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THE FACTORIAL CONJECTURE AND IMAGES OF LOCALLY NILPOTENT DERIVATIONS

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Abstract

The factorial conjecture was proposed by van den Essen *et al.* ['On the image conjecture', *J. Algebra* **340**(1) (2011), 211–224] to study the image conjecture, which arose from the Jacobian conjecture. We show that the factorial conjecture holds for all homogeneous polynomials in two variables. We also give a variation of the result and use it to show that the image of any linear locally nilpotent derivation of $\mathbb{C}[x, y, z]$ is a Mathieu–Zhao subspace.

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1. Introduction

The Jacobian conjecture (JC for short) asserts that any polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with nonzero constant Jacobian determinant is invertible (see [3] or [1]).

In 2011, van den Essen *et al.* [5] established a connection between the JC and images of derivations. They considered the following question.

Question 1.1 [5, Question 4.1]. *Let D be a derivation of $\mathbb{C}[x, y]$ with divergence zero, that is, $\partial_x(D(x)) + \partial_y(D(y)) = 0$. Is $\mathrm{Im}D$ a Mathieu–Zhao subspace of $\mathbb{C}[x, y]$?*

Throughout, a derivation always means a $\mathbb{C}$-derivation. Mathieu–Zhao subspaces (see Definition 3.1 below) are a kind of generalisation of ideals. Van den Essen *et al.* [5] showed that the two-dimensional JC holds if and only if Question 1.1 has an affirmative answer in the case $1 \in \mathrm{Im}D$.

In 2017, the second author showed in [10] that Question 1.1 has a negative answer in general (see also [4] for a generalisation of the result). However, the following conjecture is still open for any $n \geq 3$.

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CONJECTURE 1.2 [14, Conjecture 1.7]. *Let D be a locally nilpotent derivation of $A := \mathbb{C}[x_1, x_2, \ldots, x_n]$ (so that D is of divergence zero). Then* $\operatorname{Im} D$ *is a Mathieu–Zhao subspace of A.*

Conjecture 1.2 holds for $n = 2$, since by Rentschler's theorem (see [3, Theorem 1.3.48]) a locally nilpotent derivation of $\mathbb{C}[x, y]$ is conjugate by a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[x, y]$ to a derivation of the form $f(x)\partial_y$, the image of which is the ideal $(f(x))$ and thus a Mathieu–Zhao subspace of $\mathbb{C}[x, y]$.

In this paper, we focus on Conjecture 1.2 in dimension $n = 3$. We find that the problem is related to integrals of polynomial functions and, more precisely, to the following factorial conjecture.

CONJECTURE 1.3 (Factorial conjecture (FC(n)) [6, Conjecture 4.2]). *Suppose that $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is such that*

$$\int_{\mathbb{R}^n_{\geq 0}} f^m \cdot e^{-(x_1+x_2+\cdots+x_n)}\, dx_1\, dx_2 \cdots dx_n = 0$$

for all $m \geq 1$. Then $f = 0$.

The factorial conjecture was proposed by van den Essen *et al.* [6] to study the image conjecture, which also arose from the study of the Jacobian conjecture (see [6, 7, 9, 11]). Edo and van den Essen investigated a stronger version of the factorial conjecture in [2]. The factorial conjecture FC(n) has only been verified in the following cases: (1) $n = 1$; (2) $n = 2$ and f is quadratic homogeneous; (3) n is arbitrary and f has a very special form, for example, f is a power of a linear form or $f = c_1x_1^d + c_2x_2^d + \cdots + c_nx_n^d$, where $c_1, c_2, \ldots, c_n \in \mathbb{C}$ (see [6]).

In Section 2, we show that the factorial conjecture holds for all homogeneous polynomials in two variables and we also establish a variant of this result. In Section 3, we apply the result to show that Conjecture 1.2 has an affirmative answer for any linear locally nilpotent derivation in dimension $n = 3$.

2. Factorial conjecture

Let $\mathbb{C}^{[n]} := \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in n variables over $\mathbb{C}$. For any $f \in \mathbb{C}^{[n]}$, let

$$E_1(f) := \int_{\mathbb{R}^n_{\geq 0}} f \cdot e^{-(x_1+x_2+\cdots+x_n)}\, dx_1\, dx_2 \cdots dx_n,$$

$$E_2(f) := \int_{\mathbb{R}^n} f \cdot \frac{e^{-(x_1^2+x_2^2+\cdots+x_n^2)/2}}{(\sqrt{2\pi})^n}\, dx_1\, dx_2 \cdots dx_n,$$

which are the expected values of f with respect to the exponential distribution and the Gaussian normal distribution, respectively. We can restate the factorial conjecture as: *if $f \in \mathbb{C}^{[n]}$ is such that $E_1(f^m) = 0$ for all $m \geq 1$, then $f = 0$.*

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REMARK 2.1. The name 'factorial conjecture' comes from the fact that for any $i \in \mathbb{N}$, $E_1(t^i) = \int_0^\infty t^i e^{-t}\,dt = i!$ and, thus, for any monomial $M = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$,

$$E_1(M) = \int_{\mathbb{R}^n_{\geq 0}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} e^{-(x_1+x_2+\cdots+x_n)}\,dx_1\,dx_2 \cdots dx_n = i_1! i_2! \cdots i_n!.$$

LEMMA 2.2. *$E_1(f) = E_2(f\frac{1}{2}(y_1^2+z_1^2), \frac{1}{2}(y_2^2+z_2^2), \dots, \frac{1}{2}(y_n^2+z_n^2))$ for any $f \in \mathbb{C}^{[n]}$.*

PROOF. Using the polar coordinate transformation of $\mathbb{R}^2$,

$$\begin{aligned} E_2\Big(\Big(\frac{t_1^2+t_2^2}{2}\Big)^i\Big) &= \int_{\mathbb{R}^2} \Big(\frac{t_1^2+t_2^2}{2}\Big)^i \frac{e^{-(t_1^2+t_2^2)/2}}{2\pi}\,dt_1\,dt_2 = \int_0^\infty \int_0^{2\pi} \Big(\frac{r^2}{2}\Big)^i \frac{e^{-r^2/2}}{2\pi}\,r\,dr\,d\theta \\ &= \Big(\int_0^\infty \Big(\frac{r^2}{2}\Big)^i e^{-r^2/2}\,d\frac{r^2}{2}\Big)\Big(\int_0^{2\pi} \frac{1}{2\pi}\,d\theta\Big) = \int_0^\infty t^i e^{-t}\,dt = E_1(t^i) \end{aligned}$$

and it follows that $E_1(M) = E_2(M(\frac{1}{2}(y_1^2+z_1^2), \frac{1}{2}(y_2^2+z_2^2), \dots, \frac{1}{2}(y_n^2+z_n^2)))$ for any monomial $M = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. The conclusion follows. □

In what follows, we will show that FC(n) holds in dimension $n = 2$ for all homogeneous polynomials f.

We first recall the usual polar coordinate transformation of $\mathbb{R}^4$:

$$\begin{cases} x_1 = r\cos\theta_1, \\ x_2 = r\sin\theta_1\cos\theta_2, \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_4 = r\sin\theta_1\sin\theta_2\sin\theta_3, \end{cases}$$

where $0 \leq r < \infty, 0 \leq \theta_1 \leq \pi, 0 \leq \theta_2, \theta_3 < 2\pi$. This polar coordinate transformation cannot be applied successfully to prove Theorem 2.6 below. So, we need to define a new coordinate transformation of $\mathbb{R}^4$.

DEFINITION 2.3. Define a coordinate transformation of $\mathbb{R}^4$ as follows:

$$\begin{cases} x_1 = r\cos\theta_1\cos\theta_2, \\ x_2 = r\cos\theta_1\sin\theta_2, \\ x_3 = r\sin\theta_1\cos\theta_3, \\ x_4 = r\sin\theta_1\sin\theta_3, \end{cases} \tag{2.1}$$

where $0 \leq r < \infty, 0 \leq \theta_1 \leq (1/2)\pi, 0 \leq \theta_2, \theta_3 < 2\pi$. One may verify that

$$\begin{aligned} dx_1\,dx_2\,dx_3\,dx_4 &= |\det J_{r,\theta_1,\theta_2,\theta_3}(x_1,x_2,x_3,x_4)| \cdot dr\,d\theta_1\,d\theta_2\,d\theta_3 \\ &= r^3 \sin\theta_1\cos\theta_1\,dr\,d\theta_1\,d\theta_2\,d\theta_3. \end{aligned}$$

The following lemma ensures that Definition 2.3 is reasonable.

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LEMMA 2.4. *The map* $\phi : M = [0,\infty) \times [0,(1/2)\pi] \times [0,2\pi) \times [0,2\pi) \to \mathbb{R}^4$ *defined by formula (2.1) is almost one-to-one with the inverse given by*

$$\begin{cases} r = \sqrt{x_1^2+x_2^2+x_3^2+x_4^2}; \\ \theta_1 = \arccos \dfrac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}}; \\ \theta_2 = \textit{the unique angle in } [0,2\pi) \textit{ with} \cos\theta_2 = \dfrac{x_1}{\sqrt{x_1^2+x_2^2}}, \sin\theta_2 = \dfrac{x_2}{\sqrt{x_1^2+x_2^2}}; \\ \theta_3 = \textit{the unique angle in } [0,2\pi) \textit{ with} \cos\theta_3 = \dfrac{x_3}{\sqrt{x_3^2+x_4^2}}, \sin\theta_3 = \dfrac{x_4}{\sqrt{x_3^2+x_4^2}}. \end{cases} \tag{2.2}$$

More precisely, the map $\tilde{\phi}$ *induced by* ϕ *with domain* $\tilde{M}$ *defined by*

$$(0,\infty) \times (0, \tfrac{1}{2}\pi) \times [0,2\pi) \times [0,2\pi) \to \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid x_1^2+x_2^2 \neq 0, x_3^2+x_4^2 \neq 0\}$$

is one-to-one.

PROOF. Let $P = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4$ be such that $x_1^2+x_2^2 \neq 0$ and $x_3^2+x_4^2 \neq 0$. On one hand, if we take $M_0 = (r,\theta_1,\theta_2,\theta_3)$ to be the point determined by (2.2), then $M_0 \in \tilde{M}$ and one may verify by (2.1) that $\phi(M_0) = P$. On the other hand, if a point $M_0 = (r,\theta_1,\theta_2,\theta_3) \in \tilde{M}$ is such that $\phi(M_0) = P$, that is, M_0 satisfies (2.1), then one may see from (2.1) that M_0 must be of the form determined by (2.2). Therefore, the map $\tilde{\phi}$ is one-to-one. □

LEMMA 2.5 [8, Corollary 4.1]. *Suppose that* $a, b \in \mathbb{R}$ *and* $a \neq b$. *If* $f \in \mathbb{C}[t]$ *is such that* $\int_a^b f^m\,dt = 0$ *for all* $m \geq 1$, *then* $f = 0$.

THEOREM 2.6. *The factorial conjecture* FC(2) *holds for all homogeneous polynomials. More precisely, if* $f \in \mathbb{C}[x,y]$ *is homogeneous and*

$$E_1(f^m) = \int_0^\infty \int_0^\infty f^m e^{-(x+y)}\,dx\,dy = 0 \quad \textit{for all } m \geq 1,$$

then $f = 0$.

PROOF. By Lemma 2.2, for any $m \geq 1$,

$$\begin{aligned} 0 = E_1(f^m(x,y)) &= E_2\Big(f^m\Big(\frac{x_1^2+x_2^2}{2}, \frac{x_3^2+x_4^2}{2}\Big)\Big) \\ &= \int_{\mathbb{R}^4} f^m\Big(\frac{x_1^2+x_2^2}{2}, \frac{x_3^2+x_4^2}{2}\Big) \frac{e^{-(x_1^2+x_2^2+x_3^2+x_4^2)/2}}{(2\pi)^2}\,dx_1\,dx_2\,dx_3\,dx_4. \end{aligned}$$

Set $d := \deg f$. Using the coordinate transformation in Definition 2.3 with

$$dx_1\,dx_2\,dx_3\,dx_4 = r^3 \sin\theta_1 \cos\theta_1\,dr\,d\theta_1\,d\theta_2\,d\theta_3,$$

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$$\begin{aligned}0 &= \int_M f^m\Big(\frac{r^2\cos^2\theta_1}{2}, \frac{r^2\sin^2\theta_1}{2}\Big)\frac{e^{-r^2/2}}{(2\pi)^2} r^3 \sin\theta_1\cos\theta_1\, dr\, d\theta_1\, d\theta_2\, d\theta_3 \\ &= \int_M \Big(\frac{r^2}{2}\Big)^{dm} f^m(\cos^2\theta_1, \sin^2\theta_1)\,\frac{e^{-r^2/2}}{(2\pi)^2} r^3 \sin\theta_1\cos\theta_1\, dr\, d\theta_1\, d\theta_2\, d\theta_3 \\ &= c\int_0^{\pi/2} f^m(\cos^2\theta_1, \sin^2\theta_1)\sin\theta_1\cos\theta_1\, d\theta_1,\end{aligned}$$

where $M = [0,\infty)\times[0,(\pi/2)]\times[0,2\pi)\times[0,2\pi)$ and

$$\begin{aligned}c &= \Big(\int_0^\infty \Big(\frac{r^2}{2}\Big)^{dm} e^{-r^2/2} r^3\, dr\Big)\cdot\Big(\int_0^{2\pi}\int_0^{2\pi}\Big(\frac{1}{2\pi}\Big)^2 d\theta_2\, d\theta_3\Big) \\ &= \Big(2\int_0^\infty \Big(\frac{r^2}{2}\Big)^{dm+1} e^{-r^2/2}\, d\frac{r^2}{2}\Big)\cdot 1 = 2(dm+1)! \neq 0.\end{aligned}$$

It follows that

$$\begin{aligned}0 &= \int_0^{\pi/2} f^m(\cos^2\theta_1, \sin^2\theta_1)\sin\theta_1\cos\theta_1\, d\theta_1 \\ &= \frac{1}{2}\int_0^{\pi/2} f^m(\cos^2\theta_1, \sin^2\theta_1)\, d(\sin^2\theta_1) \\ &= \frac{1}{2}\int_0^1 f^m(1-t,t)\, dt\end{aligned}$$

and, by Lemma 2.5, $f(1-t,t)=0$.

Let $f = \sum_{j=0}^d c_j x^{d-j}y^j$, where $c_j \in \mathbb{C}$ for $j = 0,1,\ldots,d$. Then

$$\sum_{j=0}^d c_j(1-t)^{d-j}t^j = 0,$$

which implies that $c_0 = 0$ by setting $t = 0$. Then $\sum_{j=1}^d c_j(1-t)^{d-j}t^{j-1} = 0$ and it follows that $c_1 = 0$ for the same reason. Similarly, one may obtain that $c_2 = c_3 = \cdots = c_d = 0$. Therefore, $f = 0$. □

Now we give a variation of Theorem 2.6.

Theorem 2.7. *Let $f \in \mathbb{C}[x,y]$ be a homogeneous polynomial in two variables such that*

$$E_2\Big(f^m\Big(\frac{x_1^2+x_2^2}{2}, y^2\Big)\Big) = 0 \quad \textit{for all } m \geq 1.$$

Then $f = 0$.

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PROOF. Let $d = \deg f$. Note that for any $m \geq 1$,

$$0 = E_2\Big(f^m\Big(\frac{x_1^2 + x_2^2}{2}, y^2\Big)\Big) = \int_{\mathbb{R}^3} f^m\Big(\frac{x_1^2 + x_2^2}{2}, y^2\Big)\frac{e^{-(x_1^2+x_2^2+y^2)/2}}{(\sqrt{2\pi})^3}\, dx_1\, dx_2\, dy.$$

Using the polar coordinate transformation

$$\begin{cases} x_1 = r\sin\theta_1\cos\theta_2, \\ x_2 = r\sin\theta_1\sin\theta_2, \\ y = r\cos\theta_1, \end{cases}$$

where $0 \leq r < \infty, 0 \leq \theta_1 \leq \pi, 0 \leq \theta_2 < 2\pi$ and $dx_1\, dx_2\, dy = r^2 \sin\theta_1\, dr\, d\theta_1\, d\theta_2$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} f^m\Big(\frac{x_1^2 + x_2^2}{2}, y^2\Big)\frac{e^{-(x_1^2+x_2^2+y^2)/2}}{(\sqrt{2\pi})^3}\, dx_1\, dx_2\, dy \\ &= \int_0^\infty \int_0^\pi \int_0^{2\pi} r^{2dm} f^m\Big(\frac{\sin^2\theta_1}{2}, \cos^2\theta_1\Big)\frac{e^{-r^2/2}}{(\sqrt{2\pi})^3} r^2 \sin\theta_1\, dr\, d\theta_1\, d\theta_2 \\ &= \Big(\int_0^\infty r^{2dm+2}\frac{e^{-r^2/2}}{\sqrt{2\pi}}\, dr\Big)\Big(-\int_0^\pi f^m\Big(\frac{\sin^2\theta_1}{2}, \cos^2\theta_1\Big)\, d\cos\theta_1\Big)\Big(\int_0^{2\pi}\frac{1}{2\pi}\, d\theta_2\Big) \\ &= \frac{(2dm+1)!!}{2}\int_{-1}^1 f^m\Big(\frac{1-t^2}{2}, t^2\Big)\, dt \end{aligned}$$

for all $m \geq 1$, where $(2dm + 1)!! = (2dm + 1) \times (2dm - 1) \times \cdots \times 1$.

It follows from Lemma 2.5 that $f(\frac{1}{2}(1 - t^2), t^2) = 0$ and, as in the proof of the last theorem, we obtain $f = 0$. □

3. Images of linear locally nilpotent derivations

In this section, we show that the image of any linear locally nilpotent derivation of the polynomial algebra $A = \mathbb{C}[x, y, z]$ is a Mathieu–Zhao subspace of A.

DEFINITION 3.1 [12]. A $\mathbb{C}$-subspace M of a commutative $\mathbb{C}$-algebra S is called a Mathieu–Zhao subspace if for each pair $f, g \in S$ with $f^m \in M$ for all $m \geq 1$, we have $gf^m \in M$ for all $m \gg 0$ (that is, there exists some m_g such that $gf^m \in M$ for all $m \geq m_g$).

The definition of Mathieu–Zhao subspaces was introduced by Zhao in [12] and called ‘Mathieu subspaces’ in some early literature. For more properties of Mathieu–Zhao subspaces, we refer the reader to [7, 13].

Recall that a derivation D of $\mathbb{C}^{[n]} = \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called locally nilpotent if for each $f \in \mathbb{C}^{[n]}$, there exists an $m_f \in \mathbb{N}$ such that $D^{m_f}(f) = 0$. Also, D is called linear if each $D(x_i)$ is a linear form.

LEMMA 3.2. *Let $D = x\partial_y - y\partial_z$ and let $x^i y^k z^j$ be any monomial in $\mathbb{C}[x, y, z]$. If $i > j$, then $x^i y^k z^j \in \mathrm{Im} D$.*

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PROOF. First observe that for any $i, j, k \in \mathbb{N}$,

$$D(x^i y^k z^j) = kx^{i+1}y^{k-1}z^j - jx^i y^{k+1}z^{j-1}. \tag{3.1}$$

We use induction on j to show that $x^i y^k z^j \in \mathrm{Im}D$ if $i > j$. When $j = 0$, by setting $j = 0$ in (3.1), we have $x^{i+1}y^{k-1} \in \mathrm{Im}D$ for any $i \in \mathbb{N}$, $k \geq 1$. Now suppose that $j > 0$ and $x^i y^{k+1} z^{j-1} \in \mathrm{Im}D$ for any $i \in \mathbb{N}$, $k \geq 1$ with $i > j - 1$. Then, by (3.1), we have $x^{i+1}y^{k-1}z^j \in \mathrm{Im}D$ for any $i \in \mathbb{N}$, $k \geq 1$ with $i + 1 > j$, as desired. □

LEMMA 3.3. *Let $D = x\partial_y - y\partial_z$ and let $f \in \mathbb{C}[x, y, z]$ be such that $f \in \mathrm{Im}D$. Then $E_2(f((x - z\mathbf{i})/\sqrt{2}, y, (x + z\mathbf{i})/\sqrt{2})) = 0$.*

PROOF. Using the polar coordinate transformation

$$x = \sqrt{2}r\cos\theta, \quad z = \sqrt{2}r\sin\theta \quad \text{and} \quad dx\,dz = 2r\,dr\,d\theta,$$

we have

$$\begin{aligned}
E_2\Big(\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}\Big)^i\Big(\frac{x + z\mathbf{i}}{\sqrt{2}}\Big)^j\Big) &= \int_{\mathbb{R}^2}\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}\Big)^i\Big(\frac{x + z\mathbf{i}}{\sqrt{2}}\Big)^j \frac{e^{-(x^2+z^2)/2}}{(\sqrt{2\pi})^2}\,dx\,dz \\
&= \int_0^\infty\int_0^{2\pi}(r\cos\theta - r\mathbf{i}\sin\theta)^i(r\cos\theta + r\mathbf{i}\sin\theta)^j\frac{e^{-r^2}}{2\pi}\,2r\,dr\,d\theta \\
&= \int_0^\infty\int_0^{2\pi}(re^{-\mathbf{i}\theta})^i(re^{\mathbf{i}\theta})^j\frac{e^{-r^2}}{2\pi}\,dr^2\,d\theta \\
&= \Big(\int_0^\infty r^{i+j}e^{-r^2}\,dr^2\Big)\Big(\int_0^{2\pi}e^{\mathbf{i}\theta(j-i)}\frac{1}{2\pi}\,d\theta\Big) = \begin{cases} i! & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\end{aligned}$$

Also note that

$$E_2(y^k) = \int_{-\infty}^{\infty} y^k\frac{e^{-y^2/2}}{\sqrt{2\pi}} = \begin{cases} (k-1)!! & \text{for } k \in 2\mathbb{N}, \\ 0 & \text{for } k \notin 2\mathbb{N}, \end{cases} \tag{3.2}$$

where $(k - 1)!! = (k - 1) \times (k - 3) \times \cdots \times 1$. So, for any monomial $M = x^i y^k z^j$,

$$E_2\Big(M\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}, y, \frac{x + z\mathbf{i}}{\sqrt{2}}\Big)\Big) = \begin{cases} i!(k-1)!! & \text{for } i = j \text{ and } k \in 2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Since $f \in \mathrm{Im}D$, f is a $\mathbb{C}$-linear combination of the monomials considered in Lemma 3.2, that is, polynomials of the form

$$h_{i,j,k} := kx^{i+1}y^{k-1}z^j - jx^i y^{k+1}z^{j-1}.$$

By (3.2) and (3.3),

$$\begin{aligned}
&E_2\Big(h_{i,j,k}\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}, y, \frac{x + z\mathbf{i}}{\sqrt{2}}\Big)\Big) \\
&\quad = \begin{cases} k(i+1)!(k-2)!! - (i+1)i!k!! = 0 & \text{for } i + 1 = j \text{ and } k - 1 \in 2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore,

$$E_2\Big(f\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}, y, \frac{x + z\mathbf{i}}{\sqrt{2}}\Big)\Big) = 0.$$

□

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THEOREM 3.4. *Let D be any linear locally nilpotent derivation of $A = \mathbb{C}[x, y, z]$. Then* $\mathrm{Im}D$ *is a Mathieu–Zhao subspace of* A.

PROOF. We may assume that $D \neq 0$. Since D is a linear locally nilpotent derivation, D is nilpotent as a linear operator on $V := \mathbb{C}x + \mathbb{C}y + \mathbb{C}z$. So, there exists some linear polynomial automorphism T of A such that $(T^{-1} \circ D \circ T)|_V$ has the matrix E_{12} or $E_{12} - E_{23}$ with respect to the $\mathbb{C}$-basis x, y, z of V. That is, $T^{-1} \circ D \circ T$ is of the form $x\partial_y$ or $x\partial_y - y\partial_z$.

For any polynomial automorphism ϕ and any $\mathbb{C}$-subspace M of A, M is a Mathieu–Zhao subspace of A if and only if $\phi(M)$ is a Mathieu–Zhao subspace of A. So, we may assume without loss of generality that $D = x\partial_y$ or $D = x\partial_y - y\partial_z$. When $D = x\partial_y$, we see that $\mathrm{Im}D$ is the ideal (x) and thus is a Mathieu–Zhao subspace of A.

From now on we assume that $D = x\partial_y - y\partial_z$. Consider the $w = (-1, 0, 1)$-degree on $A = \mathbb{C}[x, y, z]$, so that $\deg x^i y^k z^j = j - i$. Suppose that $0 \neq f \in A$ is such that $f^m \in \mathrm{Im}D$ for all $m \geq 1$. We divide the discussion into two cases.

Case 1: $\deg_w f < 0$. For any $g \in A$,

$$\deg_w gf^m = \deg_w g + m \deg_w f < 0$$

for $m \gg 0$ and thus $gf^m \in \mathrm{Im}D$ for $m \gg 0$, by Lemma 3.2. It follows that $\mathrm{Im}D$ is a Mathieu–Zhao subspace.

Case 2: $s := \deg_w f \geq 0$. We show that this case does not occur. Since $x^s \in \ker D$ and $f^m \in \mathrm{Im}D$, we have $(x^s f)^m = x^{sm} f^m \in \mathrm{Im}D$ and $\deg_w x^s f = (-s) + s = 0$. So, replacing f by $x^s f$, we may assume that $\deg_w f = 0$.

Let f_i be the w-homogeneous part of f of w-degree i. Then $f = f_0 + f_{-1} + \cdots$. Note that $f^m = f_0^m + g$, where $\deg_w g < 0$. By Lemma 3.2, $g \in \mathrm{Im}D$ and thus $f_0^m \in \mathrm{Im}D$ for all $m \geq 1$. Replacing f by f_0, we may assume that f is w-homogeneous of w-degree 0.

Denote by $\overline{f}$ the highest homogeneous part of f with respect to the ordinary degree. Since D is a homogeneous derivation with respect to the ordinary degree, replacing f by $\overline{f}$, we may assume that f is homogeneous with respect to the ordinary degree. Replacing f by f^2 if necessary, we may also assume that $\deg f$ is even.

In conclusion, we may assume that f is w-homogeneous of w-degree 0 and is homogeneous of even degree with respect to the ordinary degree, that is, f is of the form $f = g(xz, y^2)$, where $g(t_1, t_2) \in \mathbb{C}[t_1, t_2]$ is homogeneous.

Since $f^m \in \mathrm{Im}D$ for all $m \geq 1$, from Lemma 3.3,

$$E_2\Big(f^m\Big(\frac{x - z\mathbf{i}}{\sqrt{2}}, y, \frac{x + z\mathbf{i}}{\sqrt{2}}\Big)\Big) = 0 \quad \text{for all } m \geq 1,$$

that is,

$$E_2\Big(g^m\Big(\frac{x^2 + z^2}{2}, y^2\Big)\Big) = 0 \quad \text{for all } m \geq 1.$$

Since $g(t_1, t_2)$ is homogeneous, it follows from Theorem 2.7 that $g(t_1, t_2) = 0$ and thus $f = 0$, which is a contradiction. □

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