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FLIP-FLOPS IN HYPOHAMILTONIAN GRAPHS

BY
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Throughout this note, we adopt the graph-theoretical terminology and notation of Harary [3]. A graph G is *hypohamiltonian* if G is not hamiltonian but the deletion of any point u from G results in a hamiltonian graph G-u. Gaudin, Herz, and Rossi [2] proved that the smallest hypohamiltonian graph is the Petersen graph. Using a computer for a systematic search, Herz, Duby, and Vigué [4] found that there is no hypohamiltonian graph with 11 or 12 points. However, they found one with 13 and one with 15 points. Sousselier [4] and Lindgren [5] constructed independently the same sequence of hypohamiltonian graphs with $6k+10$ points. Moreover, Sousselier found a cubic hypohamiltonian graph with 18 points. This graph and the Petersen graph were the only examples of cubic hypohamiltonian graphs until Bondy [1] constructed an infinite sequence of cubic hypohamiltonian graphs with $12k+10$ points. Bondy also proved that the Coxeter graph [6], which is cubic with 28 points, is hypohamiltonian.

We will construct a hypohamiltonian graph with p points for every $p \geq 26$ and a cubic hypohamiltonian graph with p points for every even $p \geq 42$. Actually, our method yields $f(p)$ hypohamiltonian and $g(p)$ cubic hypohamiltonian graphs with p points where $f(p) \to \infty$, $g(2p) \to \infty$. We also obtain new hypohamiltonian graphs with 21, 23, and 24 points and cubic ones with 26, 34, 36, and 38 points. Thus the question of existence of a hypohamiltonian graph with p points is left open only for

$$p = 14, 17, 19, 20, 25;$$

the existence of a cubic hypohamiltonian graph with p points remains uncertain only for

$$p = 14, 16, 20, 24, 30, 32, 40.$$

Definitions. Let G be graph. A pair (a, b) will be called *good* in G if G has a spanning path with endpoints a, b. Similarly, $((a, b), (c, d))$ will be called good in G if G has a spanning subgraph consisting of two disjoint paths, one of them having endpoints a, b and the other one c, d. By a *flip-flop* we mean a quintuple

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(G, a, b, c, d) where G is a graph and a, b, c, d distinct points of G such that

(i) (a, d), (b, c) and $((a, d), (b, c))$ are good in G,

(ii) none of (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$ and $((a, c), (b, d))$ is good in G,

(iii) for each point u of G, at least one of (a, c), (b, d), $((a, b), (c, d))$, $((a, c), (b, d))$ is good in G-u.

The *order* of a flip-flop is the number of points of G. A flip-flop will be called *cubic* if a, b, c, d have degree two in G and all the other points have degree three. Let $F_i=(G_i, a_i, b_i, c_i, d_i)$ be flip-flops. By (F_1, F_2) we denote the quintuple $(G_4, a_1, b_1, c_2, d_2)$ where G_4 is obtained (see Figure 1) by taking (disjoint) graphs G_1, G_2 and joining c_1 to b_2 and d_1 to a_2. By (F_1, F_2, F_3) we denote the quintuple $(G_5, a_1, b_1, c_3, d_3)$ where G_5 is obtained as follows (see Figure 2): take (pairwise disjoint) graphs G_1, G_2, G_3, add four more points u_1, v_1, u_2, v_2 and ten more lines u_1v_1, u_2v_2, c_1u_1, u_1a_2, d_1v_1, v_1b_2, c_2u_2, u_2a_3, d_2v_2, v_2b_3. Let G be a graph and a, b, c, d four distinct points of G. By the graph *based on* (G, a, b, c, d) we mean the graph H obtained (see Figure 3) from G by adding two more points u, v and five more lines uv, ua, ud, vb, and vc.

PROPOSITION 1. *Let F_1, F_2, F_3 be flip-flops. Then (F_1, F_2) and (F_1, F_2, F_3) are flip-flops. Moreover, they are cubic whenever the initial* F$_i$'s *are cubic.*

PROPOSITION 2. *Every graph based on a flip-flop is hypohamiltonian. Moreover, a graph based on a cubic flip-flop is cubic.*

PROPOSITION 3. *Let the graph based on a $(G, a\ b, c, d)$ be hypohamiltonian. Then (a, d) and (b, c) are good in G but none of (a, b), (a, c), (b, d), (c, d), $((a, b), (c, d))$; $((a, c), (b, d))$ is.*

PROPOSITION 4. *F_8, F_{11}, F_{13}, F_{14}, and F_{26} in* Figures 4–8 *are flip-flops.*

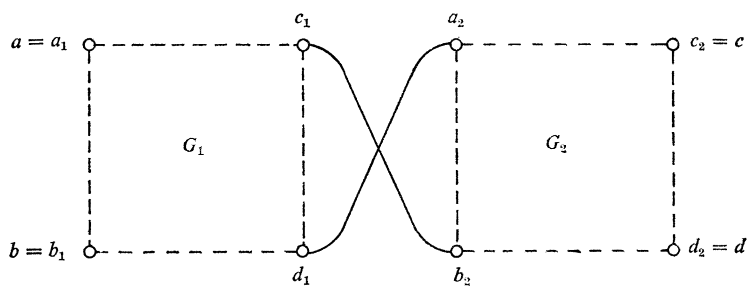

FIGURE 1

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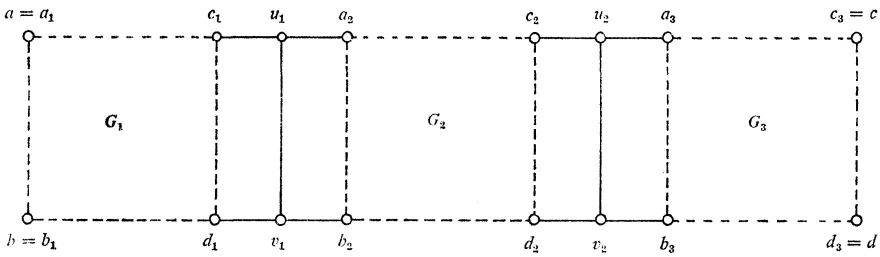

FIGURE 2

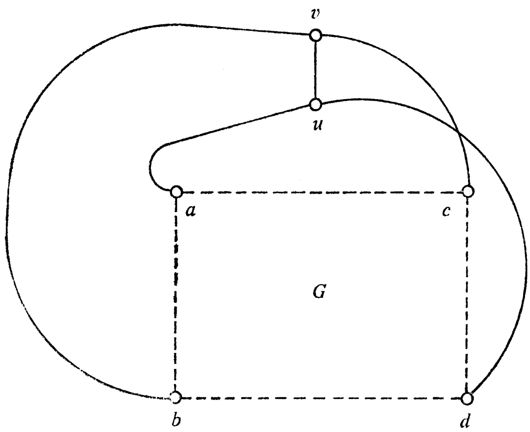

FIGURE 3

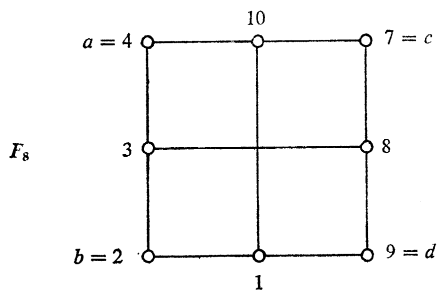

FIGURE 4

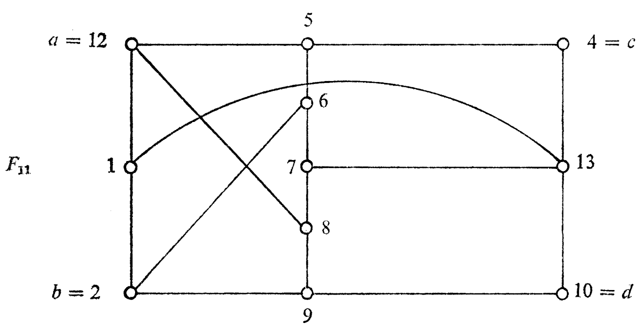

FIGURE 5

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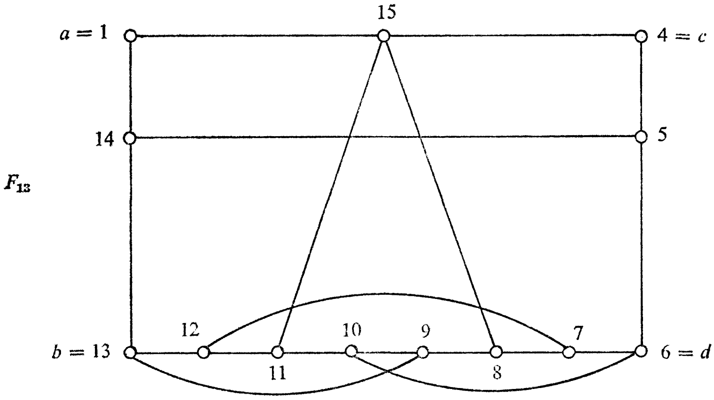

FIGURE 6

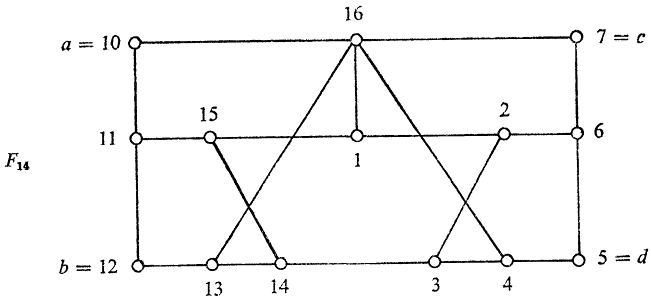

FIGURE 7

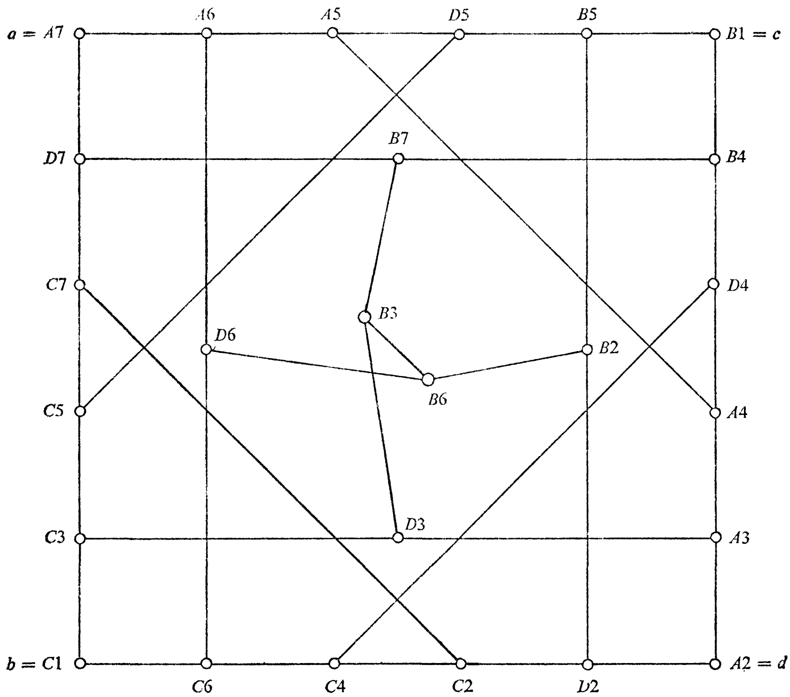

FIGURE 8

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Proof. The graph based on F_8 is the Petersen graph, the graphs based on F_{11} and F_{13} are the two hypohamiltonian graphs found by Herz, Duby, and Vigué, the graph based on F_{14} is the second term of the sequence constructed by Sousselier and Lindgren, the graph based on F_{26} is the Coxeter graph.

Actually, the labeling of points in our Figures 4–8 sets up isomorphisms with the drawings in [4] and the description in [1], [6]. In view of Proposition 3, we only have to exhibit, for each of the five F_i's, a spanning subgraph of G showing that $((a, d), (b, c))$ is good in G and also certain spanning subgraphs of G-u for each $u \in G$. The spanning subgraphs of G-u are listed only for some points $u \in G$; the others are easily obtained by symmetries (which should be obvious from Figures 4–8).

F_8: 4-3-8-9, 2-1-10-7
(u=3) 4-10-1-2, 7-8-9
(u=2) 4-3-8-9-1-10-7

F_{11}: 12-1-13-10, 2-9-8-7-6-5-4
(u=1) 12-5-6-7-8-9-2, 4-13-10
(u=5) 12-1-2-6-7-8-9-10-13-4
(u=6) 2-1-12-5-4-13-7-8-9-10
(u=7) 2-6-5-4-13-1-12-8-9-10
(u=13) 12-1-2, 4-5-6-7-8-9-10
(u=12) 2-1-13-4-5-6-7-8-9-10
(u=4) 2-1-13-7-6-5-12-8-9-10

F_{13}: 1-14-5-6, 13-12-7-8-9-10-11-15-4
(u=14) 1-15-8-7-12-11-10-9-13, 4-5-6
(u=12) 1-14-13-9-10-11-15-8-7-6-5-4
(u=11) 1-14-5-6-10-9-13-12-7-8-15-4
(u=10) 1-14-13-9-8-15-11-12-7-6-5-4
(u=15) 1-14-13-12-11-10-9-8-7-6-5-4
(u=13) 1-14-5-6-7-12-11-10-9-8-15-4
(u=1) 13-14-5-4-15-8-9-10-11-12-7-6

F_{14}: 10-11-15-14-3-4-5, 12-13-16-1-2-6-7
(u=11) 10-16-1-15-14-13-12, 7-6-2-3-4-5
(u=15) 10-11-12, 7-6-2-1-16-13-14-3-4-5
(u=13) 12-11-10-16-7-6-2-1-15-14-3-4-5
(u=14) 12-13-16-1-15-11-10, 7-6-2-3-4-5
(u=16) 10-11-15-1-2-6-7, 12-13-14-3-4-5
(u=1) 12-13-14-15-11-10-16-7-6-2-3-4-5
(u=10) 12-11-15-1-2-6-7-16-13-14-3-4-5
(u=12) 10-11-15-1-2-6-5-4-3-14-13-16-7

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F_{26}: $A7$-$A6$-$A5$-$A4$-$A3$-$D3$-$B3$-$B7$-$D7$-$C7$-$C2$-$D2$-$A2$
$C1$-$C3$-$C5$-$D5$-$B5$-$B2$-$B6$-$D6$-$C6$-$C4$-$D4$-$B4$-$B1$
$(u=D7)$ $A7$-$A6$-$D6$-$B6$-$B3$-$B7$-$B4$-$B1$
$C1$-$C6$-$C4$-$D4$-$A4$-$A5$-$D5$-$B5$-$B2$-$D2$-$C2$-$C7$-$C5$-$C3$-$D3$-$A3$-$A2$
$(u=C7)$ $A7$-$D7$-$B7$-$B4$-$D4$-$A4$-$A3$-$D3$-$B3$-$B6$-$B2$-$B5$-$B1$
$C1$-$C3$-$C5$-$D5$-$A5$-$A6$-$D6$-$C6$-$C4$-$C2$-$D2$-$A2$
$(u=D6)$ $A7$-$A6$-$A5$-$D5$-$B5$-$B1$
$C1$-$C6$-$C4$-$C2$-$D2$-$B2$-$B6$-$B3$-$D3$-$C3$-$C5$-$C7$-$D7$-$B7$-$B4$-$D4$-$A4$-$A3$-$A2$
$(u=B3)$ $A2$-$D2$-$B2$-$B6$-$D6$-$A6$-$A5$-$D5$-$B5$-$B1$
$C1$-$C6$-$C4$-$C2$-$C7$-$C5$-$C3$-$D3$-$A3$-$A4$-$D4$-$B4$-$B7$-$D7$-$A7$
$(u=A7)$ $C1$-$C3$-$C5$-$D5$-$A5$-$A6$-$D6$-$C6$-$C4$-$D4$-$A4$-$A3$-$D3$-$B3$-$B6$-$B2$-$B5$-$B1$-$B4$-$B7$-$D7$-$C7$-$C2$-$D2$-$A2$

PROPOSITION 5. *There are flip-flops of every order* $n \geq 24$. *There are cubic flip-flops of every even order* $n \geq 40$.

Proof. The flip-flops $((F_8, F_8), F_8)$, (F_{11}, F_{14}), (F_{13}, F_{13}), $((F_8, F_8), F_{11})$, (F_8, F_8, F_8), $((F_8, F_8), F_{13})$, $((F_{11}, F_{11}), F_8)$, (F_8, F_8, F_{11}) have orders 24, 25, . . . , 31. The cubic flip-flops $((((F_8, F_8), F_8), F_8), F_8)$, $((F_8, F_8), F_{26})$, $(((F_8, F_8, F_8), F_8), F_8)$, (F_8, F_8, F_{26}) have orders 40, 42, 44, 46. If F is a flip-flop of order n then (F, F_8) is a flip-flop of order $n+8$. Moreover, if F is cubic then (F, F_8) is cubic.

THEOREM. *For each positive integer* p, *let us denote by* $A(p)$, *resp.* $B(p)$, *the number of nonisomorphic hypohamiltonian*, *resp. cubic hypohamiltonian*, *graphs with* p *points. Then* $A(p)>0$ *whenever* $n \geq 26$ *and* $B(2p)>0$ *whenever* $2p \geq 42$. *Moreover* $A(p) \to \infty$ *and* $B(2p) \to \infty$.

Proof. The first assertion follows directly from Propositions 2 and 5. To prove $A(p) \to \infty$, consider graphs G based on flip-flops F where F is obtained by compositions of F_8 and F_{11} only. If exactly n flip-flops F_{11} were used to construct F then G has exactly $3n$ points of degree four (and the other ones have degree three). Therefore $A(p) \geq a(p)$ where $a(p)$ is the number of solutions of $p=8m+11n+2$ in nonnegative integers m, n. Clearly $a(p) \to \infty$ and so $A(p) \to \infty$. Similarly, let G be a cubic graph based on a flip-flop F where F is obtained by compositions of F_8 and F_{26} only (and the order of F is >16). It is easy to check that a line of G belongs to a circuit of length seven if and only if it comes from F_{26}. Therefore G determines the number of flip-flops F_{26} which have been used to construct F. Thus we have $B(p) \geq b(p)$ where $b(p)$ is the number of solutions of $p=8m+26n+2$. Since $b\,(2p) \to \infty$, we have $B(2p) \to \infty$ and the proof is finished.

We have also obtained three new hypohamiltonian graphs with less than 26 points. These are based on flip-flops (F_8, F_{11}), (F_8, F_{13}), and(F_{11}, F_{11}). Let us note

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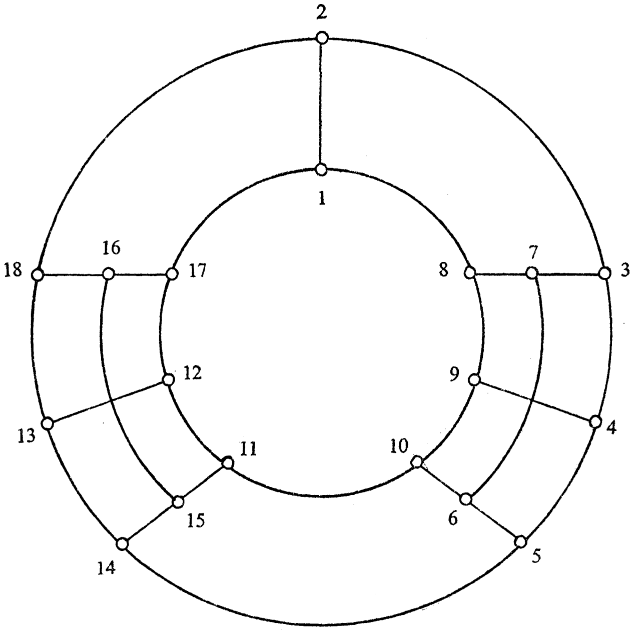

FIGURE 9

that the graph based on (F_8, F_8) is isomorphic to one found by Sousselier. Indeed, the numbering of its vertices in Figure 9 sets up an isomorphism with its drawing in [4]. Moreover, the flip-flops $((F_8, F_8), F_8)$, $(((F_8, F_8), F_8), F_8)$, (F_8, F_{26}), and $((F_8, F_8, F_8), F_8)$ give rise to new cubic hypohamiltonian graphs with less than 42 points.

We conclude with a few questions. Is there a planar hypohamiltonian graph? If so, can such a graph be cubic? Is there a hypohamiltonian graph containing a triangle? (Herz, Duby, and Vigué [4] conjecture that every hypohamiltonian graph has girth ≥ 5.) More generally, is there a graph G such that no hypohamiltonian graph has a subgraph isomorphic to G? Can a hypohamiltonian graph with p points have a point of degree $> \frac{1}{3}(p-1)$? What is the maximum number of lines among hypohamiltonian graphs with p points?

Professor J. A. Bondy noted in a conversation that, for every hypohamiltonian graph G, no two points u, v, having degree three can belong to the same triangle; a fortiori, every cubic hypohamiltonian graph has girth ≥ 4. The proof is simple: if w is the third point of a triangle containing u and v then u and v have degree two in G-w. Consequently, the hamiltonian circuit in G-w must include the line uv. But then G is hamiltonian which is a contradiction. Professor Bondy has also observed that, contrary to the impression one might get from considering the Petersen graph, there are hypohamiltonian graphs which remain hypohamiltonian after a suitable addition of a new line. For instance, adding the lines 2-12 and 5-15

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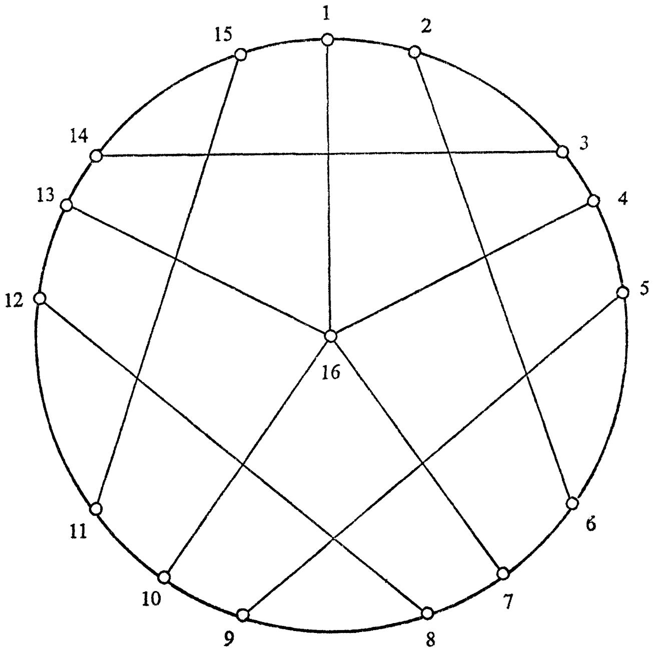

FIGURE 10

to the second graph of Sousselier's sequence (see Figure 10) one obtains a hypohamiltonian graph again. Actually, Professor Bondy made the following

CONJECTURE. If the addition of a new line to a hypohamiltonian graph with girth ≥ 5 does not create a circuit of length ≤ 4 then it does not create a hamiltonian circuit.

Finally, we conjecture that, if the deletion of a line x from a hypohamiltonian graph G does not create a point of degree two, then the line-deleted graph G-x is hypohamiltonian.

Remark Added in Proof. Professor Claude Berge kindly drew my attention to the fact that hypohamiltonian graphs were studied by Sousselier almost forty years ago (at this time, Sousselier proved, among other things, that the Petersen graph is the smallest hypohamiltonian one). See C. Berge, *Graphes et hypergraphes*, Dunod, Paris (1970), 213–217, and C. Berge, *Problèmes plaisants et délectables*, Rubrique no. 29, Revue Fr. Rech. Opérationnelle, **29** (1963), 405–406 (no. 31, 214–218).

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