\theta(W_o + I_o(i))}{\ell(|X_i|)},\ |X_i| \le |X_j|,\ j \in \mathbb{N}\biggr). \qquad (4)
\end{aligned}$$

Since F_i is exponentially distributed with mean $1/\mu$, and W_o and $I_o(i)$ are mutually independent, conditioning yields

$$\begin{aligned}
&\mathbb{P}\biggl(F_i > \frac{\theta(W_o + I_o(i))}{\ell(|X_i|)},\ |X_i| \le |X_j|,\ j \in \mathbb{N}\biggr) \\
&\quad = \mathbb{E}(\mathrm{e}^{-\mu\theta W_o/\ell(|X_i|)}\mathrm{e}^{-\mu\theta I_o(i)/\ell(|X_i|)}\mathbf{1}_{\{|X_i|\le|X_j|,\ j\in\mathbb{N}\}}) \\
&\quad = \mathbb{E}\biggl(\mathcal{L}_W\biggl(\frac{\mu\theta}{\ell(|X_i|)}\biggr)\mathbb{E}(\mathrm{e}^{-\mu\theta I_o(i)/\ell(|X_i|)} \mid \Phi)\mathbf{1}_{\{|X_i|\le|X_j|,\ j\in\mathbb{N}\}}\biggr), \qquad (5)
\end{aligned}$$

where $\mathbf{1}_A$ denotes the indicator of the set A. Furthermore, since F_j, $j \in \mathbb{N}$, are mutually independent, applying (2) yields

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{-\mu\theta I_o(i)/\ell(|X_i|)} \mid \Phi) &= \prod_{j\in\mathbb{N}\setminus\{i\}} \mathbb{E}(\mathrm{e}^{-\mu\theta F_j \ell(|X_j|)/\ell(|X_i|)} \mid \Phi) \\
&= \prod_{j\in\mathbb{N}\setminus\{i\}} \biggl(1 + \theta\biggl|\frac{X_i}{X_j}\biggr|^{2\beta}\biggr)^{-1}, \qquad (6)
\end{aligned}$$

where the Laplace transform $\mathcal{L}_F(s) = \mu/(\mu + s)$ of F_j and $\ell(r) = \alpha r^{-2\beta}$, $r > 0$, are applied in the second equality. Hence, applying (5) and (6) to (4), we obtain (3).

3. Ginibre point process

In this section we give the definition of the Ginibre point process and make a brief review of its useful properties (see, e.g. [13], [18], and [19] for details). The Ginibre point process is one of the determinantal point processes on the complex plane $\mathbb{C}$ defined as follows. Let Φ denote a simple point process on $\mathbb{C}$, and let $\rho_n : \mathbb{C}^n \to \mathbb{R}_+$, $n \in \mathbb{N}$, denote its correlation functions (joint intensities) with respect to some Radon measure ν on $\mathbb{C}$; that is, for any disjoint $C_1, C_2, \dots, C_r \in \mathcal{B}(\mathbb{C})$ and nonnegative integers $k_1, k_2, \dots, k_r$ such that $\sum_{i=1}^r k_i = n$,

$$\mathbb{E}\biggl(\frac{\Phi(C_1)!}{(\Phi(C_1)! - k_1)!}\frac{\Phi(C_2)!}{(\Phi(C_2) - k_2)!}\cdots\frac{\Phi(C_r)!}{(\Phi(C_r) - k_r)!}\biggr)$$
$$= \int_{C_1^{k_1}\times C_2^{k_2}\times\cdots\times C_r^{k_r}} \rho_n(z_1, z_2, \dots, z_n)\nu(\mathrm{d}z_1)\nu(\mathrm{d}z_2)\cdots\nu(\mathrm{d}z_n),$$

where $\Phi(C)$ represents the number of points of Φ that fall in $C \in \mathcal{B}(\mathbb{C})$. The point process Φ is said to be a determinantal point process with kernel $K : \mathbb{C}^2 \to \mathbb{C}$ with respect to ν if ρ_n, $n \in \mathbb{N}$, satisfy

$$\rho_n(z_1, z_2, \dots, z_n) = \det(K(z_i, z_j))_{1\le i,j\le n}, \qquad z_1, z_2, \dots, z_n \in \mathbb{C},\ n \in \mathbb{N}. \tag{7}$$

Determinantal point processes are negatively correlated in the sense that

$$\rho_{n+m}(z_1, z_2, \dots, z_{n+m}) \le \rho_n(z_1, z_2, \dots, z_n)\rho_m(z_{n+1}, z_{n+2}, \dots, z_{n+m})$$

for $z_1, z_2, \dots, z_{n+m} \in \mathbb{C}$.

A determinantal point process Φ is said to be a Ginibre point process when the kernel K in (7) is given by $K(z, w) = \mathrm{e}^{z\bar{w}}$, $z, w \in \mathbb{C}$, with respect to the Gaussian measure $\nu(\mathrm{d}z) = \pi^{-1}\mathrm{e}^{-|z|^2}m(\mathrm{d}z)$, where $\bar{w}$ denotes the complex conjugate of $w \in \mathbb{C}$ and m denotes the Lebesgue measure on $\mathbb{C}$. This choice of pair of K and ν is not unique. Indeed, the determinantal point process associated with the kernel $\widetilde{K}(z, w) = \pi^{-1}\mathrm{e}^{-(|z|^2+|w|^2)/2}\mathrm{e}^{z\bar{w}}$ with respect to $\tilde{\nu}(\mathrm{d}z) = m(\mathrm{d}z)$ coincides with the Ginibre point process. From this expression, it is easy to see that $\tilde{\rho}_n(z_1, z_2, \dots, z_n) = \det(\widetilde{K}(z_i, z_j))_{1\le i,j\le n}$ is motion invariant, or, equivalently, that the Ginibre point process is motion invariant. One of its useful properties comes from the radial symmetry and is described as follows (see, e.g. [13, Section 4.7] or [15]).

Proposition 1. ([15].) *Let X_i, $i \in \mathbb{N}$, denote the points of the Ginibre point process. Then the set $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$, where Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang distribution with unit-rate parameter, denoted by $Y_i \sim$* Gamma$(i, 1)$, $i \in \mathbb{N}$.

By the definition of the Ginibre point process, we see that $\mathbb{E}\Phi(C) = \pi^{-1}m(C)$ for $C \in \mathcal{B}(\mathbb{C})$, that is, the (first-order) intensity is equal to π^{-1} with respect to the Lebesgue measure. To make it possible to control the intensity, we consider a scaled version Φ_c of the Ginibre point process with scaling parameter $c > 0$ and kernel $K_c(z, w) = \mathrm{e}^{cz\bar{w}}$ with respect to the reference measure $\nu_c(\mathrm{d}z) = (c/\pi)\mathrm{e}^{-c|z|^2}m(\mathrm{d}z)$, or, equivalently, $\widetilde{K}_c(z, w) = (c/\pi)\mathrm{e}^{-c(|z|^2+|w|^2)/2}\mathrm{e}^{cz\bar{w}}$ with respect to the Lebesgue measure. The scaled Ginibre point process Φ_c has intensity c/π and, for the points X_i, $i \in \mathbb{N}$, of Φ_c, the set $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$, where Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang distribution with rate parameter c, denoted by $Y_i \sim \mathrm{Gamma}(i, c)$, $i \in \mathbb{N}$. Note here that $Y_i \sim \mathrm{Gamma}(i, c)$ has mean $\mathbb{E}Y_i = i/c$, $i \in \mathbb{N}$.

4. Performance analysis

We adopt the scaled Ginibre point process Φ_c given in Section 3 as the configuration of the base stations in the cellular network model described in Section 2, where a point $z = x + \mathrm{i}y \in \mathbb{C}$ is identified as $(x, y) \in \mathbb{R}^2$.

4.1. Integral representation of the coverage probability

Theorem 1. *Consider the cellular network model described in Section 2 with the base stations deployed according to the scaled Ginibre point process Φ_c defined in Section 3. Then the coverage probability of a typical user is given by*

$$p(\theta, \beta) = \int_0^\infty \mathrm{e}^{-v} \mathcal{L}_W\biggl(\frac{\mu\theta}{\alpha}\Bigl(\frac{v}{c}\Bigr)^\beta\biggr) M(v, \theta, \beta) S(v, \theta, \beta)\, \mathrm{d}v, \tag{8}$$

where

$$M(v, \theta, \beta) = \prod_{j=0}^\infty \frac{1}{j!} \int_v^\infty \frac{s^j \mathrm{e}^{-s}}{1 + \theta (v/s)^\beta}\, \mathrm{d}s, \tag{9}$$

$$S(v, \theta, \beta) = \sum_{i=0}^\infty v^i \biggl(\int_v^\infty \frac{s^i \mathrm{e}^{-s}}{1 + \theta (v/s)^\beta}\, \mathrm{d}s\biggr)^{-1}. \tag{10}$$

Note that the coverage probability $p(\theta, \beta)$ given in (8)–(10) is not of closed form but computable by numerical integration given the Laplace transform $\mathcal{L}_W$ of W_o.

Proof. Let $Y_j \sim \operatorname{Gamma}(j, c)$, $j \in \mathbb{N}$, be mutually independent. For the points X_i, $i \in \mathbb{N}$, of the point process Φ_c, $\{|X_i|\}_{i\in\mathbb{N}}$ has the same distribution as $\{\sqrt{Y_i}\}_{i\in\mathbb{N}}$ by the arguments in the preceding section. Thus, from (3) and the conditional independence of $\mathbf{1}_{\{Y_j \ge Y_i\}}$, $j \in \mathbb{N} \setminus \{i\}$, given Y_i, we have

$$\begin{aligned} p(\theta, \beta) &= \sum_{i=1}^\infty \mathbb{E}\biggl(\mathcal{L}_W\biggl(\frac{\mu\theta {Y_i}^\beta}{\alpha}\biggr) \prod_{j\in\mathbb{N}\setminus\{i\}} \biggl(1 + \theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\biggr)^{-1} \mathbf{1}_{\{Y_j \ge Y_i\}}\biggr) \\ &= \sum_{i=1}^\infty \int_0^\infty \frac{c^i u^{i-1} \mathrm{e}^{-cu}}{(i-1)!} \mathcal{L}_W\biggl(\frac{\mu\theta u^\beta}{\alpha}\biggr) \\ &\quad \times \prod_{j\in\mathbb{N}\setminus\{i\}} \int_u^\infty \frac{c^j y^{j-1} \mathrm{e}^{-cy}}{(j-1)!} \biggl(1 + \theta\Bigl(\frac{u}{y}\Bigr)^\beta\biggr)^{-1} \mathrm{d}y\, \mathrm{d}u, \end{aligned} \tag{11}$$

where the second equality follows from applying the density functions of Y_j, $j \in \mathbb{N}$. Hence, by the change of variables $s = cy$ and $v = cu$, we obtain (8) after some manipulations.

Remark 1. We see from (8) that in the interference-limited (or noise-free) case ($W_o \equiv 0$) the coverage probability $p(\theta, \beta)$ does not depend on the parameters c, α, and μ. This is also the case when the base stations are deployed according to a homogeneous Poisson point process. In this case, following Theorem 2 of [1] (or applying the density function of the distance to the nearest point from the origin and then the Laplace functional of the Poisson point process to (3)), the coverage probability is given by

$$p^{(\mathrm{Poi})}(\theta, \beta) = \int_0^\infty \mathcal{L}_W\biggl(\frac{\mu\theta}{\alpha}\Bigl(\frac{v}{\pi\lambda}\Bigr)^\beta\biggr) \exp\{-v(1 + \rho(\theta, \beta))\}\, \mathrm{d}v, \tag{12}$$

where $\lambda > 0$ denotes the intensity of the Poisson point process and

$$\rho(\theta, \beta) = \frac{\theta^{1/\beta}}{\beta} \int_{1/\theta}^{\infty} \frac{u^{-1+1/\beta}}{u+1}\, \mathrm{d}u. \tag{13}$$

Remark 2. As in [1], it is not difficult to generalize the distribution of fading from the interfering base stations while retaining the Rayleigh fading from the associated base station. Provided that $|X_i| \le |X_j|$ for all $j \in \mathbb{N}$, we assume that F_i is still exponentially distributed with mean μ^{-1}, but $F_j = \mu^{-1} G_j$ for $j \in \mathbb{N} \setminus \{i\}$, where the G_j are mutually independent and identically distributed nonnegative random variables with unit mean, independent of $\Phi_c = \{X_i\}_{i\in\mathbb{N}}$, F_i, and W_0. Let $\mathcal{L}_G$ denote the Laplace transform of G_j. Then the coverage probability is obtained similarly as in Theorem 1, replacing $(1+\theta(v/s)^{\beta})^{-1}$ in (9) and (10) with $\mathcal{L}_G(\theta(v/s)^{\beta})$. In this case, the coverage probability is still computable whenever $\mathcal{L}_G$ is (e.g. the probability density function of G_j is available).

Remark 3. Besides the coverage probability, Andrews *et al.* [1] evaluated the mean achievable rate $\tau(\beta) = \mathbb{E} \ln(1+\mathsf{SINR}_o)$ of a typical user, which follows from Shannon's channel capacity $B \log_2(1 + \mathsf{SNR})$ with bandwidth B and signal-to-noise ratio SNR. We also derive the numerically computable representation for the mean achievable rate from Theorem 1. Since $\ln(1 + \mathsf{SINR}_o) > 0$ a.s.,

$$\begin{aligned} \tau(\beta) &= \int_0^{\infty} \mathbb{P}(\ln(1+\mathsf{SINR}_o) > t)\, \mathrm{d}t \\ &= \int_0^{\infty} \mathbb{P}(\mathsf{SINR}_o > \mathrm{e}^t - 1)\, \mathrm{d}t \\ &= \int_0^{\infty} p(\mathrm{e}^t - 1, \beta)\, \mathrm{d}t. \end{aligned}$$

4.2. Frequency reuse

Frequency reuse is one of the ways to increase the coverage probability by reducing the number of interfering base stations. In this section we follow [1] and consider the per-cell random frequency reuse technique. The reuse factor $\delta \in \mathbb{N}$ determines the number of different frequency bands used by the network, that is, the total frequency band is divided into δ subbands and each base station chooses one of the δ subbands uniformly at random for the use of its own cell. The interfering base stations for the typical user are then those using the same frequency band as his/her associated base station. Let R_i denote the frequency band of station i, where R_i, $i \in \mathbb{N}$, are mutually independent and distributed as $\mathbb{P}(R_i = k) = 1/\delta$, $k = 1, 2, \ldots, \delta$, and also independent of $\Phi_F = \{(X_i, F_i)\}_{i\in\mathbb{N}}$ and W_o. Noting that the noise power scales with the bandwidth, the SINR of a typical user from the associated base station is given by

$$\mathsf{SINR}_o^{(\mathrm{FR})} = \frac{F_{B_o} \ell(|X_{B_o}|)}{W_o/\delta + I_o^{(FR)}(B_o)},$$

where

$$I_o^{(\mathrm{FR})}(i) = \sum_{j\in\mathbb{N}\setminus\{i\}} F_j \ell(|X_j|) \mathbf{1}_{\{R_j = R_i\}}.$$

The coverage probability then reduces to $p(\theta, \beta, \delta) = \mathbb{P}(\mathsf{SINR}_o^{(\mathrm{FR})} > \theta)$.

https://doi.org/10.1239/aap/1409319562 Published online by Cambridge University Press

Corollary 1. *Consider the cellular network model given in Theorem 1, but apply the random frequency reuse such that δ frequency bands are randomly allocated to the cells. The coverage probability is then given by*

$$p(\theta,\beta,\delta) = \int_0^\infty \mathrm{e}^{-v}\mathcal{L}_W\biggl(\frac{\mu\theta}{\delta\alpha}\Bigl(\frac{v}{c}\Bigr)^\beta\biggr) M(v,\theta,\beta,\delta)S(v,\theta,\beta,\delta)\,\mathrm{d}v, \tag{14}$$

where

$$M(v,\theta,\beta,\delta) = \prod_{j=0}^\infty \frac{1}{j!}\int_v^\infty s^j \mathrm{e}^{-s}\biggl\{1-\frac{1}{\delta}\biggl[1-\biggl(1+\theta\Bigl(\frac{v}{s}\Bigr)^\beta\biggr)^{-1}\biggr]\biggr\}\,\mathrm{d}s,$$

$$S(v,\theta,\beta,\delta) = \sum_{i=0}^\infty v^i\biggl(\int_v^\infty s^i \mathrm{e}^{-s}\biggl\{1-\frac{1}{\delta}\biggl[1-\biggl(1+\theta\Bigl(\frac{v}{s}\Bigr)^\beta\biggr)^{-1}\biggr]\biggr\}\,\mathrm{d}s\biggr)^{-1}.$$

Proof. In this case, (3) reduces to

$$p(\theta,\beta,\delta) = \mathbb{E}\biggl(\mathcal{L}_W\biggl(\frac{\mu\theta|X_{B_0}|^{2\beta}}{\delta\alpha}\biggr)\prod_{j\in\mathbb{N}\setminus\{B_0\}}\biggl\{1-\frac{1}{\delta}\biggl[1-\biggl(1+\theta\biggl|\frac{X_{B_0}}{X_j}\biggr|^{2\beta}\biggr)^{-1}\biggr]\biggr\}\biggr). \tag{15}$$

The remaining procedures are the same as those for Theorem 1 and are omitted.

Remark 4. It is clear from (15) that the coverage probability is increasing in $\delta = 1, 2, \ldots$ for the general stationary point process $\Phi = \{X_i\}_{i\in\mathbb{N}}$. Since the frequency band is divided by δ, the mean achievable rate considered in Remark 3 is now given by

$$\tau(\beta,\delta) = \frac{1}{\delta}\mathbb{E}\ln(1+\mathsf{SINR}_o^{(\mathrm{FR})}) = \frac{1}{\delta}\int_0^\infty p(\mathrm{e}^t-1,\beta,\delta)\,\mathrm{d}t.$$

5. Asymptotic analysis in the interference-limited case

By Theorem 1 we can evaluate the coverage probability $p(\theta,\beta)$ numerically. In this section we investigate its asymptotic property as $\theta\to\infty$ in the interference-limited case.

Theorem 2. *In the interference-limited case, the coverage probability derived in Theorem 1 satisfies*

$$\lim_{\theta\to\infty}\theta^{1/\beta}p(\theta,\beta) = \int_0^\infty \prod_{j=1}^\infty \frac{1}{j!}\int_0^\infty \frac{y^j\mathrm{e}^{-y}}{1+(v/y)^\beta}\,\mathrm{d}y\,\mathrm{d}v. \tag{16}$$

The right-hand side of (16) is finite and also computable by numerical integration.

Proof. In the interference-limited case, since $\mathcal{L}_W(\cdot)=1$, (11) reduces to

$$\begin{aligned}
p(\theta,\beta) &= \sum_{i=1}^\infty \mathbb{E}\biggl(\prod_{j\in\mathbb{N}\setminus\{i\}}\biggl(1+\theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\biggr)^{-1}\mathbf{1}_{\{Y_j\ge Y_i\}}\biggr) \\
&= \mathbb{E}\biggl(\prod_{j=2}^\infty\biggl(1+\theta\Bigl(\frac{Y_1}{Y_j}\Bigr)^\beta\biggr)^{-1}\mathbf{1}_{\{Y_j\ge Y_1\}}\biggr) \\
&\quad + \sum_{i=2}^\infty \mathbb{E}\biggl(\prod_{j\in\mathbb{N}\setminus\{i\}}\biggl(1+\theta\Bigl(\frac{Y_i}{Y_j}\Bigr)^\beta\biggr)^{-1}\mathbf{1}_{\{Y_j\ge Y_i\}}\biggr),
\end{aligned} \tag{17}$$

https://doi.org/10.1239/aap/1409319562 Published online by Cambridge University Press

where $Y_j \sim \text{Gamma}(j, 1)$, $j \in \mathbb{N}$, are mutually independent. We now evaluate the two terms on the right-hand side of (17) separately. First, since Y_1 is exponentially distributed with the unit mean,

$$\begin{aligned}\mathbb{E}\biggl(\prod_{j=2}^{\infty}\biggl(1+\theta\biggl(\frac{Y_1}{Y_j}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_j\ge Y_1\}}\biggr) &= \int_0^\infty \mathrm{e}^{-u}\prod_{j=2}^{\infty}\mathbb{E}\biggl(\biggl(1+\theta\biggl(\frac{u}{Y_j}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_j\ge u\}}\biggr)\,\mathrm{d}u \\ &= \theta^{-1/\beta}\int_0^\infty \mathrm{e}^{-\theta^{-1/\beta}v}\prod_{j=2}^{\infty}\mathbb{E}\biggl(\biggl(1+\biggl(\frac{v}{Y_j}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_j\ge\theta^{-1/\beta}v\}}\biggr)\,\mathrm{d}v, \qquad (18)\end{aligned}$$

where the second equality follows from changing the variable to $v = \theta^{1/\beta}u$. The right-hand side of (18) multiplied by $\theta^{1/\beta}$ converges to that of (16) as $\theta \to \infty$ by the monotone convergence theorem with $\mathrm{e}^{-\theta^{-1/\beta}v} \uparrow 1$ and $\mathbf{1}_{\{Y_j\ge\theta^{-1/\beta}v\}} \uparrow 1$, and by applying the density functions of $Y_j \sim \text{Gamma}(j, 1)$, $j = 2, 3, \ldots$.

It remains to show that the second term on the right-hand side of (17) is $o(\theta^{-1/\beta})$ as $\theta \to \infty$. Since $(1+\theta(Y_i/Y_j)^{\beta})^{-1} \le 1$,

$$\sum_{i=2}^{\infty}\mathbb{E}\biggl(\prod_{j\in\mathbb{N}\setminus\{i\}}\biggl(1+\theta\biggl(\frac{Y_i}{Y_j}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_j\ge Y_i\}}\biggr) \le \sum_{i=2}^{\infty}\mathbb{E}\biggl(\biggl(1+\theta\biggl(\frac{Y_i}{Y_1}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_1\ge Y_i\}}\biggr). \qquad (19)$$

Applying the density functions of $Y_1 \sim \text{Gamma}(1, 1)$ and $Y_i \sim \text{Gamma}(i, 1)$, $i = 2, 3, \ldots$, to the summand of (19), we have

$$\begin{aligned}\mathbb{E}\biggl(\biggl(1+\theta\biggl(\frac{Y_i}{Y_1}\biggr)^{\beta}\biggr)^{-1}\mathbf{1}_{\{Y_1\ge Y_i\}}\biggr) &= \int_0^\infty \frac{u^{i-1}\mathrm{e}^{-u}}{(i-1)!}\int_u^\infty \mathrm{e}^{-y}\biggl(1+\theta\biggl(\frac{u}{y}\biggr)^{\beta}\biggr)^{-1}\mathrm{d}y\,\mathrm{d}u \\ &= \frac{1}{(i-1)!}\int_1^\infty\biggl(1+\frac{\theta}{s^\beta}\biggr)^{-1}\int_0^\infty u^i\mathrm{e}^{-(s+1)u}\,\mathrm{d}u\,\mathrm{d}s \\ &= i\int_1^\infty \frac{1}{(s+1)^{i+1}}\biggl(1+\frac{\theta}{s^\beta}\biggr)^{-1}\mathrm{d}s, \qquad (20)\end{aligned}$$

where the second equality follows from changing the variable to $s = y/u$ and the third equality holds by the definition of gamma functions. Here, letting $\beta^* = \lfloor \beta + 1\rfloor$ and summing the right-hand side of (20) over $i = \beta^*, \beta^*+1, \ldots$, we have

$$\begin{aligned}\sum_{i=\beta^*}^{\infty} i\int_1^\infty \frac{1}{(s+1)^{i+1}}\biggl(1+\frac{\theta}{s^\beta}\biggr)^{-1}\mathrm{d}s &\le \theta^{-1}\sum_{i=\beta^*}^{\infty} i\int_1^\infty \frac{s^\beta}{(s+1)^{i+1}}\,\mathrm{d}s \\ &= \theta^{-1}\int_1^\infty \frac{(\beta^* s+1)s^{\beta-2}}{(s+1)^{\beta^*}}\,\mathrm{d}s,\end{aligned}$$

where we have used $(1+\theta/s^\beta)^{-1} \le s^\beta/\theta$ in the inequality. The last integrand is $O(s^{-\beta^*-1+\beta})$ as $s \to \infty$, so the integral is finite, that is, the last expression is $O(\theta^{-1})$ as $\theta \to \infty$. On the other hand, for $i = 2, 3, \ldots, \beta^*-1 \le \beta$, the right-hand side of (20) satisfies

$$\begin{aligned}& i\int_1^\infty \frac{1}{(s+1)^{i+1}}\biggl(1+\frac{\theta}{s^\beta}\biggr)^{-1}\mathrm{d}s \\ &\quad \le i\int_1^\infty s^{-i-1}\biggl(1+\frac{\theta}{s^\beta}\biggr)^{-1}\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \frac{i\theta^{-i/\beta}}{\beta} \int_{1/\theta}^{\infty} \frac{t^{-i/\beta}}{t+1}\,\mathrm{d}t \\
&\le \frac{i\theta^{-i/\beta}}{\beta}\bigg[\int_{1/\theta}^{1} t^{-i/\beta}\,\mathrm{d}t + \int_{1}^{\infty} t^{-i/\beta-1}\,\mathrm{d}t\bigg] \\
&= \begin{cases} \dfrac{\beta}{\beta-i}\theta^{-i/\beta} - \dfrac{i}{\beta-i}\theta^{-1} \le \dfrac{\beta}{\beta-i}\theta^{-i/\beta} & \text{for } i < \beta, \\ \theta^{-1}(\ln\theta + 1) & \text{for } i = \beta, \end{cases}
\end{aligned}
$$

where the first equality follows from changing the variable to $t = s^{\beta}/\theta$ and the next inequality follows from $1/(t+1) \le \min(1, 1/t)$. The last expressions are $o(\theta^{-1/\beta})$ as $\theta \to \infty$ for both $i < \beta$ and $i = \beta$, which completes the proof.

Remark 5. Theorem 2 states that in the interference-limited case the distribution of the SINR of a typical user has the tail of a Pareto distribution with infinite mean. This result is also the case when the base stations are deployed according to a homogeneous Poisson point process. In the interference-limited case of the Poisson base station model, (12) reduces to

$$p^{(\mathrm{Poi})}(\theta, \beta) = \frac{1}{1 + \rho(\theta, \beta)}.$$

Here we have, from (13),

$$\theta^{-1/\beta}\rho(\theta, \beta) = \frac{1}{\beta}\int_{1/\theta}^{\infty} \frac{u^{-1+1/\beta}}{u+1}\,\mathrm{d}u \to \frac{\pi}{\beta}\csc\frac{\pi}{\beta} \qquad \text{as } \theta \to \infty,$$

which implies that

$$\lim_{\theta\to\infty} \theta^{1/\beta} p^{(\mathrm{Poi})}(\theta, \beta) = \frac{\beta}{\pi}\sin\frac{\pi}{\beta}. \tag{21}$$

Remark 6. The Pareto tail asymptotics of the SINR with infinite mean are due to the unboundedness of the path-loss function at the origin. We can show that, when the path-loss function is bounded, the coverage probability decays faster than any polynomial. Now suppose that $\ell(r) \le \alpha,\ r > 0$, for a constant $\alpha > 0$. Then the first line of (17) reduces to

$$
\begin{aligned}
\mathbb{P}(\mathsf{SINR}_o > \theta) &= \mathbb{E}\bigg(\prod_{j\in\mathbb{N}\setminus\{B_o\}} \bigg(1 + \theta\frac{\ell(\sqrt{Y_j})}{\ell(\sqrt{Y_{B_o}})}\bigg)^{-1}\bigg) \\
&= (1+\theta)\mathbb{E}\bigg(\prod_{j=1}^{\infty} \bigg(1 + \theta\frac{\ell(\sqrt{Y_j})}{\ell(\sqrt{Y_{B_o}})}\bigg)^{-1}\bigg).
\end{aligned}
$$

Since $(1+a\theta)^{-1} \le 1$ and $\theta/(1+a\theta)$ is increasing in θ for $a > 0$, by the monotone convergence theorem we see that, for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\limsup_{\theta\to\infty} \theta^{k-1}\mathbb{P}(\mathsf{SINR}_o > \theta) &\le \limsup_{\theta\to\infty} \frac{1+\theta}{\theta}\mathbb{E}\bigg(\prod_{j=1}^{k} \theta\bigg(1 + \theta\frac{\ell(\sqrt{Y_j})}{\ell(\sqrt{Y_{B_o}})}\bigg)^{-1}\bigg) \\
&= \mathbb{E}\bigg(\prod_{j=1}^{k} \frac{\ell(\sqrt{Y_{B_o}})}{\ell(\sqrt{Y_j})}\bigg) \\
&\le \alpha^k \prod_{j=1}^{k} \mathbb{E}\bigg(\frac{1}{\ell(\sqrt{Y_j})}\bigg).
\end{aligned}
$$

https://doi.org/10.1239/aap/1409319562 Published online by Cambridge University Press

Hence, if $\mathbb{E}(\ell(\sqrt{Y_j})^{-1}) < \infty$ for each $j \in \mathbb{N}$, the tail of the SINR distribution decays faster than any polynomial. This condition is satisfied, for example, by $\ell(r) = \alpha \max\{1, r\}^{-2\beta}$ and $\ell(r) = \alpha(1+r)^{-2\beta}$, $r > 0$.

6. Numerical experiments

In this section we present the results of some numerical experiments for computing the coverage probabilities. We first compare the results of computing (8) for the Ginibre base station model with those of (12) for the corresponding Poisson model. In Figure 2 we plot the coverage probability for a given value of θ, where the intensity $\lambda = c/\pi$ is set at $1/\pi$ for both point processes and the thermal noise is given as a constant such that $\mathsf{SNR} = (\mu W_o)^{-1}$ ($\mathsf{SNR} = \infty$ stands for no noise). The coefficient α of the path-loss function is set at 1, and the cases $\beta = 1.25$ and $\beta = 2.0$ (that is, $\ell(r) = r^{-2.5}$ and $\ell(r) = r^{-4}$) are computed. For both the Ginibre and Poisson models, the gaps between the $\mathsf{SNR} = 10$ and $\mathsf{SNR} = \infty$ cases are small, particularly for small values of β, which implies that the thermal noise is not a very important consideration. Furthermore, comparing Figure 2 with Figure 4 of [1], we find that

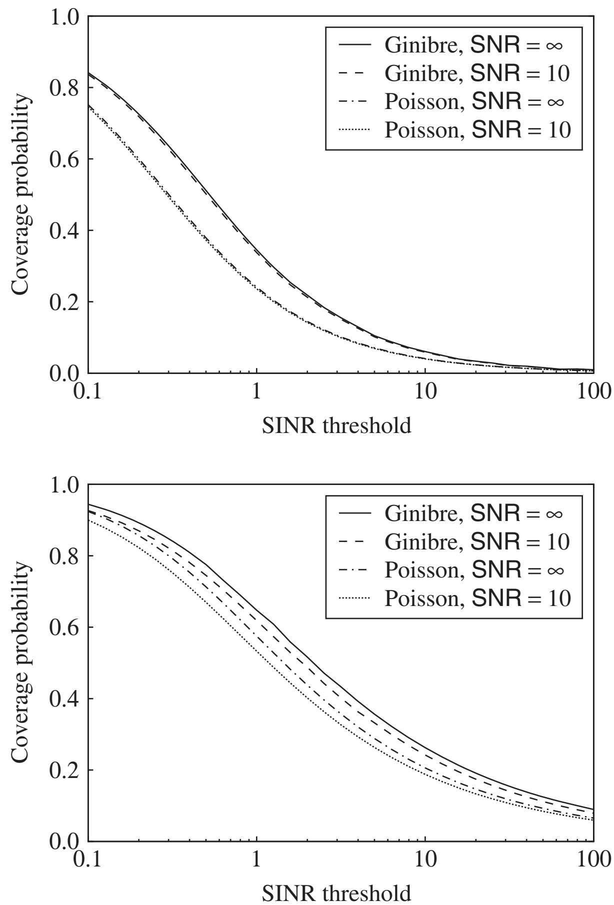

FIGURE 2: Comparison of the coverage probability between the Ginibre base station model and the corresponding Poisson model for $\beta = 1.25$ (*top*) and $\beta = 2.0$ (*bottom*).

the coverage probability for the Ginibre model is very close to that for the corresponding model with a real base station deployment by a major service provider in a relatively flat urban area. This confirms that the Ginibre base station model is a very good approximate model for real cellular networks.

Next, to see the effect of frequency reuse on the coverage probability, the results of computing $p(\theta, \beta, \delta)$ via (14) are exhibited in Figure 3, where the curves for $\delta = 1$ are the same as those for the Ginibre model with $\mathsf{SNR} = \infty$ in Figure 2. We see that the coverage probability is much improved by the frequency reuse.

In the third and final experiment, we compared the coverage probability with the corresponding asymptotics in the interference-limited case. In Figure 4, the comparison results between (8) and (16) for the Ginibre model as well as those between (12) and (21) for the Poisson model are exhibited, where the curves for the coverage probability are the same as those for $\mathsf{SNR} = \infty$ in Figure 2. Figure 4 shows that in the Poisson model the asymptotic results agree well with the coverage probability for relatively small values of θ and β. In the Ginibre model, however, the asymptotic results agree with the coverage probability only for large values of θ, particularly

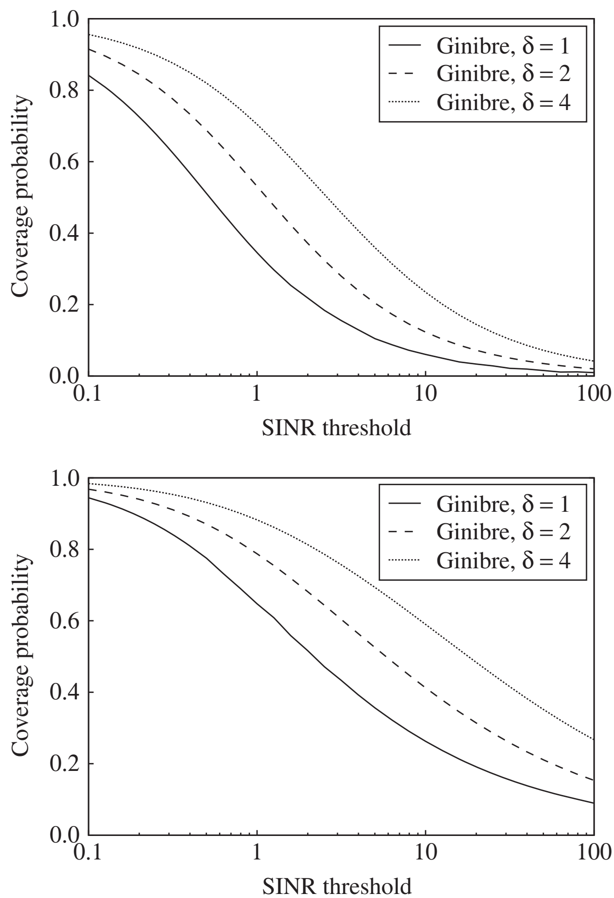

FIGURE 3: Effect of frequency reuse on the coverage probability for $\beta = 1.25$ (*top*) and $\beta = 2.0$ (*bottom*).

https://doi.org/10.1239/aap/1409319562 Published online by Cambridge University Press

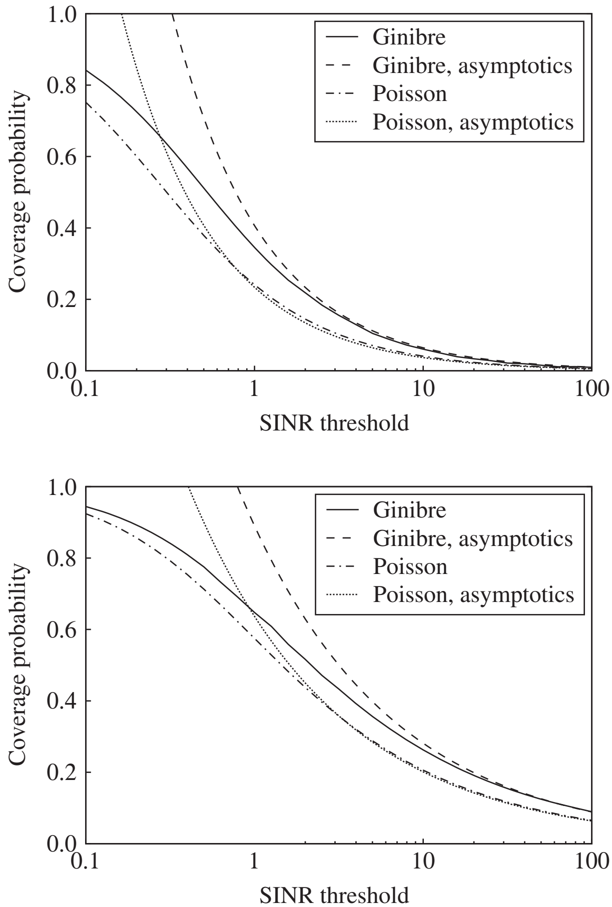

FIGURE 4: Comparison between the coverage probability and its asymptotic results for $\beta = 1.25$ (*top*) and $\beta = 2.0$ (*bottom*).

when the value of β is large. This implies that, to obtain a better approximation of the coverage probability in the Ginibre model, it is necessary to take not only the main term obtained by the asymptotic analysis but also some more terms of $o(\theta^{-1/\beta})$ as $\theta \to \infty$ into consideration.

7. Concluding remarks

We have considered a cellular network model such that the base stations are deployed according to the Ginibre point process and have derived a computable integral representation for the coverage probability. We have also investigated the asymptotic property of the coverage probability in the interference-limited case.

Possible avenues for future work are as follows. Although we have studied just a basic model, we could apply the Ginibre base station model to more practical problems, such as those developed in [6], [7], [8], [14], and [17]. Also, we could consider applications to other wireless networks, where the percolation of SINR graphs might be attractive. As extensions of the model, we could consider more general stationary point processes. Since Proposition 1 follows from the radially symmetric property of the Ginibre point process, the results obtained in the paper might be generalized to models with other radially symmetric determinantal point processes. It might also be interesting to give a general condition that the coverage probability has decay rate $\theta^{-1/\beta}$ as $\theta \to \infty$.

Acknowledgements

The authors are grateful to the anonymous referee for his/her valuable comments and suggestions. The first author wishes to thank François Baccelli for directing his interest to this research field. The first author's work was supported by the JSPS (Japan Society for the Promotion of Science) Grant-in-Aid for Scientific Research (C) 25330023. The second author's work was supported by the JSPS Grant-in-Aid for Scientific Research (B) 22340020.

References

[1] Andrews, J. G., Baccelli, F. and Ganti, R. K. (2011). A tractable approach to coverage and rate in cellular networks. *IEEE Trans. Commun.* **59,** 3122–3134.

[2] Andrews *et al.* (2010). A primer on spatial modeling and analysis in wireless networks. *IEEE Commun. Magazine* **48,** 156–163.

[3] Baccelli, F. and Błaszczyszyn, B. (2008). Stochastic geometry and wireless networks: Volume I Theory. *Foundations Trends Networking* **3,** 249–449.

[4] Baccelli, F. and Błaszczyszyn, B. (2009). Stochastic geometry and wireless networks: Volume II Applications. *Foundations Trends Networking* **4,** 1–312.

[5] Błaszczyszyn, B. and Yogeshwaran, D. (2010). Connectivity in sub-Poisson networks. In *Proc. 48th Annual Allerton Conf. Commun. Control Comput.*, IEEE, pp. 1466–1473.

[6] Decreusefond, L., Martins, P. and Vu, T.-T. (2010). An analytical model for evaluating outage and handover probability of cellular wireless networks. Preprint. Available at http://arxiv.org/abs/1009.0193v1.

[7] Dhillon, H. S., Ganti, R. K. and Andrews, J. G. (2012). Load-aware heterogeneous cellular networks: Modeling and SIR distribution. In *Proc. 2012 IEEE Global Communications Conf. (GLOBECOM)*, IEEE, pp. 4314–4319.

[8] Dhillon, H. S., Ganti, R. K., Baccelli, F. and Andrews, J. G. (2012). Modeling and analysis of K-tier downlink heterogeneous cellular networks. *IEEE J. Sel. Areas Commun.* **30,** 550–560.

[9] Ganti, R. K., Baccelli, F. and Andrews, J. G. (2012). Series expansion for interference in wireless networks. *IEEE Trans. Inf. Theory* **58,** 2194–2205.

[10] Giacomelli, R., Ganti, R. K. and Haenggi, M. (2011). Outage probability of general ad hoc networks in the high-reliability regime. *IEEE/ACM Trans. Networking* **19,** 1151–1163.

[11] Haenggi, M. (2013). *Stochastic Geometry for Wireless Networks*. Cambridge University Press.

[12] Haenggi, M. *et al.* (2009). Stochastic geometry and random graphs for the analysis and design of wireless networks. *IEEE J. Sel. Areas Commun.* **27,** 1029–1046.

[13] Hough, J. B., Krishnapur, M., Peres, Y. and Virág, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. American Mathematical Society, Providence, RI. Also available at http://research.microsoft.com/en-us/um/people/peres/GAF_book.pdf.

[14] Jo, H.-S., Sang, Y. J., Xia, P. and Andrews, J. G. (2012). Heterogeneous cellular networks with flexible cell association: a comprehensive downlink SINR analysis. *IEEE Trans. Wireless Commun.* **11,** 3484–3495.

[15] Kostlan, E. (1992). On the spectra of Gaussian matrices. Directions in matrix theory (Auburn, AL, 1990). *Linear Algebra Appl.* **162/164,** 385–388.

[16] Madhusudhanan, P., Restrepo, J. G., Liu, Y. and Brown, T. X. (2012). Downlink coverage analysis in a heterogeneous cellular network. In *Proc. 2012 IEEE Global Communications Conf. (GLOBECOM)*, IEEE, pp. 4170–4175.

[17] Mukherjee, S. (2012). Distribution of downlink SINR in heterogeneous cellular networks. *IEEE J. Sel. Areas Commun.* **30,** 575–585.

[18] Shirai, T. and Takahashi, Y. (2003). Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J. Funct. Analysis* **205,** 414–463.

[19] Soshnikov, A. (2000). Determinantal random point fields. *Russian Math. Surveys* **55,** 923–975.