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POLYNOMIAL HULLS OF SETS INVARIANT UNDER AN ACTION OF THE SPECIAL UNITARY GROUP

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1. Introduction. If K is a compact subset of $\mathbf{C}^n$, $\hat{K}$ will denote the *polynomial hull* of K:

$$\hat{K} = \{z \in \mathbf{C}^n: \ |P(z)| \leqq \sup_{w \in K} |P(w)| \text{ for all polynomials } P\}.$$

$\hat{K}$ arises in the study of uniform algebras as the maximal ideal space of the algebra $P(K)$ of uniform limits on K of polynomials (see [3]). The condition $K = \hat{K}$ (K is *polynomially convex*) is a necessary one for uniform approximation on K of continuous functions by polynomials ($P(K) = C(K)$). If K is not polynomially convex, the question of existence of analytic structure in $\hat{K}\setminus K$ is of particular interest. For $n = 1$, $\hat{K}$ is the union of K and the bounded components of $\mathbf{C}\setminus K$. The determination of $\hat{K}$ in dimensions greater than one is a more difficult problem. Among the special classes of compact sets K whose polynomial hulls have been determined are those invariant under certain group actions on $\mathbf{C}^n$. In [12] J. Wermer investigated a class of disks K in $\mathbf{C}^2$ invariant under the S^1 action

$$(1.1) \quad (z, w) \rightarrow (ze^{i\theta}, we^{-i\theta}).$$

He found that $\hat{K}\setminus K$ was foliated by a one-parameter family of analytic disks (analytic images of the unit disk in $\mathbf{C}$) with boundaries on K. Gamelin [6] later gave a description of $\hat{K}$ for arbitrary sets $K \subset \mathbf{C}^2$ invariant under the same action. He found that through each point of $\hat{K}\setminus K$ there exists an analytic disk in $\hat{K}$. Also, Björk [2] studied the general question of algebras invariant under the action of a compact group.

Recently, Debiard and Gaveau [5] investigated actions of the unitary and special unitary groups on $\mathbf{C}^3$ identified with the space of 2×2 symmetric complex matrices by

$$(z_1, z_2, z_3) \rightarrow \begin{bmatrix} z_1 & z_3 \\ z_3 & z_2 \end{bmatrix} = Z.$$

The action is then

$$(1.2) \quad Z \rightarrow gZ^t g$$

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for $g \in U(2)$ or $SU(2)$, where ${}^t g$ denotes the transpose of g. They described the hull of the $SU(2)$ orbit of any point $Z \in \mathbf{C}^3$, and gave a partial description of the $U(2)$ orbit of an arbitrary point, finding a family of analytic disks with boundaries on the orbit.

In this paper we give an explicit description (Theorem 2) of $\hat{K}$ for a class of compact sets invariant under the $SU(2)$ action (1.2). We apply this to obtain a complete description of the hull of any $U(2)$ orbit (Corollary 2). Even for "special" orbits, which are three (real) dimensional subsets of $\mathbf{C}^3$, this hull contains an open subset of $\mathbf{C}^3$. These results are presented in Section 3. In Section 2 we give a general discussion of orbits of the analogous action on $\mathbf{C}^N$, $N = n(n + 1)/2$; in particular we prove that if $Z = I$ is the point identified with the identity matrix, and K is the orbit of Z, then $P(K) = C(K)$. For our main results, we use ideas from the papers of Björk, Gamelin, and Wermer mentioned above. The key ingredient is a result of Wermer stating that a certain function on the fibers of a projection from the maximal ideal space of a uniform algebra into $\mathbf{C}$ is subharmonic.

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2. Orbits. M_n will denote the ring of $n \times n$ complex matrices. $\det(A)$ and $\operatorname{tr}(A)$ will denote the determinant and trace of $A \in M_n$, and $Gl(n)$, the general linear group of invertible elements of M_n. We denote by $O(n)$, $SO(n)$, $U(n)$, and $SU(n)$ the subgroups of $Gl(n)$ consisting of the orthogonal, special orthogonal, unitary and special unitary matrices respectively. I_n will denote the $n \times n$ identity matrix. We identify the space S_n of symmetric matrices in M_n with $\mathbf{C}^N$, $N = n(n + 1)/2$, by

$$(z_1, \dots, z_N) \to \begin{bmatrix} z_1 & z_{n+1} & z_{2n} & \cdot & \cdot & z_N \\ z_{n+1} & z_2 & z_{n+2} & \cdot & \cdot & \cdot \\ \cdot & & \cdot\ \cdot & & & \\ & & \cdot & & & \\ \cdot & & & & & z_{2n-1} \\ z_N & & & & & z_n \end{bmatrix}.$$

For G a subgroup of $Gl(n)$ we define an action of G on S_n by

(2.1) $\quad Z \to gZ{}^t g \quad$ for $g \in G$, $Z \in S_n$.

$\mathcal{O}_Z^G$ will denote the G-orbit of $Z \in S_n$,

$$\mathcal{O}_Z^G = \{W \in S_n: \quad W = gZ{}^t g \quad \text{for some } g \in G\}.$$

If G is compact, so is $\mathcal{O}_Z^G$, $\mathcal{I}_Z^G$ will denote the isotropy subgroup of Z in G, i.e.,

$$\mathcal{I}_Z^G = \{g \in G: \quad gZ{}^t g = Z\}.$$

If the coset space $G/\mathscr{I}_Z^G$ is given the quotient topology then $O_Z^G \simeq G/\mathscr{I}_Z^G$, where $\simeq$ denotes homeomorphism (see [7]). We will be concerned first and primarily with the case $G = SU(n)$, and we drop the superscript G for the remainder of the discussion. Note that if $Z \in S_n$, and $W = gZ^tg \in O_Z$, then $\det(Z) = \det(W)$. Also, since $W\bar{W} = gZ\bar{Z}g^{-1}$, $W\bar{W}$ and $Z\bar{Z}$ have the same eigenvalues; in particular $\operatorname{tr}(Z\bar{Z}) = \operatorname{tr}(W\bar{W})$. Denote by $\Delta(c_1, \ldots, c_n)$ the $n \times n$ matrix with entries $c_1, \ldots, c_n$ on the main diagonal and zeroes elsewhere. The following lemma is due to Hua [**10**].

LEMMA 1. *Let $Z \in S_n$. Then there exists $g \in U(n)$ such that*

$$gZ^tg = \Delta(c_1, \ldots, c_n)$$

with $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$.

It follows that every $SU(n)$ orbit contains a diagonal element, for if $Z \in S_n$, and $D = \Delta(c_1, \ldots, c_n) = gZ^tg$ with $g \in U(n)$ and $\det(g) = e^{i\alpha}$, set

$$g' = \Delta(e^{-i\alpha}, 1, 1, \ldots, 1).$$

Then $h = gg' \in SU(n)$ and $Z = hD'^th$, where

$$D' = (g')^{-1}D(g')^{-1} = \Delta(c_1e^{2i\alpha}, c_2, \ldots, c_n).$$

Note that if $\det(Z) = 0$, then we can take $\alpha = 0$.

LEMMA 2. *If $Z \in S_n$, then O_Z consists of all those $W \in S_n$ such that*

1. $\det(Z) = \det(W)$ *and*

2. *The set of eigenvalues of $Z\bar{Z}$ and the set of eigenvalues of $W\bar{W}$ are the same.*

Proof. We have seen that if $W \in O_Z$, then W satisfies (1) and (2). Conversely, suppose W satisfies (1) and (2). By the remarks following Hua's lemma we can choose

$$D_1 = \Delta(c_1e^{i\alpha}, c_2, \ldots, c_n) \in O_Z \quad \text{and}$$
$$D_2 = \Delta(d_1e^{i\beta}, d_2, \ldots, d_n) \in O_W$$

with $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$, $d_1 \geqq d_2 \geqq \ldots \geqq d_n \geqq 0$. The eigenvalues of $D_1\bar{D}_1$ are the same as those of $Z\bar{Z}$, and similarly for $D_2\bar{D}_2$ and $W\bar{W}$, so by (2),

$$\{c_1^2, \ldots, c_n^2\} = \{d_1^2, \ldots, d_n^2\}.$$

Since the c_i and d_i are non-negative and arranged in decreasing order, $c_i = d_i$, $i = 1, \ldots, n$. If $\det(Z) = 0$, then $\det(W) = 0$, so we can take $\alpha = \beta = 0$. Otherwise,

$$c_1c_2 \ldots c_ne^{i\alpha} = d_1d_2 \ldots d_ne^{i\beta},$$

and neither side of this equation vanishes, so that $\alpha \equiv \beta \pmod{2\pi}$. In either case, $D_1 = D_2$. Thus $O_Z = O_W$.

Now we can determine the isotropy subgroups. These are divided into types according to the multiplicities of the eigenvalues of $Z\bar{Z}$. Fix $Z \in S_n$, and choose $D \in O_Z$ of the form

$$D = \Delta(c_1 e^{i\alpha}, c_2, \ldots, c_n),$$

with $c_1 \geqq c_2 \geqq \ldots \geqq c_n \geqq 0$. Then $O_Z = O_D \simeq SU(n)/\mathscr{I}_D$. Rewrite

$$\{c_1, \ldots, c_n\} = \{\lambda_1, \ldots, \lambda_r\}$$

where the λ_i are distinct, and λ_i occurs with multiplicity l_i in the list $c_1, \ldots, c_n$. Let $E = D\bar{D}$. If $g \in \mathscr{I}_D$, then $gD^t g = D$, so $gE = Eg$. From this we see that g must have the form:

$$(2.2) \quad g = \begin{bmatrix} g_1 & & & & \\ & g_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & g_r \end{bmatrix}$$

where g_i is a block of size l_i, $i = 1, \ldots, r$. Since $g^t\bar{g} = I_n$, $g_i^t\bar{g}_i = I_{l_i}$, and so

$$(2.3) \quad g_i \in U(l_i), \quad i = 1, \ldots, r.$$

Moreover from $gD^t g = D$ we obtain $\lambda_i g_i{}^t g_i = \lambda_i I_{l_i}$, and so

$$(2.4) \quad \text{if } \lambda_i > 0, \ g_i \in O(l_i).$$

Conversely, any matrix $g \in SU(n)$ of the form (2.2) satisfying (2.3) and (2.4) is easily seen to be an element of $\mathscr{I}_D$, and so we obtain the following:

LEMMA 3. *$\mathscr{I}_D$ is the subgroup of $SU(n)$ consisting of matrices of the form* (2.2) *with $g_i \in O(l_i)$ for $i = 1, \ldots, r$, if $\lambda_r > 0$;* and *$g_i \in O(l_i)$ for $i < r$, $g_r \in U(l_r)$ if $\lambda_r = 0$.*

An interesting special case is obtained by taking $Z = I_n$. By the preceding lemma, the isotropy subgroup consists of the matrices $g \in O(n)$ which are elements of $SU(n)$, i.e., which are elements of the special orthogonal group $SO(n)$, and so

$$O_{I_n} \simeq SU(n)/SO(n)$$

which is of real dimension $n^2 - 1 - (n(n - 1)/2) = (n + 2)(n - 1)/2$. For the following discussion we fix n and write $O_{I_n} = O_I$.

LEMMA 4. *O_I is polynomially convex.*

Proof. By Lemma 2,

$$O_I \subset \{Z: \ \det(Z) = 1, \ \mathrm{tr}(Z\bar{Z}) = n\}.$$

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In fact, denoting the latter set by X, $O_I = X$, for if $Z \in X$, then we can choose

$$D = \Delta(c_1, \dots, c_n) \in O_Z$$

with

$$c_1 c_2 \dots c_n = 1 \quad \text{and} \quad \sum_{i=1}^{n} c_i^2 = n.$$

The minimum of the function $\sum_{i=1}^{n} c_i^2 = B$ subject to the constraint $c_1 c_2 \dots c_n = 1$ occurs exactly when $c_1 = c_2 = \dots = c_n = 1$, i.e., when $B = n$. Thus $D = I$, and so $X = O_I$. We can easily compute that in the coordinates of $\mathbf{C}^N$.

$$\operatorname{tr}(Z\bar{Z}) = \sum_{i=1}^{n} |z_i|^2 + 2\left(\sum_{i=n+1}^{N} |z_i|^2\right).$$

The ellipsoid $\operatorname{tr}(Z\bar{Z}) \leqq c$ for any $c > 0$ is polynomially convex. Also, $\hat{O}_I$ must be contained in $\{\det(Z) = 1\}$, so

$$\hat{O}_I \subset \{\det(Z) = 1, \operatorname{tr}(Z\bar{Z}) \leqq n\}.$$

By our previous observation, the latter set is O_I.

More is true; in fact:

THEOREM 1. $P(O_I) = C(O_I)$.

Proof. For a real submanifold M of an open subset of $\mathbf{C}^k$, the space H_pM of complex tangents to M at p can be defined as follows: Any real tangent vector $L \in T_p\mathbf{C}^k$ can be written in the form

$$L = \sum_{j=1}^{n} a_j \partial/\partial x_j + \sum_{j=1}^{n} b_j \partial/\partial y_j$$

where $z_j = x_j + iy_j$ are the coordinates on $\mathbf{C}^k$. We define a map J on $T_p\mathbf{C}^k$ by setting

$$J(L) = -\sum_{j=1}^{n} b_j \partial/\partial x_j + \sum_{j=1}^{n} a_j \partial/\partial y_j.$$

Then identifying the tangent space T_pM to M at p with a subspace of $T_p\mathbf{C}^k$,

$$H_pM = T_pM \cap J(T_pM).$$

In the natural complex structure on $T_p\mathbf{C}^k$, H_pM is the largest real subspace of T_pM which is also a complex subspace of $T_p\mathbf{C}^k$. M is said to be *totally real* if $H_pM = \{0\}$ for each $p \in M$. By a theorem of Hormander and

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Wermer [9], if M is totally real, and K is a compact polynomially convex subset of M, then $P(K) = C(K)$. By Lemma 4, it thus suffices to show that $M = O_I$ is totally real. The map $Z \to gZ{}^tg$ for fixed g is nonsingular and complex linear, which implies that the dimension of H_pM is constant on M. Thus it suffices to check that $H_pM = \{0\}$ for $p = I$. We consider $SU(n)$ as a submanifold of $Gl(n)$. Let T be the map $T(g) = g{}^tg$ of $Gl(n)$ into itself. The image of T restricted to $SU(n)$ is M, and the image of T_* restricted to the tangent space of $SU(n)$ at I is T_IM. $Gl(n)$ we identify with a subset of $\mathbf{C}^{n^2}$ by using the coordinates $z_{kl} = x_{kl} + iy_{kl}$ for the (k, l)-th entry of $A \in Gl(n)$. The tangent space to $Gl(n)$ at I can be identified with M_n by assigning to the tangent vector

$$L = \sum_{k,l=1}^{n} a_{kl}\partial/\partial x_{kl} + b_{kl}\partial/\partial y_{kl}$$

the matrix

$$\tilde{L} = [a_{kl} + ib_{kl}].$$

Under this identification, $JL = i\tilde{L}$. The tangent space to $SU(n)$ at I is then identified with the space $su(n)$ of skew-Hermitian matrices of trace zero (see [1]). It is easy to compute that for $A \in M_n$, $T_*A = A + {}^tA$ and so $T_*A = A - \bar{A}$ for $A \in su(n)$. In particular, T_*A is purely imaginary. It follows that $J(T_IM) \cap T_IM = \{0\}$, and so M is totally real.

Remark. According to a theorem of A. Browder [4], for a compact manifold M, $C(M)$ requires at least $\dim(M) + 1$ generators. If $M = O_{I_n}$,

$$\dim(M) + 1 = n(n + 1)/2 = N,$$

and so $C(M)$ in this case has the minimum possible number of generators $(z_1, \dots, z_N)$.

We have not yet determined hulls of orbits with more complicated isotropy groups for $n > 2$. In what follows, we review the case $n = 2$ in preparation for the work of Section 3. Most of these results are contained in [5].

$SU(2)$ can be identified with the unit sphere in $\mathbf{C}^2$ by associating to the point (λ_1, λ_2) with $|\lambda_1|^2 + |\lambda_2|^2 = 1$ the matrix

$$\begin{bmatrix} \lambda_1 & -\bar{\lambda}_2 \\ \lambda_2 & \bar{\lambda}_1 \end{bmatrix}.$$

For $n = 2$, $N = 3$, the eigenvalues of $Z\bar{Z}$ are uniquely determined by $\mathrm{tr}(Z\bar{Z})$ and $\det(Z\bar{Z}) = |\det(Z)|^2$, so that by Lemma 2, if $\det(Z) = A$ and $\mathrm{tr}(Z\bar{Z}) = B$,

$$O_Z = \{W \in \mathbf{C}^3: \quad \det(W) = A, \mathrm{tr}(W\bar{W}) = B\}.$$

Note that since $\det(Z)$ is a polynomial and the set $\operatorname{tr}(Z\bar{Z}) \leqq c$ is polynomially convex, for any Z,

(2.5) $\hat{O}_Z = \{W \in \mathbf{C}^n: \ \det(W) = A, \operatorname{tr}(W\bar{W}) \leqq B\}$.

Since the eigenvalues of $Z\bar{Z}$ are real, the equation

$$\det(Z\bar{Z} - \lambda I) = \lambda^2 - B\lambda + |A|^2 = 0$$

has real roots, so that $B^2 \geqq 4|A|^2$, i.e.,

(2.6) $B \geqq 2|A|$.

Equality holds if and only if the roots are repeated, which by Lemma 5 implies that $O_Z = O_{cI}$ for some $c \in \mathbf{C}$. If $c = 0$, then $O_Z = (0, 0, 0) = \hat{O}_Z$. If $c \neq 0$, then O_Z is biholomorphic by a simple dilation to O_I, and by the previous discussion, $\hat{O}_Z = O_Z$ and $P(O_Z) = C(O_Z)$. Note that $O_Z \simeq SU(2)/SO(2)$. This quotient is obtained by identifying the matrices

$$X_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

to I. It is essentially equivalent to that obtained from the Hopf fibration $S^3/S^1 \simeq P^1 \simeq S^2$ which identifies (z_1, z_2) with $(\lambda z_1, \lambda z_2)$ if $|\lambda| = 1$; in terms of $SU(2)$ this amounts to identifying the matrices

$$Y_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

to I. Since $gX_\theta g^{-1} = Y_\theta$, where

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix} \in SU(2),$$

the two quotients are homeomorphic and $O_Z \simeq S^2$.

If $B > 2|A|$, then the eigenvalues of $Z\bar{Z}$ are distinct, which by Lemma 2.3 implies that the associated isotropy subgroup is just $\{\pm I\}$. (The same conclusion is reached whether or not $Z\bar{Z}$ has a zero eigenvalue.) In this case $O_Z \simeq SU(2)/\{\pm I\}$, which is homeomorphic to real projective three-space. Following **[5]** we refer to an orbit for which $B = 2|A|$ as *special*, and orbits for which $B > 2|A|$ as *general*.

It is helpful to visualize the parameter space (A, B) of the orbits in $\mathbf{C} \times \mathbf{R} \simeq \mathbf{R}^3$ (see Fig. 1).

The cone $B = 2|A|$ represents the special orbits, its interior $B > 2|A|$ the general orbits. By (2.5) the hull of a given general orbit O consists of a vertical segment joining O to the cone. Note that the hull of a given general orbit contains a special orbit. Also, to determine the hull of an

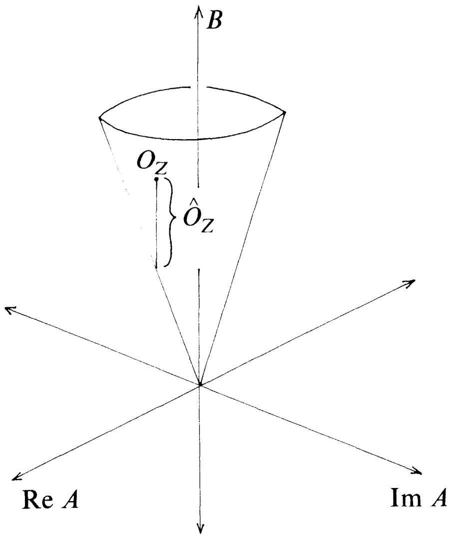

Figure 1

arbitrary set $\hat{K}$ invariant under the $SU(2)$ action (1.2) we need only determine the following:

(i) $Y =$ the set of all $\zeta \in \mathbf{C}$ with

$$\{Z \in \hat{K}: \quad \det(Z) = \zeta\} = F_{\hat{K}(\zeta)}$$

nonempty, and for each $\zeta \in Y$,

$$\text{(ii)} \quad t(\zeta) = \sup_{Z \in F_{\hat{K}}(\zeta)} \{\operatorname{tr}(Z\bar{Z})\}.$$

Then

$$\hat{K} = \{Z \in \mathbf{C}^3: \quad \det(Z) = \zeta \in Y, \operatorname{tr}(Z\bar{Z}) \leqq t(\zeta)\}.$$

In Section 3 we first determine Y for any invariant K and then we find $t(\zeta)$ for a particular class of invariant K. As a preliminary, the following lemma describes $M = t(\zeta)$ for a single orbit $K = O_{Z^0}$.

LEMMA 6. *For* $Z^0 \in \mathbf{C}^3$, *let*

$$M = M(Z^0) = \max\{|z_1|: \quad Z \in O_{Z^0}\}.$$

Then

(a) $M = \max\{|z_1|: \quad Z \in O_{Z^0}$ *and* $z_3 = 0\}$.

(b) $M > 0$ *unless* $Z^0 = (0, 0, 0)$.

(c) $\operatorname{tr}(Z^0\bar{Z}^0) = M^2 + |A|^2/M^2$, *where* $A = \det(Z^0)$, *and this function is strictly increasing in* M *for* $|A|$ *fixed.*

(d) *If* $W \in \mathbf{C}^3$, $\det(W) = \det(Z^0)$, *and* $M(W) \leqq M(Z^0)$, *then* $W \in \hat{O}_{Z^0}$.

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Proof. By the remarks following Lemma 1, there exists $D \in O_{Z^0}$ of the form

$$D = \begin{bmatrix} xe^{i\alpha} & 0 \\ 0 & y \end{bmatrix}$$

with $x \geqq y \geqq 0$. If $g \in SU(2)$ we can write

$$g = \begin{bmatrix} \cos(\theta)\, e^{i\beta} & -\sin(\theta)\, e^{-i\gamma} \\ \sin(\theta)\, e^{i\gamma} & \cos(\theta)\, e^{-i\beta} \end{bmatrix}.$$

Set $Z = gD^t g \in O_{Z^0}$. Then

$$z_1 = x \cos^2(\theta)\, e^{i(\alpha + 2\beta)} + y \sin^2(\theta)\, e^{-2i\gamma}$$

and so

$$\begin{aligned} |z_1|^2 &= x^2 \cos^4(\theta) + y^2 \sin^4(\theta) \\ &\quad + 2xy \cos^2(\theta) \sin^2(\theta) \cos(\alpha + 2\beta + 2\gamma) \\ &\leqq (x \cos^2(\theta) + y \sin^2(\theta))^2 \leqq x^2 \leqq M^2 \end{aligned}$$

which proves the first assertion. If Z is any element of O_{Z^0} with $z_3 = 0$, then $|z_1|^2$ and $|z_2|^2$ are eigenvalues of $Z\bar{Z}$, so

$$\{ |z_1|^2, |z_2|^2 \} = \{x^2, y^2\},$$

and since $x \geqq y$, $|z_1| \leqq x$. So $x = M$. Note that $\operatorname{tr}(Z\bar{Z}^0) > 0$ unless $Z^0 = (0, 0, 0)$; since

$$\operatorname{tr}(Z^0\bar{Z}^0) = \operatorname{tr}(D\bar{D}) = x^2 + y^2,$$

and $x \geqq y$, $x = M$ is positive if $Z^0 \neq (0, 0, 0)$, which proves (b). Since $\det(Z^0) = \det(D)$, $y = |A|/M$. Thus

$$\operatorname{tr}(Z^0\bar{Z}^0) = M^2 + |A|^2/M^2,$$

and (c) is proved. Note that $x \geqq y$ implies that $M^2 \geqq |A|$. It is easily verified that this function is strictly increasing in M for $M^2 \geqq |A|$. It follows that under the conditions in part (d), $\operatorname{tr}(W\bar{W}) \leqq \operatorname{tr}(Z^0\bar{Z}^0)$, and so $W \in \hat{O}_{Z^0}$.

3. Hulls of invariant sets. Let K be a compact subset of $\mathbf{C}^3$ invariant under the $SU(2)$ action (2.1) and let $\hat{K}$ denote the polynomial hull of K. Since the map T_g: $Z \to gZ^t g$ for fixed $g \in SU(2)$ is non-singular and complex linear, $\hat{K}$ is also invariant under this action. Let $i(K)$ denote the closed subalgebra of $P(K)$ consisting of all $f \in P(K)$ such that $f \circ T_g = f$ for all $g \in SU(2)$, the *invariant algebra*. For $f \in P(K)$ define the projection of f onto $i(K)$ by

$$\mathscr{P}(f)(z) = \int_{SU(2)} f(T_g(z))d\mu(g)$$

where μ denotes normalized Haar measure on $SU(2)$. Then $\mathscr{P}(f) \in i(K)$, $\mathscr{P}(f) = f$ if $f \in i(K)$, and

$$\|\mathscr{P}(f)\|_K \leqq \|f\|_K.$$

Moreover, if Q is a polynomial, so is $\mathscr{P}(Q)$. (See [**2**].) Set $F(Z) = \det(Z) \in i(K)$, and let $X = F(K)$. X is a compact subset of $\mathbf{C}$. Note that if $X = \{c_1, \dots, c_s\}$ is a finite set, then $\hat{X}$ is easy to describe: if

$$B_i = \max\{\mathrm{tr}(Z\bar{Z}): \quad F(Z) = c_i, Z \in K\},$$

then

$$\hat{K} = \bigcup_{i=1}^{n} \{Z: \quad \mathrm{tr}(Z\bar{Z}) \leqq B_i, F(Z) = c_i\}.$$

Henceforth we assume that X is infinite.

LEMMA 7. *$i(K)$ is generated by $F(Z)$.*

Proof. First we claim that if P is a polynomial in $i(K)$, then P is a polynomial in F. The proof is by induction on the degree of P. The claim is true for polynomials of degree 0. Assume $\deg(P) > 0$. Fix $Z^0 \in K$, $Z^0 \neq (0, 0, 0)$. Let $c = F(Z^0)$. P is constant on O_{Z^0}, say $P = d$. Then P is constant $= d$ on $\hat{O}_{Z^0}$. Since the hull of each orbit contains a special orbit, P is constant on a special orbit Y: $F(Z) = c$, $\mathrm{tr}(Z\bar{Z}) = 2|c|$. By the proof of Theorem 1 any special orbit is totally real. It is well-known (see [**11**]) that if M is a complex manifold of complex dimension n, Y a totally real submanifold of M of real dimension n, and f is any holomorphic function on M vanishing on Y, then $f \equiv 0$ on M. Applying this to $M = \{F(Z) = c\}$ of complex dimension 2 and the special orbit Y of real dimension 2, we find that P is constant on M. It follows from the Nullstellensatz that

$$P(Z) - d = (F(Z) - c)^m Q(Z)$$

for some polynomial Q with $\deg(Q) < \deg(P)$, and some integer $m > 0$. Note that Q is invariant on the set $K' = K \backslash \{F(Z) = c\}$. Since $F(K')$ is infinite, by induction we may assume that Q is a polynomial in F, and the claim is proved. Now suppose $f \in i(K)$ and choose polynomials P_n converging uniformly on K to f. Since

$$\|\mathscr{P}(P_n) - f\| = \|\mathscr{P}(P_n - f)\| \leqq \|P_n - f\|,$$

we can assume $P_n \in i(K)$ for all n, and the proof of the lemma is complete.

In the following lemma we use an argument of Wermer [**12**].

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LEMMA 8. $F(\hat{K}) = \hat{X}$.

Proof. It is immediate that $F(\hat{K}) \subset \hat{X}$. Fix $\zeta_0 \in \hat{X}\backslash X$. If $\zeta_0 \notin F(\hat{K})$, then

$$g(Z) = (F(Z) - \zeta_0)$$

does not vanish on $\hat{K}$, so $g^{-1} \in P(K)$. Clearly then $g^{-1} \in i(K)$. By the previous lemma, there exists a sequence of polynomials in $\zeta = F(Z)$ converging to g^{-1} on K, so $P_n(\zeta)$ converges to $(\zeta - \zeta_0)^{-1}$ on X, implying that $(\zeta - \zeta_0)^{-1} \in P(X)$, which is a contradiction.

Now we turn to the problem of determining $\hat{K}$. We assume that:

(3.1) $F(K) = X$ is a simple closed curve in $\mathbf{C}$, given as the image of a one-to-one continuous map γ: $[0, 1] \to \mathbf{C}$ with $\gamma(0) = \gamma(1)$ and

(3.2) $0 \notin X$.

Let Ω be the bounded component of $\mathbf{C}\backslash X$. By Lemma 8, $F(\hat{K}) = \hat{X} = \bar{\Omega}$. Let $\mathscr{F}_K$ and $\mathscr{F}_{\hat{K}}$ be the K and $\hat{K}$ fibers of the projection F, respectively:

$$\mathscr{F}_K(\zeta) = \{Z \in K: \quad F(Z) = \zeta\}, \quad \mathscr{F}_{\hat{K}}(\zeta) = \{Z \in \hat{K}: \quad F(Z) = \zeta\}$$

and set

$$M_K(\zeta) = \max\{\, |z_1|: \quad Z \in \mathscr{F}_K(\zeta)\,\},$$

$$M_{\hat{K}}(\zeta) = \max\{\, |z_1|: \quad Z \in \mathscr{F}_{\hat{K}}(\zeta)\,\},$$

$$\Psi_K(\zeta) = \log(M_K(\zeta)\,), \quad \Psi_{\hat{K}}(\zeta) = \log(M_{\hat{K}}(\zeta).$$

For any function f on X denote by H_f the Perron solution to the Dirichlet problem with boundary values f, i.e.,

$$H_f = \sup\{u(\zeta): \quad u \in \mathscr{L}_f\}$$

where $\mathscr{L}_f$ is the class of functions subharmonic or identically $-\infty$ in Ω, bounded above on Ω, and satisfying

$$\limsup_{\zeta \in \Omega \to \zeta^0} u(\zeta) \leqq f(\zeta^0), \quad \text{all } \zeta^0 \in X.$$

We use the following facts from potential theory (see **[8]**): If f is bounded on X, then H_f is harmonic and bounded on Ω. Moreover the existence of a barrier at each point $\zeta^0 \in X$ implies that

(3.3) $\displaystyle\limsup_{\zeta \in \Omega \to \zeta^0} H_f(\zeta) \leqq \limsup_{\zeta \in X \to \zeta^0} f(\zeta).$

Take $f = \Psi_K$ and let $H = H_{\Psi_K}$. By assumption (3.2) combined with Lemma 6 (b), Ψ_K is bounded, and so H is harmonic on Ω.

THEOREM 2. *Let K be a compact set in $\mathbf{C}^3$ invariant under the $SU(2)$ action* (2.1), *and assume that* (3.1) *and* (3.2) *hold. Then*

$$(3.4) \quad \hat{K} = \{Z \in \mathbf{C}^3: \quad F(Z) = \zeta \in \hat{X}, \operatorname{tr}(Z\bar{Z}) \leqq t(\zeta)\}$$

where

$$t(\zeta) = \beta^2 + |\zeta|^2\beta^{-2}$$

and

$$\beta(\zeta) = e^{2H(\zeta)} \text{ for } \zeta \in \Omega, \ \beta(\zeta) = M_K(\zeta) \text{ for } \zeta \in X.$$

We begin the proof with the following lemma:

LEMMA 9. $\limsup_{\zeta \in \Omega \to \zeta^0} H(\zeta) \leqq \Psi_K(\zeta^0)$, *all* $\zeta^0 \in X$.

Proof. By (3.3) it suffices to show that

$$(3.5) \quad \limsup_{\zeta \in X \to \zeta^0} \Psi_K(\zeta) \leqq \Psi_K(\zeta^0), \quad \text{all } \zeta^0 \in X.$$

Choose a sequence $\{Z_j\}$, $Z_j \in K$, all j, with $\zeta_j = F(Z_j) \in X$ converging to ζ^0. If $Z_{j'}$ is a subsequence converging to $Z^0 \in K$, then

$$F(Z^0) = \zeta^0 \quad \text{and} \quad |(z_1)_{j'}| \to |z_1^0|,$$

from which (3.5) follows.

Next we make use of the following result of Wermer [**12**]:

Let A be a uniform algebra on a compact Hausdorff space K, and let M_A denote the maximal ideal space of A. Fix $F \in A$, and let $\hat{F}$ denote the Gelfand transform of F. For each $\zeta \in \mathbf{C}$, let $\mathscr{F}(\zeta)$ be the fiber of the projection $\hat{F}$,

$$\mathscr{F}(\zeta) = \{x \in M_A: \quad \hat{F}(x) = \zeta\}.$$

Then for any $g \in A$, the function

$$\Psi(\zeta) = \log\left(\max_{x \in \mathscr{F}(\zeta)} |g(x)|\right)$$

is subharmonic on $\mathbf{C} \backslash F(K)$.

Applying this result to the algebra $A = P(K)$, where $M_A = \hat{K}$, and taking $F(Z) = \det(Z) = \hat{F}(Z)$, $g(Z) = z_1$, we see that the function $\Psi_{\hat{K}}(\zeta)$ is subharmonic on $\mathbf{C} \backslash X$.

LEMMA 10. $\limsup_{\zeta \in \Omega \to \zeta^0} \Psi_{\hat{K}}(\zeta) \leqq \Psi_K(\zeta^0)$, *all* $\zeta^0 \in X$.

Proof. We follow very closely the proof of a similar statement in [**12**]. Suppose the assertion of the lemma is false. Then by an argument similar to that in Lemma 9, there exists a point $Z^0 \in \hat{K}$ with $\det(Z^0) = \zeta^0 \in X$, and

$$|z_1^0| > \sup_{Z \in \mathscr{F}_K(\zeta^0)} |z_1|.$$

Let B denote the maximum of $\mathrm{tr}(Z\bar{Z})$ on $\mathscr{F}_K(\zeta^0)$. By Lemma 6 (c), $\mathrm{tr}(Z^0\bar{Z}^0) > B$. Let

$$Y = \{Z: \quad F(Z) = \zeta, \ \mathrm{tr}(Z\bar{Z}) \leqq B\}.$$

Then $Y \subset \hat{K}$, Y is polynomially convex, and $Z^0 \notin Y$. Choose a polynomial P with $|P| < 1$ on a neighborhood N of Y in $\hat{K}$, and $|P(Z^0)| > 2$. Then $X_1 = F(K \setminus N)$ is a closed subset of $X \setminus \{\zeta\}$. Since each point of X is a peak point for the algebra $P(X)$, there exists a polynomial h with $h(\zeta) = 1$, $|h| < 1$ on $X \setminus \{\zeta\}$. Choose $\rho > 0$ so that $|h| < 1 - \rho$ on X_1. Then $|h \circ F| < 1 - \rho$ on $K \setminus N$, and $|h \circ F| \leqq 1$ on K. Choose n so that

$$(1 - \rho)^n \max_K |P| < 1,$$

and set $Q = (h \circ F)^n P$. Then $|Q| < 1$ on K, but

$$|Q(Z^0)| = |h(\zeta^0)|\,|P(Z^0)| > 2,$$

which contradicts $Z^0 \in K$, and we are done.

It follows from the preceding lemma that $\Psi_{\hat{K}}$ belongs to the class $\mathscr{L}_{\Psi_K}$, and so

(3.6) $\quad H(\zeta) \geqq \Psi_{\hat{K}}(\zeta) \quad$ for $\zeta \in \Omega$.

Let H^* denote the harmonic conjugate of H in Ω, and set

$$\varphi = e^{H + iH^*}.$$

Then φ is analytic, bounded, and non-vanishing on Ω. Let D be the image of the map G: $\Omega \to \mathbf{C}^3$ given by

$$G(\zeta) = (\varphi(\zeta), \zeta/\varphi(\zeta), 0).$$

Note that $F(G(\zeta)) = \zeta$. Also, for $\zeta \in \Omega$, by (3.6),

$$|\varphi(\zeta)| = e^{H(\zeta)} \geqq e^{\Psi_{\hat{K}}(\zeta)} = M_{\hat{K}}(\zeta)$$

so that

$$|\zeta|/|\varphi(\zeta)| \leqq |\zeta|/M_{\hat{K}}(\zeta).$$

Choosing $Z \in \mathscr{F}_{\hat{K}}(\zeta)$ with $|z_1| = M_{\hat{K}}(\zeta)$, by the proof of Lemma 6,

$$M_{\hat{K}}(\zeta)^2 = M^2(Z) \geqq |\zeta|,$$

where

$$M(Z) = \{\max|w_1|: \quad W \in O_Z\}$$

which implies that

(3.7) $\quad |\zeta|/|\varphi(\zeta)| \leqq M_{\hat{K}}(\zeta).$

If $Z^0 \in \partial D$, then there exists a sequence $\{\zeta_n\}$, $\zeta_n \in \Omega$, converging to $\zeta^0 \in X$, with

$$Z_n = (\varphi(\zeta_n), \zeta_n/\varphi(\zeta_n), 0)$$

converging to Z^0. It follows that $F(Z^0) = \zeta^0$, and so by Lemma 9,

$$|z_1^0| \leqq \limsup_{\zeta \to \zeta^0} |\varphi(\zeta)| = \limsup_{\zeta \to \zeta^0} e^{H(\zeta)} \leqq e^{\Psi_K(\zeta^0)} = M_K(\zeta^0)$$

and also

$$|z_2^0| \leqq \limsup_{\zeta \in \Omega \to \zeta^0} |\zeta^n|/|\varphi(\zeta^n)| \leqq \limsup_{\zeta \in \Omega \to \zeta^0} M_{\hat{K}}(\zeta) \leqq M_K(\zeta^0)$$

by (3.7) and Lemma 10. Since $z_3 = 0$, as in the proof of Lemma 6,

$$\max\{ |z_1^0|, |z_2^0| \} = M(Z^0),$$

and so $M(Z^0) \leqq M_K(\zeta^0)$. By Lemma 6 (d), $O_{Z^0} \subset \hat{K}$. It follows that $\partial D \subset \hat{K}$, and so $D \subset \hat{K}$. Thus for $\zeta \in \Omega$, by definition of $M_{\hat{K}}$,

$$e^{H(\zeta)} = |\varphi(\zeta)| \leqq M_{\hat{K}}(\zeta) = e^{\Psi_{\hat{K}}(\zeta)},$$

so $H(\zeta) \leqq \Psi_{\hat{K}}(\zeta)$. Combining this with (3.6) gives

$$H(\zeta) = \Psi_{\hat{K}}(\zeta) \quad \text{for all } \zeta \in \Omega.$$

If $F(Z) = \zeta \in \Omega$, and $Z \in \hat{K}$, then $M(Z) \leqq e^{H(\zeta)}$ implies by Lemma 6 (c) that (3.4) holds. Conversely, if $\det(Z) = \zeta$, and (3.4) holds, by Lemma 6 (c),

$$M(Z) \leqq \beta(\zeta) = e^{H(\zeta)} = M_{\hat{K}}(\zeta)$$

and so by Lemma 2.6 (d), $O_Z \subset \hat{O}_W \subset \hat{K}$. If $F(Z) = \zeta \in X$, a similar argument shows that $Z \in \hat{K}$ if and only if (3.4) holds. The proof is complete.

COROLLARY 1. *Under the assumptions in Theorem* 2, *each point in* $\hat{K} \setminus K$ *lies on an analytic disk in* $\hat{K}$.

Proof. If $Z \in \hat{K} \setminus K$, and $\det(Z) = \zeta$, then by Theorem 2, and equation (2.6), one of the following holds:

(i) $\operatorname{tr}(Z\bar{Z}) = t(\zeta)$
(ii) $2|\zeta|^2 < \operatorname{tr}(Z\bar{Z}) < t(\zeta)$
(iii) $\operatorname{tr}(Z\bar{Z}) = 2|\zeta|^2 < t(\zeta)$.

Let D be the disk constructed in the proof of Theorem 2, the image of

$$\zeta \to (\varphi(\zeta), \zeta/\varphi(\zeta), 0) = Z(\zeta) \subset \hat{K}.$$

Then $\det(Z(\zeta)) = \zeta$, and if (i) holds,

$$\operatorname{tr} Z(\zeta)\bar{Z}(\zeta) = t(\zeta) = \operatorname{tr}(Z\bar{Z}),$$

so that $Z = \lambda_0$ for some $g \in SU(2)$, so Z belongs to the disk $\zeta \to \lambda_0$. If (ii) holds then an open neighborhood of Z in $\mathbf{C}^3$ belongs to $\hat{K}$. If (iii) holds then Z belongs to the orbit of $\lambda_0 I$ for some $\lambda_0 \in \mathbf{C}$; so

$$Z = g(\lambda_0, \lambda_0, 0)^t g \quad \text{for some } g \in SU(2).$$

Moreover by (iii) for some $\epsilon > 0$, the disk $\lambda \to g(\lambda, \lambda, 0)^t g$ lies in $\hat{K}$ for $|\lambda - \lambda_0| < \epsilon$, and contains Z.

We now apply Theorem 2 to the study of orbits under the action (2.1) where the group G is taken to be $U(2)$. Such orbits are *a fortiori* invariant under the same action with $G = SU(2)$. Since each element g of $U(2)$ can be written as $g = e^{i\alpha}g'$ with $g' \in SU(2)$, we see that

$$O_Z^{U(2)} = \{W \in \mathbf{C}^3: \ |\det(W)| = |\det(Z)|, \operatorname{tr}(W\bar{W}) = \operatorname{tr}(Z\bar{Z})\},$$

and

$$O_Z^{U(2)} \simeq O_Z^{SU(2)} \times S^1.$$

In particular, if $O_Z^{SU(2)}$ is special, $O_Z^{U(2)} \simeq S^2 \times S^1$ is three dimensional.

Fix $Z^0 \in \mathbf{C}^3$ with

$$A = \det(Z^0), \quad B = \operatorname{tr}(Z^0\bar{Z}^0), \quad M = \max_{Z \in O_{Z^0}} |z_1|.$$

Set $K = O_Z^{U(2)}$. Then

$$X = F(K) = \{\zeta \in \mathbf{C}: \ |\zeta| = A\}, \quad \text{and} \quad F(\hat{K}) = \{\, |\zeta| \leqq A\}.$$

By Lemma 2.6, M depends only on B and $|A|$, in fact, we easily compute that

$$(3.8) \quad M = \left[\frac{B + (B^2 - 4|A|^2)^{1/2}}{2}\right]^{1/2}$$

so that $\Psi_K(\zeta) = \log(M)$ is constant on X. Thus $H \equiv \log(M)$ on $\hat{X}$, and we have:

COROLLARY 2. *If* $\det(Z) = A$, $\operatorname{tr}(Z\bar{Z}) = B$, *and* $K = O_Z^{U(2)}$, *then*

$$\hat{K} = \{Z \in \mathbf{C}^3: \ \det(Z) = |\zeta| \leqq |A|, \operatorname{tr}(Z\bar{Z}) \leqq M^2 + |\zeta|^2/M^2\}$$

where M is given by (3.8).

The hull of the $U(2)$ orbit of I (for which $|A| = 1$, $B = 2$, $M = 1$) is shown (in the parameter space) in Fig. 2.

Note in particular that $\hat{K}\setminus K$ is an open subset of $\mathbf{C}^3$: If $\zeta \in \Omega$, then

$$\mathscr{F}_{\hat{K}}(\zeta) = \{Z{:}2|\zeta|^2 \leqq \operatorname{tr}(Z\bar{Z}) \leqq M^2 + |\zeta|^2/M^2\}.$$

As noted in the proof of Lemma 6, $M \geqq |A|/M$, so if $|\zeta| < |A|$,

$$(M - |\zeta|/M)^2 > 0$$

and thus

$$M^2 - |\zeta|^2/M^2 > 2|\zeta|^2.$$

It is likely that these methods could be used to determine the hulls of more general sets invariant under the $SU(2)$ action (2.1).

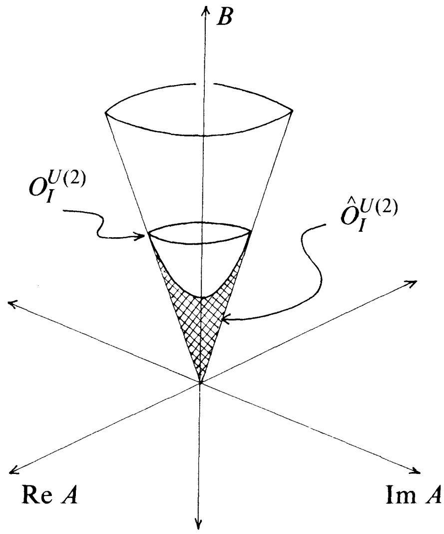

Figure 2

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