UNBOUNDED APPROXIMATE IDENTITIES IN NORMED ALGEBRAS

by P. G. DIXON

(Received 26 November, 1990)

1. Introduction. The object of this paper is to consider two easy propositions concerning bounded approximate identities and show that they do not extend to unbounded approximate identities. The propositions are as follows.

PROPOSITION 1.1. *Every bounded left approximate identity in a normed algebra is a left approximate identity for the completion.*

PROPOSITION 1.2. *Every bounded left approximate identity in a separable normed algebra has a subsequence which is a left approximate identity.*

In both cases, the proofs are easy and in both cases we shall give counterexamples to the proposition obtained by omitting the word "bounded". In fact, our counterexamples will be stronger; showing the absence of any left approximate identities of the desired type. Both counterexamples will be commutative and the second will be complete.

Let us begin by recalling some definitions. We say that a normed algebra A has a (*bounded*) *left approximate identity* if (there exists $K>0$ such that) for every finite set $\{x_1, x_2, \ldots, x_m\} \subseteq A$ and every $\varepsilon > 0$ there exists $u \in A$ with $\|x_i - ux_i\| < \varepsilon$ $(1 \leq i \leq m)$ (and $\|u\| \leq K$). We define *right* and *two-sided approximate identities* similarly and in commutative algebras we simply speak of *approximate identities*. If u can be selected from some fixed countable set, independent of $\{x_1, x_2, \ldots, x_m\}$ and ε, we say the left approximate identity is *countable*.

An equivalent view, used in the statements of the propositions above, is to regard an approximate identity as a net $(u_\lambda)_{\lambda \in \Lambda}$ in A with $u_\lambda x \to x$ for each $x \in A$. The approximate identity is bounded (countable) if $\{u_\lambda : \lambda \in \Lambda\}$ is bounded (countable). The approximate identity is *sequential* if $\Lambda = \mathbb{N}$. We shall see that an algebra can have a countable approximate identity without having a sequential approximate identity.

We say that a normed algebra A has *left approximate units* if, for every $x \in A$ and $\varepsilon > 0$, there exists $u \in A$ with $\|ux - x\| < \varepsilon$. It is known that a normed algebra has bounded left approximate units if and only if it has a bounded left approximate identity, (see [**3**, §9]), but it is not known whether the existence of unbounded approximate units implies the existence of an (unbounded) approximate identity.

We define the *multiplier norm* $\|.\|_M$ on a normed algebra A by

$$\|a\|_M = \sup\{\|ax\| : x \in A, \|x\| \leq 1\}.$$

Clearly $\|a\|_M \leq \|a\|$ $(a \in A)$, but it is not necessarily true that the two norms are equivalent. Indeed, the easiest way to manufacture examples of algebras with unbounded approximate identities but no bounded approximate identities is to take normed algebras with bounded approximate identities and modify the norms so that the approximate identities becomes unbounded in norm, while remaining bounded in multiplier norm. Such a construction may be seen in [**3**, Example (9.6)]. However, this technique is of limited use since most applications of bounded approximate identities, including the two

https://doi.org/10.1017/S0017089500008703 Published online by Cambridge University Press

propositions presently under consideration, hold under the weaker assumption that the approximate identity is bounded in multiplier norm. The proofs of Propositions 1.1 and 1.2, thus extended, are easy and are left to the reader.

REMARK 1.3. Given a left approximate identity in a separable normed algebra, we can easily deduce the existence of a *countable* left approximate identity, simply by approximating elements of the approximate identity by elements of the given countable dense subset; but this does not, in general, produce a *sequential* left approximate identity, as we shall see in Section 3.

2. Unbounded approximate identities in completions. We show that Proposition 1.1 does not generalize to unbounded approximate identities: indeed, we have the following slightly stronger result.

THEOREM 2.1. *There is a commutative separable normed algebra with an unbounded approximate identity whose completion does not have approximate units.*

Proof. Let A_0 be the commutative algebra with generators e_n, x_n $(n = 1, 2, 3, \ldots)$ and the following relations for $i, j \in \mathbb{N}$:

$$e_i e_j = e_{\min\{i,j\}},$$
$$e_i x_j = x_j \qquad \text{if } i \geq j,$$
$$x_i x_j = 0.$$

A typical element $x \in A_0$ has the form

$$x = \sum_i \alpha_i e_i + \sum_j \beta_j x_j + \sum_{i<j} \gamma_{ij} e_i x_j, \tag{1}$$

the sums being finite. To see this, i.e. to check that the multiplication of such expressions is associative, we note that:

(i) $e_i e_j e_k = e_{\min\{i,j,k\}}$, however the product is bracketed;

(ii) every product of e_i, e_j and x_k is

$$\begin{aligned} e_{\min\{i,j\}} x_k &\qquad \text{if } \min\{i,j\} < k \\ x_k &\qquad \text{if } \min\{i,j\} \geq k; \end{aligned}$$

(iii) every product containing two or more x's vanishes.

We define an algebra norm on A_0 by saying that if x is as in (1) then

$$\|x\| = \sum_i |\alpha_i|\, 2^i + \sum_j |\beta_j| + \sum_{i<j} |\gamma_{ij}|\, 2^i. \tag{2}$$

Then A_0 is a commutative normed algebra with a sequential unbounded approximate identity (e_i), since if x is as in (1) and n is greater than all the i, j occurring in the (finite) sums in (1), then $e_n x = x$.

The completion A of A_0 is the algebra of all x of the form (1) where the sums are infinite, subject to the restriction that $\|x\|$, as defined by formula (2), be finite. Consider the element $a \in A$ given by

$$a = \sum_{n=1}^{\infty} 2^{-n} x_n.$$

https://doi.org/10.1017/S0017089500008703 Published online by Cambridge University Press

We shall show that there is no $x \in A$ with $\|a - xa\| < 1$. Let $x \in A$ be as in (1), (with infinite sums and finite norm (2)). Then, using 1 to denote a formal identity element,

$$\begin{aligned}
\|a - xa\| &= \left\|\left(\sum_{n=1}^{\infty} 2^{-n}x_n\right)\left(1 - \sum_i \alpha_i e_i - \sum_j \beta_j x_j - \sum_{i<j} \gamma_{ij} e_i x_j\right)\right\| \\
&= \left\|\left(\sum_{n=1}^{\infty} 2^{-n}x_n\right)\left(1 - \sum_i \alpha_i e_i\right)\right\| \\
&= \left\| - \sum_{n=2}^{\infty} \sum_{i=1}^{n-1} 2^{-n}\alpha_i e_i x_n + \sum_{n=1}^{\infty} 2^{-n}\left(1 - \sum_{i=n}^{\infty} \alpha_i\right)x_n\right\| \\
&= \sum_{n=2}^{\infty} \sum_{i=1}^{n-1} |\alpha_i|\, 2^{i-n} + \sum_{n=1}^{\infty} 2^{-n} \left|1 - \sum_{i=n}^{\infty} \alpha_i\right| \\
&\geqslant \sum_{i=1}^{\infty} \sum_{n=i+1}^{\infty} |\alpha_i|\, 2^{i-n} + \sum_{n=1}^{\infty} 2^{-n}\left(1 - \sum_{i=n}^{\infty} |\alpha_i|\right) \\
&= \sum_{i=1}^{\infty} |\alpha_i| + 1 - \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} 2^{-n}\, |\alpha_i| \\
&= \sum_{i=1}^{\infty} |\alpha_i| + 1 - \sum_{i=1}^{\infty} (1 - 2^{-i})\, |\alpha_i| \\
&\geqslant 1.
\end{aligned}$$

Thus A does not have approximate units.

3. Sequential approximate identities. Our counterexample to the extension of Proposition 1.2 to the unbounded case consists of some simple observations about a sophisticated example of George Willis [**4**]. It gives us an example which is both commutative and complete.

THEOREM 3.1. *There is a commutative separable Banach algebra with a countable approximate identity and no sequential approximate identity.*

Proof. We use Willis's example ([**4**, Example 5]) of a commutative separable Banach algebra such that:

(a) the multiplier norm is equivalent to the original norm;
(b) null sequences factor; and
(c) there is no bounded approximate identity for A.

We need not explain the meaning of (b) except to remark that, by [**2**, Theorem 3.1], factorization of null sequences in a commutative separable normed algebra implies the existence of a (not necessarily bounded) approximate identity, and hence, by Remark 1.3 above, the existence of a countable approximate identity.

If there were a sequential approximate identity $(e_n)_{n=1}^{\infty}$ in A, then the collection of operators $T_n : x \mapsto e_n x$ ($n \in \mathbb{N}$) on A would be pointwise bounded and therefore uniformly bounded. Property (a) would then imply that $\sup_n \|e_n\| < \infty$, contradicting (c).

Notice that we needed to use Willis's Example 5, which has property (a), rather than his much less technical Example 3 which has all the desired properties except (a). In fact, it is easy to see that his Example 3 does have a sequential approximate identity.

https://doi.org/10.1017/S0017089500008703 Published online by Cambridge University Press

As a final remark, we note that this paper forms a corrigendum to [**1**] where, in the paragraph after Definition (2.2), it was erroneously asserted that the result of Proposition 1.2 above held without any boundedness condition on the approximate identity. The author apologises to readers of that paper.

REFERENCES

1. P. G. Dixon, Approximate identities in normed algebras II, *J. London Math. Soc.* (2), **17** (1978), 141–151.

2. P. G. Dixon, Factorization and unbounded approximate identities in Banach algebras, *Math. Proc. Camb. Philos. Soc.*, **107** (1990), 557–571.

3. R. S. Doran and J. Wichmann, *Approximate identities and factorization in Banach modules.* Lecture Notes in Mathematics **768,** (Springer, 1979).

4. G. Willis, Examples of factorization without bounded approximate units, *Proc. London Math. Soc.* to appear.

DEPT. OF PURE MATHEMATICS,
UNIVERSITY OF SHEFFIELD,
SHEFFIELD, S3 7RH,
ENGLAND.