

CENTRAL COHOMOLOGY OPERATIONS AND K -THEORY

IMMA GÁLVEZ-CARRILLO¹ AND SARAH WHITEHOUSE

¹*Departament de Matemàtica Aplicada III, Escola D'Enginyeria de Terrassa,
Universitat Politècnica de Catalunya, C. Colom 1, 08222 Terrassa, Spain*
(m.immaculada.galvez@upc.edu)

²*School of Mathematics and Statistics, University of Sheffield,
Sheffield S3 7RH, UK* (s.whitehouse@sheffield.ac.uk)

(Received 17 April 2012)

Abstract For stable degree 0 operations, and also for additive unstable operations of bidegree $(0, 0)$, it is known that the centre of the ring of operations for complex cobordism is isomorphic to the corresponding ring of connective complex K -theory operations. Similarly, the centre of the ring of BP operations is the corresponding ring for the Adams summand of p -local connective complex K -theory. Here we show that, in the additive unstable context, this result holds with BP replaced by $BP\langle n \rangle$ for any n . Thus, for all chromatic heights, the only central operations are those coming from K -theory.

Keywords: K -theory; operations; cobordism

2010 *Mathematics subject classification:* Primary 55S25
Secondary 55N22; 19L41

1. Introduction

We study cohomology operations for various cohomology theories related to complex cobordism and show that, in a suitable context, the central cohomology operations are precisely those coming from complex K -theory. Specifically, we consider the ring of additive unstable bidegree $(0, 0)$ operations for the Adams summand of p -local complex K -theory, and we show that this ring maps via an injective ring homomorphism to the corresponding ring of operations for the theory $BP\langle n \rangle$ for all $n \geq 1$. The image of this map is the centre of the target ring.

Previously, results of this type were established with target BP (which may be regarded as the $n = \infty$ case) in both the stable and additive unstable contexts (see [3, 5]).

The $BP\langle n \rangle$ result that we give here is quite a simple consequence of combining certain unstable BP splittings due to Wilson [7] with the results of [5]. Nonetheless, we think it is interesting since it shows that the central operations are precisely those arising from K -theory at every chromatic height.

Let p be an odd prime and let BP be the p -local Brown–Peterson spectrum, a summand of the p -local complex bordism spectrum $MU_{(p)}$. For each $n \geq 0$, there exists a connective

commutative ring spectrum $\mathrm{BP}\langle n \rangle$ with coefficient groups

$$\mathrm{BP}\langle n \rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n] = \mathrm{BP}_* / (v_{n+1}, v_{n+2}, \dots) = \mathrm{BP}_* / J_n.$$

Here, $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, where the v_i are Hazewinkel's generators, with v_i in degree $2(p^i - 1)$ and $J_n = (v_{n+1}, v_{n+2}, \dots)$. These theories were introduced by Wilson in [7] and further studied by Johnson and Wilson in [4]. They fit into a tower of BP-module spectra:

$$\mathrm{BP} \rightarrow \dots \rightarrow \mathrm{BP}\langle n \rangle \rightarrow \mathrm{BP}\langle n-1 \rangle \rightarrow \dots$$

In particular, $\mathrm{BP}\langle 0 \rangle = H\mathbb{Z}_{(p)}$ and $\mathrm{BP}\langle 1 \rangle = g$, the Adams summand of connective p -local complex K -theory.

Recall that, for a cohomology theory E , the bidegree $(0, 0)$ unstable operations are given by $E^0(\underline{E}_0)$, where \underline{E}_0 denotes the 0th space of the Ω -spectrum representing the cohomology theory E . Within $E^0(\underline{E}_0)$ we have $PE^0(\underline{E}_0)$, the additive bidegree $(0, 0)$ unstable operations, which we denote by $\mathcal{A}(E)$. This is a ring, with multiplication given by the composition of operations.

Using unstable BP splittings due to Wilson, we define an injective ring homomorphism $\hat{\iota}_n: \mathcal{A}(g) \rightarrow \mathcal{A}(\mathrm{BP}\langle n \rangle)$. Our main result, Theorem 5.3, is that the image of $\hat{\iota}_n$ is the centre of $\mathcal{A}(\mathrm{BP}\langle n \rangle)$.

The situation is analogous to that of matrix rings, where the diagonal matrices form the centre of the $n \times n$ -matrices for all n . Indeed, we see that all operations considered are determined by the matrices giving their actions on homotopy groups. Of course, not all matrices arise as actions of operations; there are complicated constraints. Essentially, we show that

- (1) at every height n , enough matrices arise that central operations are forced to act diagonally (in a suitable sense), and
- (2) the constraints on the diagonal operations that can occur are the same for all n .

This paper has the following structure. In §2 we explain some of Wilson's results on unstable BP splittings and deduce faithfulness of the actions of additive $\mathrm{BP}\langle n \rangle$ operations of bidegree $(0, 0)$ on homotopy groups. In §3 we recall some results on additive operations for the Adams summand g of connective p -local complex K -theory. We also define our map of operations $\mathcal{A}(g) \rightarrow \mathcal{A}(\mathrm{BP}\langle n \rangle)$ and give its basic properties. In §4 we define and study diagonal operations. In §5 we prove our main result, Theorem 5.3, which is that the image of the map coincides with the centre of the target ring.

2. Unstable splittings

In this section we begin by recalling some results on unstable BP splittings. These results are due to Wilson [7] (we use [2] as our main reference). We then deduce some straightforward consequences for operations.

As usual, let \underline{E}_k denote the k th space of the Ω -spectrum representing the cohomology theory E . For $n \geq 0$, write $\pi_n: \underline{BP}_0 \rightarrow \underline{BP}\langle n \rangle_0$ for the map coming from the map of BP-module spectra $BP \rightarrow BP\langle n \rangle$. The induced map on homotopy,

$$(\pi_n)_*: BP_* \rightarrow BP\langle n \rangle_* = BP_* / J_n,$$

is the canonical projection.

The following lemma is the special case of [2, Lemma 22.1] for zero spaces.

Lemma 2.1 (Boardman et al. [2, Lemma 22.1]). *For all $n \geq 0$, there exists an H -space splitting $\theta_n: \underline{BP}\langle n \rangle_0 \rightarrow \underline{BP}_0$ of π_n . Let $e_n = \theta_n \pi_n$ denote the corresponding additive idempotent BP operation; the choices can be made compatibly, so that $e_n e_m = e_m e_n = e_m$ for $m < n$.*

These splittings immediately allow us to compare operations.

Lemma 2.2. *We have maps*

$$i_n: \mathcal{A}(BP\langle n \rangle) \rightleftarrows \mathcal{A}(BP) : p_n$$

such that

- (1) $i_n p_n: \mathcal{A}(BP) \rightarrow \mathcal{A}(BP)$ is given by $[f] \mapsto [e_n f e_n]$,
- (2) p_n splits i_n (so i_n is injective and p_n is surjective),
- (3) i_n is a non-unital ring homomorphism,
- (4) p_n is an additive group homomorphism.

Proof. We have the maps

$$\begin{aligned} [\theta_n \circ - \circ \pi_n]: BP\langle n \rangle^0(\underline{BP}\langle n \rangle_0) &\rightarrow BP^0(\underline{BP}_0) \\ [f] &\mapsto [\theta_n f \pi_n] \end{aligned}$$

and

$$\begin{aligned} [\pi_n \circ - \circ \theta_n]: BP^0(\underline{BP}_0) &\rightarrow BP\langle n \rangle^0(\underline{BP}\langle n \rangle_0) \\ [f] &\mapsto [\pi_n f \theta_n]. \end{aligned}$$

Since π_n and θ_n are H -space maps, these maps restrict to maps on the additive operations, which we denote by i_n and p_n , respectively.

The first property follows from $\theta_n \pi_n = e_n$. The remaining properties are easy to check using that $\pi_n \theta_n \simeq \text{id}$ and that θ_n is a map of H -spaces. \square

Remark 2.3. It follows that we may identify $\mathcal{A}(BP\langle n \rangle)$ with the subring $e_n \mathcal{A}(BP) e_n$ of $\mathcal{A}(BP)$.

In the following lemma, we record some information about actions in homotopy. Note that we can regard $\mathrm{BP}\langle n \rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ as both a subring and a quotient ring of BP_* . We will abuse notation by just writing $\mathrm{BP}\langle n \rangle_*$ when this is viewed as a subring of BP_* , without explicit notation for the inclusion, and we use $[\cdot]$ to denote classes in $\mathrm{BP}\langle n \rangle_* = \mathrm{BP}_*/J_n$.

Lemma 2.4.

- (1) For $x \in \mathrm{BP}\langle n \rangle_*$, $(\theta_n)_*(x) \equiv x \pmod{J_n}$. In particular, $(\theta_n)_*$ is the identity in degrees less than $2(p^{n+1} - 1)$.
- (2) Let $\phi \in \mathcal{A}(\mathrm{BP}\langle n \rangle)$. Then,
 - (a) for $y \in \mathrm{BP}_*$, $(i_n\phi)_*(y) \equiv \phi_*([y]) \pmod{J_n}$, and,
 - (b) for $y \in J_n$, $(i_n\phi)_*(y) = 0$.
- (3) Let $\varphi \in \mathcal{A}(\mathrm{BP})$ and suppose that $\varphi_*(J_n) \subseteq J_n$. Then, for $z \in \mathrm{BP}\langle n \rangle_*$, $(p_n\varphi)_*(z) = [\varphi_*(z)]$.

Proof. (1) is immediate from $\pi_n\theta_n \simeq \mathrm{id}$. Then, for (2), for $y \in \mathrm{BP}_*$,

$$(i_n\phi)_*(y) = (\theta_n)_*\phi_*(\pi_n)_*(y) = (\theta_n)_*\phi_*([y]) \equiv \phi_*([y]) \pmod{J_n},$$

and, for $y \in J_n$,

$$(i_n\phi)_*(y) = (\theta_n)_*\phi_*(\pi_n)_*(y) = (\theta_n)_*\phi_*(0) = 0.$$

Finally, for (3), we have that

$$\begin{aligned} (p_n\varphi)_*(z) &= (\pi_n)_*\varphi_*(\theta_n)_*(z) \\ &= (\pi_n)_*\varphi_*(z + w) && \text{for some } w \in J_n \\ &= (\pi_n)_*(\varphi_*(z) + \varphi_*(w)) && \text{since } \varphi \text{ is additive} \\ &= [\varphi_*(z)] && \text{since, by hypothesis, } \varphi_*(w) \in J_n. \end{aligned}$$

□

Remark 2.5. It is worth noting that $(\theta_n)_*$ is *not* the obvious splitting on homotopy groups with image $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$ (and it is not a ring homomorphism). See [2, p817] for an example.

Another important consequence of the splitting is that the action of the additive $\mathrm{BP}\langle n \rangle$ operations of bidegree $(0, 0)$ on homotopy groups is faithful. As we will see, the splitting allows us to deduce this from the corresponding result for BP , which was proved in [5, Proposition 1]. (Key ingredients for the BP case are that BP theory has good duality and that everything is torsion free.)

Given an unstable E -operation $\theta \in E^0(E_0) \cong [E_0, E_0]$, we may consider the induced homomorphism of graded abelian groups $\theta_*: \pi_*(E_0) \rightarrow \pi_*(E_0)$ given by the action of θ on homotopy groups. For a graded abelian group M , we write $\mathrm{End}(M)$ for the ring of homomorphisms of graded abelian groups from M to itself.

Sending an operation to its action on homotopy groups gives a map

$$E^0(\underline{E}_0) \rightarrow \text{End}(\pi_*(\underline{E}_0))$$

$$\phi \mapsto \phi_*$$

The restriction of this map to the additive E -operations $\mathcal{A}(E)$ is a ring homomorphism, and we denote this by β_E :

$$\beta_E: \mathcal{A}(E) \rightarrow \text{End}(\pi_*(\underline{E}_0))$$

$$\phi \mapsto \phi_*$$

Proposition 2.6. *For all $n \geq 0$, the ring homomorphism*

$$\beta_{\text{BP}\langle n \rangle}: \mathcal{A}(\text{BP}\langle n \rangle) \rightarrow \text{End}(\pi_*(\underline{\text{BP}}\langle n \rangle_0))$$

is injective.

Proof. Let $\phi \in \mathcal{A}(\text{BP}\langle n \rangle)$ and suppose that $\beta_{\text{BP}\langle n \rangle}(\phi) = \phi_* = 0$. Then,

$$\beta_{\text{BP}}(i_n(\phi)) = (i_n(\phi))_* = (\theta_n \phi \pi_n)_* = (\theta_n)_* \phi_* (\pi_n)_* = 0.$$

But β_{BP} is injective (see [5, Proposition 1]), and so $i_n(\phi) = 0$, so $\phi = 0$. □

3. The comparison map

In this section we begin with some reminders about the additive operations for the Adams summand g of p -local connective complex K -theory, and we recall the main result of [5]. We then go on to define the main map to be studied in this paper, $\hat{i}_n: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP}\langle n \rangle)$, and we discuss its basic properties.

A description of the ring of additive operations $\mathcal{A}(g)$ for the Adams summand can be deduced from the corresponding result for integral complex K -theory (see [1, Lecture 4]). Another description can be found in [6]: Theorems 3.3 and 4.2 of [6] together give a topological basis for this ring, where the basis elements are certain polynomials in the Adams operations Ψ^0, Ψ^p and Ψ^q (where q is primitive modulo p^2 and, thus, a topological generator for the p -adic units). The precise details of the description are not needed here; what is important to note is that all operations can be described in terms of Adams operations.

The main result of [5] (in the split case) is the following.

Theorem 3.1 (Strong and Whitehouse [5, Theorem 19]). *There exists an injective ring homomorphism $\hat{i}: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP})$ such that the image is precisely the centre of the ring $\mathcal{A}(\text{BP})$.*

It is worth noting that \hat{i} is different from the ring homomorphism $i_1: \mathcal{A}(\text{BP}\langle 1 \rangle) = \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP})$ provided by Lemma 2.2. Indeed, \hat{i} sends the identity operation of g to the identity operation of BP , whereas i_1 sends the identity operation of g to e_1 . More generally, it is instructive to consider the effects of these two maps on Adams operations: \hat{i} takes the Adams operation Ψ_g^k of the Adams summand to the corresponding Adams operation Ψ_{BP}^k for BP ; this operation acts as multiplication by $k^{(p-1)n}$ on each element

of the group $\pi_{2(p-1)n}(\text{BP})$. On the other hand, i_1 sends Ψ_g^k to an operation which acts as 0 on the ideal J_1 .

The two maps share the property of being split by p_1 .

Lemma 3.2. *The map p_1 also splits \hat{i} .*

Proof. All elements of the topological ring $\mathcal{A}(g)$ can be explicitly expressed as certain (infinite) linear combinations of Adams operations (see [5, Proposition 18]). By Lemma 2.2, p_1 is additive, and it is straightforward to see that it is continuous with respect to the profinite filtrations on the rings of operations. Thus, it is enough to check that $p_1 \hat{i}(\Psi_g^k) = \Psi_g^k$ for all $k \in \mathbb{Z}_{(p)}$. Now, $\hat{i}(\Psi_g^k) = \Psi_{\text{BP}}^k$ and this BP Adams operation acts as multiplication by $k^{(p-1)r}$ on $\text{BP}_{2(p-1)r}$ (and so, in particular, preserves J_1). Then, by Lemma 2.4 (3), $p_1 \hat{i}(\Psi_g^k)$ acts as multiplication by $k^{(p-1)r}$ on $\pi_{2(p-1)r}(g) = \mathbb{Z}_{(p)}\langle v_1^r \rangle$. But, by [5, Proposition 1], this completely characterizes Ψ_g^k . \square

The main map we consider comes from composing the map $\hat{i}: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP})$ of Theorem 3.1 with the map $p_n: \mathcal{A}(\text{BP}) \rightarrow \mathcal{A}(\text{BP}\langle n \rangle)$ of Lemma 2.2.

Definition 3.3. Define $\hat{i}_n = p_n \hat{i}: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP}\langle n \rangle)$.

Note that this gives us our map of operations without explicitly mentioning Adams operations for $\text{BP}\langle n \rangle$. On the other hand, we can define such Adams operations as follows.

Definition 3.4. Define unstable Adams operations for $\text{BP}\langle n \rangle$ as the images of the corresponding BP operations:

$$\Psi_{\text{BP}\langle n \rangle}^k := p_n(\Psi_{\text{BP}}^k)$$

for $k \in \mathbb{Z}_{(p)}$.

Using Lemma 2.4 (3), we see that this definition gives unstable Adams operations for $\text{BP}\langle n \rangle$ with the expected actions on homotopy (namely, $\Psi_{\text{BP}\langle n \rangle}^k(z) = k^{(p-1)r}z$ for $z \in \text{BP}\langle n \rangle_{2(p-1)r}$).

Since $\hat{i}(\Psi_g^k) = \Psi_{\text{BP}}^k$, it follows from this definition of the Adams operations for $\text{BP}\langle n \rangle$ and the description of $\mathcal{A}(g)$ in terms of Adams operations that the map \hat{i}_n is determined by mapping g Adams operations to the corresponding $\text{BP}\langle n \rangle$ Adams operations and extending to (suitable infinite) linear combinations.

Our main result is that the analogue of Theorem 3.1 holds for $\hat{i}_n: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP}\langle n \rangle)$. We begin with some basic properties of \hat{i}_n ; in particular, it is a ring homomorphism (even though p_n is not).

Proposition 3.5. *For all $n \geq 1$, the map $\hat{i}_n: \mathcal{A}(g) \rightarrow \mathcal{A}(\text{BP}\langle n \rangle)$ is an injective unital ring homomorphism whose image is contained in the centre of $\mathcal{A}(\text{BP}\langle n \rangle)$.*

Proof. First, we check that \hat{i}_n is a ring homomorphism. For $a, b \in \mathcal{A}(g)$, we have that

$$\begin{aligned}
 i_n(\hat{i}_n(a)\hat{i}_n(b)) &= i_n(\hat{i}_n(a))i_n(\hat{i}_n(b)) && (i_n \text{ ring homomorphism}) \\
 &= i_n p_n \hat{i}(a) i_n p_n \hat{i}(b) && (\text{definition of } \hat{i}_n) \\
 &= e_n \hat{i}(a) e_n^2 \hat{i}(b) e_n && (\text{by Lemma 2.2}) \\
 &= e_n^3 \hat{i}(a) \hat{i}(b) e_n && (\text{image of } \hat{i} \text{ central}) \\
 &= e_n \hat{i}(a) \hat{i}(b) e_n && (e_n \text{ idempotent}) \\
 &= e_n \hat{i}(ab) e_n && (\hat{i} \text{ ring homomorphism}) \\
 &= i_n p_n \hat{i}(ab) && (\text{by Lemma 2.2}) \\
 &= i_n \hat{i}_n(ab) && (\text{definition of } \hat{i}_n).
 \end{aligned}$$

But i_n is injective, so $\hat{i}_n(a)\hat{i}_n(b) = \hat{i}_n(ab)$.

Similarly, we find that $i_n(\hat{i}_n(1)) = e_n = i_n(1)$, so $\hat{i}_n(1) = 1$ and \hat{i}_n is unital.

Next, we show injectivity. Let $\phi \in \mathcal{A}(g)$, with $\phi \neq 0$. By [5, Proposition 1], the action of operations in $\mathcal{A}(g)$ on homotopy groups is faithful. Thus, there exists some r such that ϕ acts on $\pi_{2(p-1)r}(g_0)$ as multiplication by some non-zero element λ of $\mathbb{Z}_{(p)}$. But, then, the action of $\hat{i}_n(\phi)$ is given by multiplication by $\lambda \neq 0$ on $\pi_{2(p-1)r}(\underline{\text{BP}}\langle n \rangle_0) \neq 0$, and so $\hat{i}_n(\phi) \neq 0$.

Finally, we need to show that the image is central. The image consists of certain infinite linear combinations of Adams operations for $\text{BP}\langle n \rangle$. It is clear from the action of $\Psi_{\text{BP}\langle n \rangle}^k$ on homotopy that $\beta_{\text{BP}\langle n \rangle}(\Psi_{\text{BP}\langle n \rangle}^k) = (\Psi_{\text{BP}\langle n \rangle}^k)_*$ commutes with all elements of $\text{End}(\pi_*(\text{BP}\langle n \rangle_0))$. So the same holds for the image under $\beta_{\text{BP}\langle n \rangle}$ of (suitable infinite) linear combinations of the Adams operations. But, by Proposition 2.6, $\beta_{\text{BP}\langle n \rangle}$ is injective, so any element of the image of $\mathcal{A}(g)$ commutes with all elements of $\mathcal{A}(\text{BP}\langle n \rangle)$. \square

As a consequence of the definitions, we have the following commutative diagram of abelian groups, for $m \leq n$, giving the compatibility between the various \hat{i} maps:

$$\begin{array}{ccc}
 & \mathcal{A}(\text{BP}\langle n \rangle) & \xrightarrow{\cong} e_n \mathcal{A}(\text{BP}) e_n \\
 \hat{i}_n \nearrow & \uparrow p_n & \downarrow e_m \circ - \circ e_m \\
 \mathcal{A}(g) & \xrightarrow{\hat{i}} \mathcal{A}(\text{BP}) & \\
 \hat{i}_m \searrow & \downarrow p_m & \\
 & \mathcal{A}(\text{BP}\langle m \rangle) & \xrightarrow{\cong} e_m \mathcal{A}(\text{BP}) e_m
 \end{array}$$

Remark 3.6. It is natural to ask if one can obtain the ring $\mathcal{A}(\text{BP})$ as any kind of limit over the $\mathcal{A}(\text{BP}\langle n \rangle)$, but this does not seem to be the case. On the one hand, we can put the $\mathcal{A}(\text{BP}\langle n \rangle)$ into a direct system of injective ring homomorphisms and produce an injective ring homomorphism $\lim_{\rightarrow, n} \mathcal{A}(\text{BP}\langle n \rangle) \rightarrow \mathcal{A}(\text{BP})$. However, this is not surjective; for example, the identity operation on BP is not in the image. On the other hand, the maps in the other direction are not ring homomorphisms, so the inverse limit $\lim_{\leftarrow, n} \mathcal{A}(\text{BP}\langle n \rangle)$ can only be formed in the category of abelian groups.

4. Diagonal operations

We define unstable diagonal operations for $\text{BP}\langle n \rangle$, just as was done for BP in [5].

Definition 4.1. Write $\mathcal{D}(\text{BP}\langle n \rangle)$ for the subring of $\mathcal{A}(\text{BP}\langle n \rangle)$ consisting of operations whose action on each homotopy group $\pi_{2(p-1)r}(\text{BP}\langle n \rangle_0)$ is multiplication by an element μ_r of $\mathbb{Z}_{(p)}$. We call elements of $\mathcal{D}(\text{BP}\langle n \rangle)$ *unstable diagonal operations*.

The main result of this section is that the central operations coincide with the diagonal operations. One inclusion is easy.

For a ring R , we write $Z(R)$ for its centre.

Lemma 4.2. *We have $\mathcal{D}(\text{BP}\langle n \rangle) \subseteq Z(\mathcal{A}(\text{BP}\langle n \rangle))$.*

Proof. The action on homotopy of $\phi \in \mathcal{D}$ commutes with the action of any operation in $\mathcal{A}(\text{BP}\langle n \rangle)$, so the inclusion $\mathcal{D}(\text{BP}\langle n \rangle) \subseteq Z(\mathcal{A}(\text{BP}\langle n \rangle))$ follows from the faithfulness of the action (see Proposition 2.6). \square

Our proof of the reverse inclusion amounts to finding enough operations in order to force a central operation to act diagonally. Our strategy is to start from stable BP operations, over which we have better control, and then to view these as additive unstable operations and project them to $\mathcal{A}(\text{BP}\langle n \rangle)$.

First, we need some notation for sequences indexing monomials. We write v^α for the monomial $v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_m^{\alpha_m}$, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a sequence of non-negative integers, with $\alpha_m \neq 0$. We order such sequences right lexicographically; explicitly for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, we have $\alpha < \beta$ if $m < n$ or if $m = n$ and there exists some j , with $1 \leq j \leq m$, such that $\alpha_k = \beta_k$ for all $k > j$ but $\alpha_j < \beta_j$.

We add sequences placewise: $(\alpha + \beta)_i = \alpha_i + \beta_i$, so that $v^\alpha v^\beta = v^{\alpha + \beta}$. It is straightforward to check that the ordering behaves well with respect to the addition: if $\alpha \leq \alpha'$ and $\beta \leq \beta'$, then $\alpha + \beta \leq \alpha' + \beta'$.

The degree of v^α is $2 \sum_{i=1}^m \alpha_i (p^i - 1)$, and we write this as $|\alpha|$.

Lemma 4.3. *Let α, β, γ denote sequences indexing monomials in the same degree, $|\alpha| = |\beta| = |\gamma|$.*

- (1) *There exists a stable BP operation ϕ_β in $\text{BP}^{|\alpha|}(\text{BP})$ whose action*

$$\text{BP}_{|\alpha|} \rightarrow \text{BP}_0 = \mathbb{Z}_{(p)}$$

has the property that $(\phi_\beta)_(v^\gamma) = \mu_{\gamma, \beta}$, where*

$$\begin{aligned} \mu_{\beta, \beta} &\neq 0, \\ \mu_{\gamma, \beta} &= 0 \quad \text{if } \gamma < \beta. \end{aligned}$$

- (2) *There exists a stable BP operation $\phi_{\alpha, \beta}$ in $\text{BP}^0(\text{BP})$ whose action $\text{BP}_{|\alpha|} \rightarrow \text{BP}_{|\alpha|}$ has the property that $(\phi_{\alpha, \beta})_*(v^\gamma) = \mu_{\gamma, \beta} v^\alpha$, where*

$$\begin{aligned} \mu_{\beta, \beta} &\neq 0, \\ \mu_{\gamma, \beta} &= 0 \quad \text{if } \gamma < \beta. \end{aligned}$$

Proof. The second part follows immediately from the first, by taking $\phi_{\alpha,\beta} = v^\alpha \phi_\beta$.

For the first part, we recall that BP has good duality, so a stable operation ϕ in $BP^*(BP)$ corresponds to a degree 0 BP_* -linear functional $\bar{\phi}: BP_*(BP) \rightarrow BP_*$. The action of the operation on coefficient groups is recovered from the functional by precomposition with the right unit map, $\eta_R: BP_* \rightarrow BP_*(BP)$, that is, $\phi_* = \bar{\phi}\eta_R$. We have that $BP_*(BP) = BP_*[t_1, t_2, \dots]$, so a functional as described above is determined by any choice of its value on each monomial in the ts .

The map η_R is of course very complicated, but we need only exploit some basic information about its form. We have that

$$\eta_R(v_m) = pt_m + \sum \lambda_\gamma t^\gamma + \sum \mu_{\delta,\delta'} v^\delta t^{\delta'}$$

where $\lambda_\gamma, \mu_{\delta,\delta'} \in \mathbb{Z}_{(p)}$, $\delta \neq \emptyset$, and γ runs over sequences other than $(0, \dots, 0, 1)$ in the degree of v_m . (The point is the form of the top term, of course.) Now, η_R is a ring map and it follows from the properties of the ordering on monomials described above that

$$\eta_R(v^\gamma) = \lambda t^\gamma + \sum_{\gamma' < \gamma} \lambda'_{\gamma'} t^{\gamma'} + \sum_{\delta \neq \emptyset} \mu'_{\delta,\delta'} v^\delta t^{\delta'}$$

for some $\lambda, \lambda'_{\gamma'}, \mu'_{\delta,\delta'} \in \mathbb{Z}_{(p)}$ with $\lambda \neq 0$.

Now consider the functional $\bar{\phi}_\beta: BP_*(BP) \rightarrow BP_*$, which is 0 on all monomials except t^β , and sends t^β to 1. By construction, the corresponding operation ϕ_β has the required property. □

Lemma 4.4 now follows as a matter of elementary linear algebra.

Let $E_{\alpha,\beta}$ denote the elementary matrix with 1 in the (α, β) position and 0 everywhere else.

Lemma 4.4. *For all α, β with $|\alpha| = |\beta|$, there exist some non-zero $\bar{\mu}_{\alpha,\beta} \in \mathbb{Z}_{(p)}$ and an operation $\varphi_{\alpha,\beta}$ in $BP^0(BP)$ such that the matrix of its action on $BP_{|\alpha|}$ is $\bar{\mu}_{\alpha,\beta} E_{\alpha,\beta}$.*

Proof. The preceding lemma gives the operation $\phi_{\alpha,\beta}$. Using the $\mathbb{Z}_{(p)}$ -basis of monomials in the vs , ordered as above, this operation acts on coefficients in the given degree by the matrix

$$M_{\alpha,\beta} = \sum_{\gamma \geq \beta} \mu_{\gamma,\beta} E_{\alpha,\gamma}$$

where $\mu_{\beta,\beta} \neq 0$.

If we order the elementary matrices by $E_{\beta,\gamma} < E_{\beta',\gamma'}$ if $\gamma < \gamma'$ or $\gamma = \gamma'$ and $\beta < \beta'$, then the above shows that the matrix writing the $M_{\alpha,\beta}$ in terms of the $E_{\alpha,\beta}$ is non-singular lower triangular. Hence, for some $\bar{\mu}_{\alpha,\beta} \neq 0$, we can write $\bar{\mu}_{\alpha,\beta} E_{\alpha,\beta}$ as a $\mathbb{Z}_{(p)}$ -linear combination of the $M_{\alpha,\beta}$. We take $\varphi_{\alpha,\beta}$ to be the corresponding linear combination of the $\phi_{\alpha,\beta}$. □

Theorem 4.5. *We have $Z(\mathcal{A}(BP\langle n \rangle)) = \mathcal{D}(BP\langle n \rangle)$.*

Proof. We noted the inclusion $\mathcal{D}(\mathrm{BP}\langle n \rangle) \subseteq Z(\mathcal{A}(\mathrm{BP}\langle n \rangle))$ in Lemma 4.2, so it remains to show the reverse inclusion.

As in the proof of [5, Lemma 11], there exists an injection $\mathrm{BP}^0(\mathrm{BP}) \hookrightarrow \mathcal{A}(\mathrm{BP})$ from the stable degree 0 BP operations to the additive unstable bidegree $(0, 0)$ operations, given by sending a stable operation to its zero component (that is, applying Ω^∞). This allows us to view the operation $\varphi_{\alpha, \beta}$ constructed above as an element of $\mathcal{A}(\mathrm{BP})$, still acting on coefficients in the specified degree as some non-zero multiple of the elementary matrix $E_{\alpha, \beta}$.

We next map these operations to $\mathcal{A}(\mathrm{BP}\langle n \rangle)$: consider $p_n(\varphi_{\alpha, \beta}) \in \mathcal{A}(\mathrm{BP}\langle n \rangle)$. We consider the action of this operation on coefficients in degree $|\alpha|$. We can now write $\mathrm{BP}_{|\alpha|}$ as a direct sum of $\mathbb{Z}_{(p)}$ -modules $R \oplus J$, where $R = \mathrm{BP}_{|\alpha|} \cap \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ and $J = \mathrm{BP}_{|\alpha|} \cap J_n$. Note that any monomial in the vs lying in R is lower in the ordering than any monomial lying in J . So, when we write the action of an operation as a matrix with respect to the monomial basis, this splits into blocks, according to the decomposition into R and J .

Now let α, β index monomials in R . Using that $(p_n(\varphi_{\alpha, \beta}))_* = (\pi_n)_*(\varphi_{\alpha, \beta})_*(\theta_n)_*$, it is easy to check the action of $p_n(\varphi_{\alpha, \beta})$ is given by $\bar{\mu}_{\alpha, \beta} E_{\alpha, \beta}$ on $\mathrm{BP}\langle n \rangle_{|\alpha|}$.

So, now, suppose that we have a central operation $\phi \in \mathcal{A}(\mathrm{BP}\langle n \rangle)$. Since it commutes with each operation $\varphi_{\alpha, \beta}$, its action on $\mathrm{BP}\langle n \rangle_{|\alpha|}$ commutes with the action of some non-zero multiple of each elementary matrix. Hence, the matrix of its action in this degree is diagonal with all diagonal entries equal. That is, $\phi \in \mathcal{D}(\mathrm{BP}\langle n \rangle)$. \square

5. Congruences

Consider the map

$$\mathcal{A}(g) \rightarrow \prod_{i=0}^{\infty} \mathbb{Z}_{(p)}$$

given by sending an additive operation θ to the sequence (μ_i) such that θ acts on $\pi_{2(p-1)i}(g) = \mathbb{Z}_{(p)}$ as multiplication by μ_i . We define S_g to be the image of this map. As detailed in [5, §4], S_g is completely characterized by a system of congruences.

The congruences can be described as follows. Let G denote the periodic Adams summand and let $\{\hat{f}_n \mid n \geq 0\}$ be a $\mathbb{Z}_{(p)}$ -basis for $QG_0(\underline{G}_0)$, where Q denotes the indecomposable quotient for the \star -product. These basis elements can be written as rational polynomials in the variable $\hat{w} = \hat{u}^{-1}\hat{v}$, where $G_* = \mathbb{Z}_{(p)}[\hat{u}^{\pm 1}]$ and $\hat{v} = \eta_R(\hat{u})$. The n th congruence is the condition that the rational linear combination of the μ_i obtained from \hat{f}_n by sending \hat{w}^i to μ_i lies in $\mathbb{Z}_{(p)}$. Different choices of basis lead to equivalent systems of congruences with the same solution set S_g . (Explicit choices, involving Stirling numbers, are known, but we do not need these here.)

The following proposition is a stronger version of the congruence result of [5]. The proof closely follows that of [5, Proposition 16].

Proposition 5.1. *Fix $n \geq 1$. Suppose that an operation $\theta \in \mathcal{A}(\mathrm{BP})$ is such that its action on homotopy $\theta_*: \mathrm{BP}_* \rightarrow \mathrm{BP}_*$ satisfies the following conditions. For each $i \geq 0$, there exists some $\mu_i \in \mathbb{Z}_{(p)}$ such that*

- (1) $\theta_*(x) \equiv \mu_i x \pmod{J_n}$ if $x \notin J_n$, $|x| = 2(p-1)i$, and
- (2) $\theta_*(x) = 0$ if $x \in J_n$.

Then, $(\mu_i)_{i \geq 0} \in S_g$.

Proof. Under the isomorphism $PBP^0(\underline{BP}_0) \cong \text{Hom}_{BP_*}(QBP_*(\underline{BP}_0), BP_*)$, the operation θ corresponds to a BP_* -linear functional $\bar{\theta}: QBP_*(\underline{BP}_0) \rightarrow BP_*$ of degree 0.

Let $V_\mu: QBP_*(\underline{BP}_0) \rightarrow \mathbb{Z}_{(p)}$ be the composite $\pi\bar{\theta}$, where $\pi: BP_* \rightarrow \mathbb{Z}_{(p)}$ is defined to be the ring map determined by

$$\begin{aligned} v_1 &\mapsto 1, \\ v_i &\mapsto 0 \quad \text{for } i > 1. \end{aligned}$$

Thus, we have a commutative diagram:

$$\begin{array}{ccc} QBP_*(\underline{BP}_0) & \xrightarrow{\bar{\theta}} & BP_* \\ & \searrow V_\mu & \downarrow \pi \\ & & \mathbb{Z}_{(p)} \end{array}$$

Recall from [2] that $QBP_*(\underline{BP}_0)$ is torsion free and rationally generated by elements of the form $v^\alpha e^{2(p-1)h} \eta_R(v^\beta)$, where $v^\alpha \in BP_*$, $v^\beta \in BP_{2(p-1)h}$, $e \in QBP_1(\underline{BP}_1)$ is the suspension element and η_R is the right unit map. By [2, 12.4], the action of an operation θ on homotopy can be recovered from the corresponding functional $\bar{\theta}$ via $\theta_*(v^\beta) = \bar{\theta}(e^{2(p-1)h} \eta_R(v^\beta))$ for $v^\beta \in BP_{2(p-1)h}$.

We have that

$$\begin{aligned} V_\mu(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) &= \pi \bar{\theta}(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) \\ &= \pi(v^\alpha \theta_*(v^\beta)) \\ &= \begin{cases} \pi(v^\alpha (\mu_h v^\beta + y)) & \text{for some } y \in J_n \text{ if } v^\beta \notin J_n, \\ 0 & \text{if } v^\beta \in J_n, \end{cases} \\ &= \begin{cases} \mu_h & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, for each $x \in QBP_*(\underline{BP}_0)$, $V_\mu(x)$ is some rational linear combination of the μ_i and, since this lies in $\mathbb{Z}_{(p)}$, this gives congruences that must be satisfied by the μ_i .

Consider the standard map of ring spectra $BP \rightarrow G$. This induces a map of Hopf rings $BP_*(\underline{BP}_*) \rightarrow G_*(\underline{G}_*)$ and, thus, a ring map on indecomposables $QBP_*(\underline{BP}_0) \rightarrow QG_*(\underline{G}_0)$, which we denote by ϕ .

We now claim that we can factorize V_μ as $\pi_\mu \tilde{\phi}$, where $\tilde{\phi}: QBP_*(\underline{BP}_0) \rightarrow \text{Im}(\phi)$ is the map given by restricting the codomain of $\phi: QBP_*(\underline{BP}_0) \rightarrow QG_*(\underline{G}_0)$, and

$$\pi_\mu: \text{Im}(\phi) \rightarrow \mathbb{Z}_{(p)}$$

is the \mathbb{Q} -linear map determined by

$$\hat{u}^\alpha e^{2(p-1)b} \hat{v}^b \mapsto \mu_b.$$

To prove the claim, it is enough to check on rational generators:

$$\begin{aligned} \pi_\mu \tilde{\phi}(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)) &= \begin{cases} \pi_\mu(\hat{u}^{\alpha_1} e^{2(p-1)h} \hat{v}^h) & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots), \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \mu_h & \text{if } \alpha = (\alpha_1, 0, 0, \dots) \text{ and } \beta = (h, 0, 0, \dots), \\ 0 & \text{otherwise,} \end{cases} \\ &= V_\mu(v^\alpha e^{2(p-1)h} \eta_R(v^\beta)). \end{aligned}$$

Just as in [5, proof of Theorem 19], up to some shift by a power of \hat{u} , each basis element of $QG_0(\underline{G}_0)$, say \hat{f}_n for $n \geq 0$, is in the image of the map from $QBP_*(\underline{BP}_0)$. So, we have $x_n \in QBP_*(\underline{BP}_0)$ such that $\phi(x_n) = \hat{u}^{c_n} \hat{f}_n$ for some $c_n \in \mathbb{Z}$. Then, $V_\mu(x_n) = \pi_\mu \tilde{\phi}(x_n) = \pi_\mu(\hat{u}^{c_n} \hat{f}_n)$.

But $V_\mu(x_n) \in \mathbb{Z}_{(p)}$, and $\pi_\mu(\hat{u}^{c_n} \hat{f}_n) \in \mathbb{Z}_{(p)}$ is exactly the n th congruence condition for g . Hence, $(\mu_i)_{i \geq 0} \in S_g$. □

We now show how this applies to $BP\langle n \rangle$ operations.

Proposition 5.2. *Let $n \geq 1$ and $\phi \in \mathcal{D}(BP\langle n \rangle)$. Then, $i_n(\phi) \in \mathcal{A}(BP)$ satisfies the hypotheses of Proposition 5.1.*

Proof. Let $\phi \in \mathcal{D}(BP\langle n \rangle)$, where the action of ϕ_* on $\pi_{2(p-1)i}(\underline{BP}\langle n \rangle_0)$ is multiplication by μ_i . Then, using $[\cdot]$ to denote classes modulo J_n , for $v^\alpha \in BP_*$, by Lemma 2.4 (2),

$$\begin{aligned} (i_n(\phi))_*(v^\alpha) &= \begin{cases} \phi_*([v^\alpha]) \bmod J_n & \text{if } v^\alpha \notin J_n, \\ 0 & \text{if } v^\alpha \in J_n, \end{cases} \\ &= \begin{cases} \mu_{\|\alpha\|} v^\alpha \bmod J_n & \text{if } v^\alpha \notin J_n, \\ 0 & \text{if } v^\alpha \in J_n, \end{cases} \end{aligned}$$

where $\|\alpha\| = |\alpha|/2(p-1)$. □

Putting everything together gives the following.

Theorem 5.3. *For all $n \geq 1$, the image of the injective ring homomorphism $\hat{i}_n: \mathcal{A}(g) \hookrightarrow \mathcal{A}(BP\langle n \rangle)$ is the centre $Z(\mathcal{A}(BP\langle n \rangle))$ of $\mathcal{A}(BP\langle n \rangle)$.*

Proof. We have that $\text{Im}(\hat{i}_n) \subseteq Z(\mathcal{A}(BP\langle n \rangle)) = \mathcal{D}(BP\langle n \rangle)$, by Proposition 3.5 and Theorem 4.5. Now let $\phi \in \mathcal{D}(BP\langle n \rangle)$, where ϕ acts on $\pi_{2(p-1)i}(\underline{BP}\langle n \rangle_0)$ as multiplication by μ_i . By Proposition 2.6, ϕ is completely determined by the sequence $(\mu_i)_{i \geq 0}$. By Proposition 5.2, $i_n(\phi)$ satisfies the hypotheses of Proposition 5.1 for the same sequence $(\mu_i)_{i \geq 0}$,

and so, by Proposition 5.1, $(\mu_i)_{i \geq 0} \in S_g$. Thus, we have the following commutative diagram:

$$\begin{array}{ccccc}
 \mathcal{A}(g) & \xrightarrow[\cong]{i_n} & \text{Im}(\hat{i}_n) & \hookrightarrow & Z(\mathcal{A}(\text{BP}\langle n \rangle)) & \xrightarrow{=} & \mathcal{D}(\text{BP}\langle n \rangle) \\
 \downarrow \cong & & & & & & \downarrow \\
 S_g & \xrightarrow{=} & & & & & S_g
 \end{array}$$

So the inclusions are equalities. □

Acknowledgements. I.G.-C. gratefully acknowledges support from the FEDER-Ministerio de Economía y Competitividad (Grants MTM2010-15831 ‘Homotopía de orden superior en álgebra y geometría’, MTM2010-20692 ‘Análisis local en grupos y espacios topológicos’ and MTM2012-38122-C03-01 ‘Geometría algebraica, simpléctica, aritmética y aplicaciones’), and from AGAUR, Generalitat de Catalunya (Grant 2009SGR-1092 ‘Grup de Topologia Algebraica de Barcelona’). S.W. gratefully acknowledges support from the former two grants during visits to Barcelona.

References

1. J. F. ADAMS, Lectures on generalised cohomology, in *Category theory, homology theory and their applications, III*, Lecture Notes in Mathematics (Springer, 1969).
2. M. J. BOARDMAN, D. C. JOHNSON AND W. S. WILSON, Unstable operations in generalized cohomology, in *Handbook of algebraic topology*, pp. 687–828 (North-Holland, Amsterdam, 1995).
3. I. GÁLVEZ AND S. WHITEHOUSE, Infinite sums of Adams operations and cobordism, *Math. Z.* **251** (2005), 475–489.
4. D. C. JOHNSON AND W. S. WILSON, Projective dimension and Brown–Peterson homology, *Topology* **12** (1973), 327–353.
5. M.-J. STRONG AND S. WHITEHOUSE, Infinite sums of unstable Adams operations and cobordism, *J. Pure Appl. Alg.* **214** (2010), 910–918.
6. M.-J. STRONG AND S. WHITEHOUSE, Integer-valued polynomials and K -theory operations, *Proc. Am. Math. Soc.* **138**(6) (2010), 2221–2233.
7. W. S. WILSON, The Ω -spectrum for Brown–Peterson cohomology, II, *Am. J. Math.* **97** (1975), 101–123.