The Irreducible Covariants belonging to the Concomitant System of Three Quadrics.

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INTRODUCTION.

In the following paper I give a complete list of the types of covariants belonging to the concomitant system of three quaternary quadrics, where covariant is used in its restricted sense and refers solely to a concomitant involving the variable x alone. A complete list of the types, 62 in number, is given in § 1. In §§ (6–10) the covariants are determined, and in §§ (11–12) a list of the identities used in the reduction of the covariants is given, along with typical examples of the process.

The paper is based on a paper of Professor H. W. Turnbull (*Proc. Lond. Math. Soc.*, 2, 20 (1921) 465–489), in which he has worked out a complete system for the concomitants of three quadrics. In his paper he gives a complete list of the bracket factors from combinations of which all the concomitants can be formed. I have simply found out all the possible covariants, and have then reduced as many as possible, that is have expressed them in terms of simpler ones. The following list contains all types of irreducible covariants or covariants which cannot be expressed in terms of simpler ones.

Types of the Covariants of Three Quadrics showing the degree in the co-efficients of the quadrics and the order in x.

The K_1 Group.

	a_x^2	(1, 0, 0)	2,

The K_2 Group.

	$({}_x a_\beta c_x)$	(1, 3, 1)	2,
	$({}_x a_\beta c_\alpha b_x)$	(4, 4, 1)	2,
	$(A\beta x)^2$	(2, 3, 0)	2,
(5)	$(A_\gamma B)(AB)$	(2, 2, 3)	2,
	$(A_\gamma B)(AC)(CB)$	(2, 2, 5)	2,
	$({}_\beta A_\gamma)({}_\beta c_\alpha b_\gamma)$	(5, 4, 4)	2,
	$(A_\beta)(B_\gamma)(AB)({}_\beta a_\gamma)$	(3, 5, 3)	2,
	$(A_\beta)(B_\gamma)(AC)(CB)({}_\beta a_\gamma)$	(3, 5, 5)	2,

(10)	$(A_\beta)(B_\gamma)(AB)({}_\beta c_\alpha b_\gamma)$	(5, 6, 4)	2,
	$(A_\beta)(B_\gamma)(AC)(CB)({}_\beta c_\alpha b_\gamma)$	(5, 6, 6)	2,
	$(A_\beta)(B_\gamma)(AB)({}_\beta c_x b_\gamma)$	(2, 6, 4)	4,
	$(A_\beta)(B_\gamma)(AB)({}_\beta a_x b_\gamma)$	(3, 6, 3)	4,
	$(C_\beta)(B_\gamma)(BC)({}_\beta a_\gamma)$	(1, 5, 5)	2,
(15)	$(C_\beta)(B_\gamma)(BC)({}_\beta c_\alpha b_\gamma)$	(3, 6, 6)	2,
	$(C_\beta)(B_\gamma)(BA)(AC)({}_\beta c_\alpha b_\gamma)$	(5, 6, 6)	2,
	$(C_\beta)(B_\gamma)(BC)({}_\beta c_x b_\gamma)$	(0, 6, 6)	4,
	$(C_\beta)(B_\gamma)(BC)({}_\beta a_x b_\gamma)$	(1, 6, 5)	4,
	$(A_\beta C_\alpha B)(AB)$	(5, 5, 2)	4,
(20)	$({}_\beta C_\alpha B)(A_\gamma)(AB)({}_\beta a_\gamma)$	(6, 5, 5)	4,

The K_3 Group.

	$(\alpha\beta\gamma x)({}_\beta a_x)({}_\gamma b_x)({}_\alpha c_x)$	(4, 4, 4)	4,
	$(\alpha\beta\gamma x)({}_\alpha b_\gamma)({}_\beta c_x)$	(3, 4, 4)	2,
	$(\alpha\beta\gamma x)(A\beta\gamma)(A\beta)({}_\alpha c_\beta)$	(5, 6, 4)	2,
	$(Bca)(B\alpha)(a_\beta c)({}_\alpha b_x)$	(4, 6, 1)	2,
(25)	$(Bca)(B\alpha)(a_\beta c)({}_\alpha c_x)$	(4, 5, 2)	2,
	$(Bca)(B\alpha)(a_\beta c)({}_\alpha b_\gamma a_x)$	(5, 6, 4)	2,
	$(Bca)(B\alpha)(a_\gamma b_\alpha)c_x$	(4, 3, 4)	2,
	$(Bca)(B\alpha)c_\alpha a_x$	(4, 2, 1)	2,
	$(Bca)(B\alpha)c_\alpha(a_\gamma b_x)$	(4, 3, 4)	2,
(30)	$(Bca)(B\alpha)(a_x c)\,{}_\alpha b_x)$	(4, 3, 1)	4,
	$(Bca)(A\gamma)(BA)(c_\beta a)({}_\gamma b_x)$	(3, 6, 4)	2
	$(Bca)(A\gamma)(BC)(CA)(c_\beta a)({}_\gamma b_x)$	(3, 6, 6)	2,
	$(Bca)(A\gamma)(BA)(c_\alpha b_\gamma)a_x$	(6, 3, 4)	2,
	$(Bca)(A\gamma)(BC)(CA)(c_\alpha b_\gamma)a_x$	(6, 3, 6)	2,
(35)	$(Bca)(A\gamma)(BA)a_\gamma c_x$	(3, 2, 4)	2,
	$(Bca)(A\gamma)(BC)(CA)a_\gamma c_x$	(3, 2, 6)	2,
	$(B\alpha\gamma)(B\alpha)({}_\gamma a_x)$	(4, 2, 3)	2,
	$(B\alpha\gamma)(A\gamma)(BA)({}_\alpha c_x)$	(5, 2, 4)	2,
	$(B\alpha\gamma)(A\gamma)(BA)({}_\alpha b_\gamma a_x)$	(6, 3, 6)	2,
(40)	$(B\alpha\gamma)(A\gamma)(BA)({}_\alpha b_\gamma a_\beta c_x)$	(6, 6, 7)	2,
	$(B\alpha\gamma)(A\beta)(BA)({}_\beta a_\gamma)({}_\alpha c_x)$	(6, 5, 4)	2,
	$(B\alpha\gamma)(A\beta)(BC)(CA)({}_\beta a_\gamma)({}_\alpha c_x)$	(6, 5, 6)	2,
	$(Bac)(\beta\alpha\gamma)({}_\alpha b_\gamma)(a_x c)$	(4, 3, 4)	2,
	$(Bac)(B\alpha\gamma)({}_\alpha B_\gamma)(a_\beta c)$	(4, 7, 4)	2,

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(45)	$(Bac)(Cab)(BC)(b_x c)$	(1, 3, 3)	2,
	$(Bac)(Cab)(BA)(AC)(b_x c)$	(3, 3, 3)	2,
	$(Bac)(Cab)(BC)(b_x a_\beta c)$	(2, 6, 3)	2,
	$(Bac)(Cab)({}_\gamma A_\beta)(BC)b_\gamma c_\beta$	(3, 6, 6)	2,
	$(Bac)(Cab)(A_\gamma B)(AC)(b_x c)$	(3, 3, 6)	4,
(50)	$(Bac)(Cab)(B\gamma)(C\alpha)({}_\gamma a_x b)c_\alpha$	(5, 3, 6)	4,
	$(B\alpha\gamma)(C\alpha\beta)({}_\gamma A_\beta)(BC)$	(5, 5, 5)	2,
	h_1^2	(1, 2, 2)	2,
	$h_1(BC)a_x$	(1, 2, 2)	2,
	$h_1(BA)(AC)a_x$	(3, 2, 2)	2,
(55)	$h_1(B_\alpha C)a_x$	(4, 2, 2)	4,
	$h_1(Bac)(B\alpha)(BC)c_\alpha$	(4, 4, 3)	2.
	$h_1(B\alpha\gamma)(C\alpha)a_\gamma$	(4, 2, 5)	2,
	$h_1(B\alpha\gamma)(B\alpha)(BC)a_\gamma$	(4, 4, 5)	2,
	$h_1(B\alpha\gamma)(B\alpha)(BA)(AC)a_\gamma$	(6, 4, 5)	2,
(60)	$h_1(Bac)(Cab)\, a_x b_x c_x$	(2, 3, 3)	4,
	$h_1 h_2 h_3 a_x b_x c_x$	(3, 3, 3)	6,
	k^2	(2, 2, 2)	2.

§ (2) The corollary follows:—*The system of contravariants is correlative with this list of covariants and is immediately deducible by the principle of duality* (§ 13).

§ (3) NOTATION.—In symbolic form let the point, plane, and line equations of the three quadrics be :—

$$f_1 = a_x^2 = a_x'^2 \dots, \quad \phi_1 = u_\alpha^2 = u_{\alpha'}^2 \dots,$$
$$f_2 = b_x^2 = b_x'^2 \dots, \quad \phi_2 = u_\beta^2 = u_{\beta'}^2 \dots,$$
$$f_3 = c_x^2 = c_x'^2 \dots, \quad \phi_3 = u_\gamma^2 = u_{\gamma'}^2 \dots,$$

and,

$$\pi_1 = (A_{12}p_{34} + A_{13}p_{42} + A_{14}p_{23} + A_{34}p_{12} + A_{42}p_{13} + A_{23}p_{14})^2 = (Ap)^2,$$
$$\pi_2 = (B_{12}p_{34} + B_{13}p_{42} + B_{14}p_{23} + B_{34}p_{12} + B_{42}p_{13} + B_{23}p_{14})^2 = (Bp)^2,$$
$$\pi_3 = (C_{12}p_{34} + C_{13}p_{42} + C_{14}p_{23} + C_{34}p_{12} + C_{42}p_{13} + C_{23}p_{14})^2 = (Cp)^2,$$

where $a_x = \Sigma a_i x_i$, $u_\alpha = \Sigma u_i \alpha_i = (u\alpha)$, $i = 1, 2, 3, 4$.

$A_{ij} = (a_1 a_2)_{ij} = (a_{1i} a_{2j} - a_{1j} a_{2i})$ or briefly $A = (a_1 a_2)$.

$\alpha_{ijk} = (a_1 a_2 a_3)_{ijk} = (a_{1i} a_{2j} a_{3k})$ or briefly $\alpha = (a_1 a_2 a_3)$.

$\alpha = (Aa_3) = (a_3 A)$ etc., with similar meanings for B, β, C, γ and dashed letters.

§(4). All possible covariants are products P of factors chosen from the following four groups of factors F_1, F_2, F_3 and F_4.

(1) Three of type $F_1 : a_x, b_x, c_x$.

(2) Fifteen of type $F_2 : a_\beta, a_\gamma, b_\gamma, b_\alpha, c_\alpha, c_\beta, (A\beta x), (A\gamma x), (B\gamma x), (B\alpha x), (C\alpha x), (C\beta x), (AB), (BC), (CA)$.

(3) Eleven of type $F_3 : (\alpha\beta\gamma x), (Abc), (Bca), (Cab), (A\beta\gamma), (B\gamma\alpha), (C\alpha\beta), h_1, h_2, h_3, k$.

(4) Three of type $F_4 : (BCa\alpha) = F_4$, $(CAb\beta) = F_4'$, $(ABc\gamma) = F_4''$ where $h_1 = (BCxa) = (b_2Ca)\, b_{1x} - (b_1Ca)\, b_{2x} = \Omega_B(b_2Ca)\, b_{1x}$

$$k = (ABCxx) = (Ab_1\, c_1)\, b_{2x}\, c_{2x} - (Ab_2c_1)\, b_{1x}\, c_{2x}$$
$$+ (Ab_2\, c_2)\, b_{1x}\, c_{1x} - (Ab_1c_2)\, b_{2x}\, c_{1x} = \Omega_{BC}(Ab_1\, c_1)\, b_{2x}\, c_{2x},$$

$F_4 = (BCa\alpha) = \Omega_B(b_1\, Ca)\, b_{2\alpha}$ the rest being of type $(abcd)$.

Definition of chain symbols :—

$$(a_\beta c_\alpha b) = a_\beta c_\beta c_\alpha b_\alpha\,,\ (A_\beta C_\alpha B) = (A\beta x)(C\beta x)(C\alpha x)(B\alpha x)$$

and $(A, B) = (AC)(CB)$ or (AB). $(A\beta)$ or (A_β) is used for $(A\beta x)$.

§(5). The notation $F \equiv \phi$ means that $F - \phi$ is reducible, *i.e.* can be expressed in terms of simpler forms.

A form is reduced, (1) when its currency is raised, (2) if F_i factors are present and no F_{i+1} factors are present when the number of F_i factors is decreased, (3) if the number of F_i factors remain the same, when the maximum number of symbols A, B, C convolved in one F_i factor is decreased.

§(6). Throughout this paper the covariants have been considered in order of simplicity following the order of Professor Turnbull in his paper. They are divided into four groups K_1, K_2, K_3, K_4 where any group K_i contains at least one factor F_i and no factor F_k where $k > i$.

If MN is a covariant where N is a product of F_1 and F_2 factors, then N contains an even or an odd number of factors of type $(A\beta x)$ according as M contains an even or an odd number of capital letters. For the total number of capital letters in any product must be even, and the only two types of F_2 factors involving capital letters are (AB) and $(A\beta x)$. When finding the possible forms of N in any particular case, the covariants were graded according to the number of brackets of type $(A\beta x)$ appearing in N.

§(7). K_1 Group. The only covariants belonging to this group are the quadrics themselves of which a_x^2 is typical.

§(8). K_2 Group. The brackets which may appear in a covariant in this group are the fifteen F_2 and the three F_1 factors. There must be present in such a covariant no factors, two, four or six of type $(A\beta x)$.

No factors of type $(A\beta x)$.

The only two types of covariants are :—$({}_x a_\beta c_x)$, $({}_x a_\gamma b_\alpha c_x)$.

Two factors of type $(A\beta x)$.

These factors may be :—(1) $(A\beta x)^2$, (2) $(B_\gamma A)$, (3) $({}_\beta A_\gamma)$, (4) $(A_\beta)(B_\gamma)$, (5) $(C_\beta)(B_\gamma)$.

The types of covariants are :—

(1) $(A\beta x)^2$.

(2) $(B_\gamma A)(BA)$, $(B_\gamma A)(BC)(CA)$.

(3) $({}_\beta A_\gamma)({}_\beta a_\gamma)$,* $({}_\beta A_\gamma)({}_\beta c_\alpha b_\gamma)$, $({}_\beta A_\gamma)({}_\beta a_x b_\gamma)$,* $({}_\beta A_\gamma)({}_\beta c_x b_\gamma)$,* $({}_\beta A_\gamma)({}_\beta a_x c_\alpha b_\gamma)$ *

(4) $(A_\beta)(B_\gamma)(A, B)({}_\beta a_\gamma)$, $(A_\beta)(B_\gamma)(A, B)({}_\beta c_\alpha b_\gamma)$, $(A_\beta)(B_\gamma)(A, B)({}_\beta a_x b_\gamma)$,** $(A_\beta)(B_\gamma)(A, B)({}_\beta c_x a_\gamma)$,* $(A_\beta)(B_\gamma)(A, B)({}_\beta c_\alpha b_x a_\gamma)$,* $(A_\beta)(B_\gamma)(A, B)({}_\beta a_x c_\alpha b_\gamma)$,* $(A_\beta)(B_\gamma)(A, B)({}_\beta c_x b_\gamma)$.**

(5) $(C_\beta)(B_\gamma)(C, B)({}_\beta a_\gamma)$,** $(C_\beta)(B_\gamma)(C, B)({}_\beta c_\alpha b_\gamma)$, $(C_\beta)(B_\gamma)(C, B)({}_\beta a_x b_\gamma)$,** $(C_\beta)(B_\gamma)(C, B)({}_\beta a_x c_\alpha b_\gamma)$,* $(C_\beta)(B_\gamma)(C, B)({}_\beta c_x b_\gamma)$.**

Four factors of type $(A\beta x)$.

These factors may be :—

(1) $(A_\beta C_\alpha B)$, (2) $({}_\beta C_\alpha B_\gamma)$, (3) $({}_\beta C_\alpha B)(A_\gamma)$, (4) $({}_\beta C_\alpha)(B_\gamma A)$.

The types of covariants are :—

(1) $(A_\beta C_\alpha B)(AB)$ only, since $(A_\beta C_\alpha B)(AC)(CB)$ would split up into factors.

(2) $({}_\beta C_\alpha B_\gamma)\ M$ where $M = ({}_\beta a_\gamma)$,* $({}_\beta c_x b_\gamma)$,* $({}_\beta c_x a_\gamma)$.*

(3) $({}_\beta C_\alpha B)(A_\gamma)(AB)\ M$ where $M = ({}_\beta a_\gamma)$, $({}_\beta a_x b_\gamma)$,* $({}_\beta c_x b_\gamma)$,* $({}_\beta c_x a_\gamma)$.*

(4) None, because of the factor $(B, A)(B_\gamma A)$.

Six factors of type $(A\beta x)$.

There is only one covariant $(A_\beta C_\alpha B_\gamma A)$.*

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* The covariant is reducible.

** The covariant is reducible unless (A, B), (B, C), $(C, A) = (AB)$, (BC) (CA) respectively.

§(9). K_3 GROUP. Let N denote throughout a product of F_1 and F_2 factors. The F_3 brackets which enter into the consideration of covariants are of the following types†:—A $(\alpha\beta\gamma x)$, B (Abc) with its dual $(A\beta\gamma)$, C $h_1 = (BCax)$, D $k = (ABCxx)$.

A From the table, *loc. cit.* 484, any covariant in this group must be one of two types:—

$$(\alpha) \quad (\alpha\beta\gamma x)\, N, \qquad\qquad (\beta) \quad (\alpha\beta\gamma x)(A\beta\gamma)\, N.$$

Type (α). $(\alpha\beta\gamma x)$ N. N may contain any F_1 or F_2 factors except (AB), (BC), (CA) ‡. It may contain no factors, two or four of type $(A\beta x)$.

No factors of type $(A\beta x)$.

The types of covariants are:—

$$(\alpha\beta\gamma x)({}_\beta a_x)({}_\gamma b_x)({}_\alpha c_x), \quad (\alpha\beta\gamma x)({}_\alpha b_\gamma)({}_\beta c_x).$$

Two factors of type $(A\beta x)$.

The factors must be $({}_\beta A_\gamma)$ and the types of covariants are:—

$(\alpha\beta\gamma x)({}_\beta A_\gamma)\, M$ where $M = ({}_\alpha c_x)$,* $({}_\alpha c_\beta\, a_x)$,* $({}_\alpha c_\beta\, a_\gamma\, b_x)$.*

Four factors of type $(A\beta x)$.

The factors must be $({}_\beta A_\gamma B_\alpha)$ and the only type of covariant is

$$(\alpha\beta\gamma x)({}_\beta A_\gamma B_\alpha)({}_\gamma b_x).*$$

Type (β). $(\alpha\beta\gamma x)(A\beta\gamma)\, N$. N may not contain a_x, (AB), (BC) or (CA)‡ but may have one, three or five factors of type $(A\beta x)$.

One factor of type $(A\beta x)$.

The factor must be $(A\beta)$ and the types of covariants are:—

$(\alpha\beta\gamma x)(A\beta\gamma)(A\beta)\, M$ where $M = ({}_\alpha c_\beta)$, $({}_\alpha b_\gamma\, a_\beta)$,* $({}_\alpha c_x b_\gamma\, a_\beta)$,* $({}_\alpha b_x c_\beta)$.*

Three factors of type $(A\beta x)$.

These factors must be (1) $(A_\gamma B_\alpha)$, (2) $(A_\beta)({}_\alpha B_\gamma)$. The types of covariants are:—

(1) $(\alpha\beta\gamma x)(A\beta\gamma)(A_\gamma B_\alpha)$.*

(2) $(\alpha\beta\gamma x)(A\beta\gamma)(A_\beta)({}_\alpha B_\gamma)\, M$ where $M = ({}_\beta a_\gamma)$,* $({}_\beta c_\alpha b_\gamma)$* and $({}_\beta c_x b_\gamma)$*

Five factors of type $(A\beta x)$.

These factors must be $(A_\beta C_\alpha B_\gamma)$ and each Covariant of this type has the factor $(\alpha\beta\gamma x)(A\beta\gamma)(A_\beta C_\alpha)$.

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† *Cf. Proc. L.M.S., loc. cit.*, 488.

‡ *Cf. loc. cit.* 477.

B This group can be divided (*loc. cit.* 485) into (α) $(Bac)N$, (β) $(B\alpha\gamma)N$, (γ) $(Bac)(C\alpha\beta)N$, (δ) $(Bac)(B\alpha\gamma)N$, (ϵ) $(Bac)(Cab)N$, (ζ) $(B\alpha\gamma)(C\alpha\beta)N$.

Type α. $(Bac)N$. N may contain any of the F_1 and F_2 factors and must have one, three or five factors of type $(A\beta x)$.

One factor of type $(A\beta x)$.

This factor may be (1) $(B\alpha)$, (2) $(A\gamma)$, (3) $(A\beta)$. The types of covariants are:—

(1) $(Bca)(B\alpha)M$ where $M = (a_\beta c)({}_\alpha b_x)$, $(a_\beta c)({}_\alpha c_x)$, $c_\alpha a_x$, $(a_\beta c)({}_\alpha b_\gamma a_x)$, $(a_\gamma b_\alpha)c_x$, $c_\alpha(a_\gamma b_x)$, $(a_x c)({}_\alpha b_x)$.

(2) $(Bca)(A\gamma)(B, A)M$ where $M = (c_\beta a)({}_\gamma b_x)$, $(c_\beta a)({}_\gamma a_x)$.* $a_\gamma c_x$, $a_\gamma(c_\alpha b_x)$,* $(c_\beta a)({}_\gamma b_\alpha c_x)$,* $(c_\alpha b_\gamma)a_x$, $(a_x c)({}_\gamma b_x)$.*

(3) $(Bca)(A\beta)(B, A)M$ where $M = (c_\beta a_x)$,* $c_\beta(a_\gamma b_x)$,* $c_\beta(a_\gamma b_\alpha c_x)$* since $(Bca)(A\beta)a_\beta \equiv 0$.

Three factors of type $(A\beta x)$.

These factors may be (1) $({}_\gamma B_\alpha)(C_\beta)$, (2) $({}_\gamma B_\alpha C)$, (3) $({}_\beta A_\gamma B)$, (4) $({}_\gamma A_\beta)(B_\alpha)$, (5) $({}_\gamma A_\beta C)$, (6) $({}_\gamma A_\beta)(C_\alpha)$, (7) $(B_\gamma)(C_\alpha)(A_\beta)$, (8) $(B_\gamma A)(C_\alpha)$, (9) $(B_\gamma)(A_\beta C)$, (10) $(B_\gamma A)(C_\beta)$.

In cases (3), (4), (5), (6), (7) and (9), a_β may not occur, and in cases (1), (5), (9) and (10) c_β may not occur. In case (5) (B, C) must be (BC), and in cases (8) and (10) (A, C) must be (AC). Any covariants of types (2) and (9) involve the factors $(B_\alpha C)(B, C)$ and $(A_\beta C)(A, C)$ respectively.

Since $({}_\alpha B_\gamma)({}_\alpha b_\gamma)$, $({}_\alpha B_\gamma)({}_\alpha b_x)$, $({}_\alpha B_\gamma)({}_\gamma b_x)$ are all reducible, the types of covariants are:—

(1) $(Bca)({}_\gamma B_\alpha)(C_\beta)(B,C)a_\beta c_\alpha({}_\gamma a_x)$.*

(3) $(Bca)(B_\gamma A_\beta)M$ where $M = c_\beta a_x$*, $c_\beta(a_\gamma b_x)$*, $c_\beta(a_\gamma b_\alpha c_x)$.*

(4) $(Bca)({}_\gamma A_\beta)(B_\alpha)M$ where $M = c_\beta a_\gamma({}_\alpha b_x)$,* $c_\beta a_\gamma({}_\alpha c_x)$*, $c_\beta a_x({}_\alpha b_\gamma)$*, $c_\beta(a_x c_\alpha)({}_\gamma b_x)$*

(5) $(Bca)({}_\gamma A_\beta C)(BC)M$ where $M = a_\gamma c_x$,*$a_\gamma(c_\alpha b_x)$*, $a_x(c_\alpha b_\gamma)$,* $(a_x c)({}_\gamma b_x)$.*

(6) $(Bca)({}_\gamma A_\beta)(C_\alpha)(B,C)M$ where $M = c_\beta a_\gamma({}_\alpha b_x)$,*$c_\beta a_\gamma({}_\alpha c_x)$,* $c_\beta a_x({}_\alpha b_\gamma)$*, $c_\beta a_x({}_\gamma b_x c_\alpha)$.*

(7) $(Bca)(B_\gamma)(C_\alpha)(A_\beta)(A,C)M$ where $M = c_\beta a_\gamma({}_\alpha b_x)$*, $c_\beta a_\gamma({}_\alpha c_x)$*, $c_\beta a_x({}_\alpha b_\gamma)$*, $c_\beta({}_\gamma b_x)({}_\alpha c_x a)$*.

(8) $(Bca)(B_\gamma A)(C_\alpha)(AC)M$ where $M = (a_\beta c)({}_\alpha b_x)$*, $(a_\beta c)({}_\alpha c_x)$,* $(a_\beta c)({}_\alpha b_\gamma a_x)$*, $(a_\gamma b_\alpha)c_x$*, $c_\alpha a_x$*, $(a_x c)({}_\alpha b_x)$*, $c_\alpha(a_\gamma b_x)$.*

(10) $(Bca)(B_\gamma A)(C_\beta)(AC)M$ where $M = a_\beta c_x$,* $a_\beta(c_\alpha b_x)$ *

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Five factors of type $(A\beta x)$.

These factors may be (1)$(B_\gamma A_\beta C_\alpha)$, (2)$(A_\beta C_\alpha B_\gamma)$, (3)$(A_\gamma B_\alpha C_\beta)$.

Since any covariant of type (2) or (3) factorises the types of covariants are:—

$(Bac)(B_\gamma A_\beta C_\alpha)$ M where $M = a_x c_\alpha$,* $(a_\gamma b_x)$ c_α,* $c_x(a_, b_\alpha)$,* $(a_x c)({}_\alpha b_x)$.*

Type (β) . $(\beta\alpha\gamma) N$. Since $(\beta\alpha\gamma)$ $b_x \equiv 0$, N may not contain b_x but it may contain any other F_1 or F_2 factors. The factors of type $(A\beta x)$ are the same in this case as for (Bac) N.

One factor of type $(A\beta x)$. The types of covariants are:—

(1) $(B\alpha\gamma)(B\alpha)M$ where $M = ({}_\gamma a_x)$, $({}_\gamma a_\beta c_x)$,* $({}_\gamma b_\alpha c_x)$,* $({}_\gamma b_\alpha c_\beta a_x)$.*

(2) $(B\alpha\gamma)(A\gamma)(A, B) M$ where $M = ({}_\alpha c_x)$ ** $({}_\alpha c_\beta a_x)$,* $({}_\alpha b_\gamma a_x)$,** $({}_\alpha b_\gamma a_\beta c_x)$.**

(3) $(B\alpha\gamma)(A\beta)(A, B) M$ where $M = ({}_\alpha c_\beta)({}_\gamma a_x)$,* $({}_\beta a_\gamma)({}_\alpha c_x)$, $({}_\gamma b_\alpha)({}_\beta c_x)$,* $({}_\gamma b_\alpha)({}_\beta a_x)$.*

Three factors of type $(A\beta x)$.

Since $(B\alpha\gamma)(B\gamma) C\beta)(B, C) \equiv 0$ the types of covariants are:

(3) $(B\alpha\gamma)(B_\gamma A_\beta) M$ where $M = ({}_\alpha c_\beta)({}_\gamma a_x)$,* $({}_\beta a_\gamma)({}_\alpha c_x)$,* $({}_\gamma b_\alpha)({}_\beta c_x)$,* $({}_\gamma b_\alpha)({}_\beta a_x)$.*

(4) $(B\alpha\gamma)({}_\gamma A_\beta)(B_\alpha)({}_\beta c_x)$.*

(5) $(B\alpha\gamma)({}_\gamma A_\beta C)(BC) M$ where $M = ({}_\alpha c_x)$,* $({}_\alpha c_\beta a_x)$,* $({}_\alpha b_\gamma a_x)$,* $({}_\alpha b_\gamma a_\beta c_x)$.*

(6) $(B\alpha\gamma)({}_\gamma A_\beta)(C\alpha)(B, C) M$ where $M = ({}_\beta a_x)$,* $({}_\beta c_x)$,* $({}_\beta c_\alpha b_\gamma a_x)$,* $({}_\beta a_\gamma b_\alpha c_x)$.*

(7) $(B\alpha\gamma)(B\gamma)(C\alpha)(A\beta)(A, C) M$ where $M = ({}_\beta a_x)$,* $({}_\beta c_x)$.*

(8) $(B\alpha\gamma)(B_\gamma A)(C\alpha)(AC) M$ where $M = ({}_\gamma a_x)$,* $({}_\gamma a_\beta c_x)$.*

(10) $(B\alpha\gamma)(B_\gamma A)(C\beta)(AC) M$ where $M = ({}_\alpha c_\beta)({}_\gamma a_x)$,* $({}_\beta a_\gamma)({}_\alpha c_x)$,* $({}_\gamma b_\alpha)({}_\beta c_x)$,* $({}_\gamma b_\alpha)({}_\beta a_x)$.*

Five factors of type $(A\beta x)$.

The types of covariants are:—

(1) $(B\alpha\gamma)(B_\gamma A_\beta C_\alpha) M$ where $M = ({}_\gamma a_x)$,* $({}_\gamma a_\beta c_x)$,* $({}_\gamma b_\alpha c_x)$,* $({}_\gamma b_\alpha c_\beta a_x)$.*

Type (γ). $(Bac)(C\alpha\beta) N$. The following are some reductions

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which considerably shorten the work in this section. They are at once deducible from the formulae in § (11).

$$(Bac)(Ca\beta)c_\beta \equiv 0, \quad (Bac)(C\alpha\beta)(A\gamma) \equiv 0,$$
$$(Bac)(C\alpha\beta)(A\beta) \equiv 0, \quad (Bac)(Ca\beta)(C\beta)c_\alpha \equiv 0.$$

N must therefore be formed from the factors $a_x, b_x, a_\beta, c_\alpha, b_\alpha, b_\gamma, a_\gamma, (B\gamma), (B\alpha)$, and $(C\alpha)$. $(C\beta)$ is impossible as then the c in (Bca) could not be paired off. N may have no factors of type $(A\beta x)$ or two factors of that type.

No factors of type $(A\beta x)$.

There are no covariants of this type.

Two factors of type $(A\beta x)$.

These factors may be, (1) $({}_\alpha B_\gamma)$, (2) $(B_\alpha C)$, (3) $(B_\gamma)(C_\alpha)$. The types of covariants are :—

(1) $(Bac)(C\alpha\beta)({}_\gamma B_\alpha)(B,C)\ M$ where $M = a_\beta(c_\alpha b_\gamma)$,* $a_\beta(c_\alpha b_x a_\gamma)$.*

(2) $(Bac)(C\alpha\beta)(B_\alpha C)\ a_\beta c_\alpha$.*

(3) $(Bac)(C\alpha\beta)(B_\gamma)(C_\alpha)\ M$ where $M = a_\beta(c_\alpha b_\gamma)$,* $a_\beta(c_\alpha\ b_x a_\gamma)$.*

Type δ. $(Bac)(B\alpha\gamma)N$. The following reductions considerably shorten the work in this section.

$$(Bac)(C\beta)c_\beta \equiv 0,\ (Bac)(A\beta)a_\beta \equiv 0,\ (Bac)(B\alpha\gamma)(A\gamma)(C\alpha) \equiv 0,$$
$$(Bac)(B\alpha\gamma)(A\gamma)(B\alpha) \equiv 0,\ (Bac)(B\alpha\gamma)(C\alpha)(B\gamma) \equiv 0.$$

N may contain any F_1 or F_2 factor, except b_x and must have no factors, two or four of type $(A\beta x)$.

No factors of type $(A\beta x)$.

The types of covariants are :—

$(Bac)(B\alpha\gamma)(a_x c)({}_\alpha b_\gamma)$, $(Bac)(B\alpha\gamma)(a_x c)({}_\alpha c_\beta a_\gamma)$.*

Two factors of type $(A\beta x)$.

These factors may be, (1) $({}_\alpha B_\gamma)$, (2) $({}_\alpha C_\beta)$, (3) $(B_\alpha)(C_\beta)$, (4) $(B_\alpha C)$, (5) $(B_\alpha)(A_\beta)$, (6) $(C_\alpha)(A_\beta)$, (7) $(C_\beta A)$.

Any covariant of types (4) or (7) has a factor

$$(B_\alpha C)(B,C) \text{ or } (C_\beta A)(C,A).$$

The types of covariants are :—

(1) $(Bac)(B\alpha\gamma)({}_\alpha B_\gamma)\ M$ where $M = (a_\beta c)$, $(a_x c)$,* $(a_\gamma b_\alpha c)$.*

(2) $(Bac)(B\alpha\gamma)({}_\alpha C_\beta)\ M$ where $M = a_\beta({}_\gamma a_x c)$,* $a_\beta({}_\gamma b_\alpha c)$.*

(3) $(Bac)(B\alpha\gamma)(B_\alpha)(C_\beta)(B,C)\ M$ where $M = a_\beta({}_\gamma a_x c)$,* $a_\beta({}_\gamma b_\alpha c)$.*

(5) $(Bac)(B\alpha\gamma)(B_\alpha)(A_\beta)(B,A)\ c_\beta(a,\gamma)$.*

(6) $(Bac)(B\alpha\gamma)(C_\alpha)(A_\beta)(A,C)\ M$ where $M = c_\beta a_\gamma$,* $c_\beta(a_x c_\alpha b_\gamma)$.*

Four factors of type $(A\beta x)$.

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These factors may be, (1) $(B_\gamma A_\beta C)$, (2) $({}_\alpha B_\gamma)(A_\beta C)$, but any covariant of either of those types involves a factor.

Type (ϵ). $(Bac)(Cab)N$. N may contain any of the F_1 or F_2 factors but many of the covariants in this section reduce because of identities of the three types.

$(Bac)(A_\beta)\,a_\beta \equiv 0$, $(Bac)(Cab)(B\alpha)\,c_\beta \equiv 0$, $(Bac)(Cab)(C_\beta)\,b_\gamma \equiv 0$.

N may have no factors, two or four of type $(A\beta x)$.

No factors of type $(A\beta x)$.

The types of covariants are :—

$(Bac)(Cab)(B,C)(b_x c)$, $(Bac)(Cab)(B,C)(b_x a_\beta c)$.**

Two factors of type $(A\beta x)$.

Since $(Bac)(Cab)(B\gamma)(C\beta)\equiv 0$ the factors may be (1) $({}_\beta A_\gamma)$, (2) $({}_\gamma B_\alpha)$, (3) $(A\gamma)(B\alpha)$, (4) $(A_\gamma B)$, (5) $(A\gamma)(C\alpha)$, (6) $(A\gamma)(C\beta)$, (7) $(B\gamma)(C\alpha)$, (8) $(B_\alpha C)$.

The types of covariants are :—

(1) $(Bac)(Cab)({}_\beta A_\gamma)(B,C)\,b_\gamma c_\beta$.**

(2) $(Bac)(Cab)({}_\gamma B_\alpha)(B,C)$ M where $M = ({}_\gamma a_x b)\,c_\alpha$,* $({}_\gamma a_x c)\,b_\alpha$.*

(3) $(Bac)(Cab)(A_\gamma B)(A\,C)$ M where $M = (b_\alpha c)$,* $(b_x c)$, $(b_x a_\beta c)$.*

(4) $(Bac)(Cab)(A\gamma)(B\alpha)(A,C)$ M where $M = b_\gamma c_\alpha$,* $b_\gamma({}_\alpha b_x c)$,* $b_\gamma({}_\alpha b_x a_\beta c)$.*

(7) $(Bac)(Cab)(B\gamma)(C\alpha)$ M where $M = ({}_\gamma a_x b)\,c_\alpha$, $({}_\gamma a_x c)\,b_\alpha$.*

(8) $(Bac)(Cab)(B_\alpha C)$ M where $M = (b_\alpha c)$,* $(b_x c)$,* $(b_x a_\beta c)$.*

Four factors of type $(A\beta x)$.

These factors may be, (2) $({}_\gamma A_\beta C_\alpha)$, (4) $(A_\beta C_\alpha B)$, (6) $({}_\gamma B_\alpha C)(A_\beta)$, (7) $(C_\beta A_\gamma)(B\alpha)$, (9) $(C_\alpha B)(A_\gamma)$.

All the covariants of these types reduce either because they involve factors or because of the three identities quoted above.

Type (ξ). $(B\alpha\gamma)(C\alpha\beta)N$. N may contain any F_1 or F_2 factors except b_x and c_x and may have no factors, two or four of type $(A\beta x)$.

No factors of type $(A\beta x)$.

There are no covariants of this type.

Two factors of type $(A\beta x)$.

Since $(B\alpha\gamma)(C\alpha\beta)(B_\alpha C)\equiv 0$ the factors are the same as for $(Bac)(Cab)N$ where $(C\beta)(B\gamma)$ takes the place of $(B_\alpha C)$ in (8).

The types of covariants are :—

(1) $(B\alpha\gamma)(C\alpha\beta)({}_\gamma A_\beta)(B, C)$.**

(2) $(B\alpha\gamma)(C\alpha\beta)({}_\gamma B_\alpha)(B, C)(\alpha, \beta, \gamma)$.*

(3) $(B\alpha\gamma)(C\alpha\beta)(A\gamma)(B\alpha)(A, C)M$, where $M = ({}_\beta c_\alpha)$,* $({}_\beta a_\gamma b_\alpha)$.*

(4) $(B\alpha\gamma)(C\alpha\beta)(A\gamma B)(AC)M$, where $M = ({}_\beta a_\gamma)$,* $({}_\beta c_\alpha b_\gamma)$,*

(5) $(B\alpha\gamma)(C\alpha\beta)(A\gamma)(C\alpha)(A, B)M$, where $M = ({}_\alpha c_\beta)$,* $({}_\alpha b_\gamma a_\beta)$.

(6) $(B\alpha\gamma)(C\alpha\beta)(A\gamma)(C\beta)(A, B)$.*

(7) $(B\alpha\gamma)(C\alpha\beta)(B\gamma)(C\alpha)M$ where $M = ({}_\alpha c_\beta)$,* $({}_\alpha b_\gamma a_\beta)$.*

(8) $(B\alpha\gamma)(C\alpha\beta)(B\gamma)(C\beta)$.*

Four factors of type $(A\beta x)$.

Because type (8) above reduces thes factors may be

(1) $({}_\alpha C_\beta A_\gamma)$, (2) $(C_\beta A_\gamma)(B_\alpha)$.

The types of covariants are :—

(1) $(B\alpha\gamma)(C\alpha\beta)({}_\alpha C_\beta A_\gamma)(B, C)(\alpha, \beta)$.*

(2) $(B\alpha\gamma)(C\alpha\beta)(C_\beta A_\gamma)(B\alpha)$ M where $M = ({}_\alpha c_\beta)$,*$({}_\alpha b_\gamma a_\beta)$.*

$C\ h_1 = (BCxa)$.

From *loc. cit.* 488, we see that possible covariants with an h factor are : —

(a) $h_1 h_2 h_3 a_x b_x c_x$, (b) $h_1 h_2 (a_x b)(A, B)$,

(c) h_1^2, (d) $h_1 (B\gamma\alpha)(C\alpha\beta)({}_\gamma a_\beta) a_x$,*

(e) $h_1 (Bac)(Cab) a_x b_x c_x$, (f) $h_1 (B\gamma\alpha) N$,

(g) $h_1 (Bca) N$, (h) $(Abc) N$, (i) $h_1 N$.

Of these (a), (c), (d) and (e) are particular covariants.

Type (b) . $h_1 h_2 (a_x b)(A, B)$. (A, B) may not be $(AC)(CB)$ or else $h_1 (BC) a_x$ would be a factor and if (A, B) were of the form $(A\beta)(B\alpha)(\alpha, \beta)$ the covariant would be reducible mod k^3. Therefore $h_1 h_2 (AB)(a_x b)$* is the only possible covariant in this group.

Type (f) $h_1 (B\alpha\gamma) N$. N may not contain $b_x, b_\gamma, c_\beta, (A\beta), (A\gamma)$, or $(B\gamma)$. It may have one or three factors of type $(A\beta x)$.

One factor of type $(A\beta x)$.

This factor may be (1) $(B\alpha x)$, (2) $(C\alpha x)$, (3) $(C\beta x)$.

The types of covariants are :—

(1) $h_1 (B\gamma\alpha)(B\alpha)(B, C) a_\gamma$.

(2) $h_1 (B\gamma\alpha)(C\alpha) a_\gamma$.

(3) $h_1 (B\gamma\alpha)(C\beta) a_\gamma ({}_\beta a_x)({}_\alpha c_x)$.*

Three factors of type $(A\beta x)$.

The only three possible factors are $(B_\alpha C_\beta)$ and any covariant of this type has the factor $(B_\alpha C)(B, C)$.

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Type (g). $h_1\,(Bca)\,N$. In this section the following identities were used :—

$$h_1\,(Bac)\,(C\beta) \equiv 0, \quad h_1\,(A\gamma)\,a_\gamma \equiv 0, \quad h_1\,(A\beta)\,a_\beta \equiv 0,$$
$$h_1\,(A\gamma)\,c_\alpha \equiv 0\ , \quad h_1\,(A\beta)\,b_\alpha \equiv 0, \quad h_1\,(A\beta)\,c_\alpha \equiv 0,$$
$$h_1\,(A\gamma)\,b_\alpha \equiv 0\ , \quad h_1\,b_\gamma \equiv 0\ , \quad h_1\,c_\beta \equiv 0\ ,$$
$$h_1\,b_\alpha\,c_x \equiv h_1\,c_\alpha\,b_x\ , \quad h_1\,b_\alpha\,b_x \equiv 0\ , \quad h_1\,c_\alpha\,c_x \equiv 0\ .$$

N may have one three or five factors of type $(A\beta x)$.

One factor of type $(A\beta x)$.

The factor may be (1) $(C\alpha)$, (2) $(B\alpha)$, (3) $(B\gamma)$.

The types of covariants are :—

(1) $h_1\,(Bac)\,(C\alpha)\,c_\alpha$.*

(2) $h_1\,(Bac)\,(B\alpha)\,(B,\,C)\,c_\alpha$.**

(3) $h_1\,(Bac)\,(B\gamma)\,(B,\,C)\,({}_\gamma\,a_x\,c)$.*

Three factors of type $(A\beta x)$.

These factors may be (1) $(C_\alpha B_\gamma)$, (2) $(A_\gamma B_\alpha)$, (3) $(A_\gamma B)\,(C_\alpha)$.

The only possible type of covariant is :—

(1) $h_1\,(Bac)\,(C_\alpha B_\gamma)\,({}_\gamma\,a_x\,c)$.*

Five factors of type $(A\beta x)$.

There is no possible covariant of this type

Type (h) . $h_1\,(Abc)\,N$. N may have one, three or five factors of type $(A\beta x)$.

One factor of type $(A\beta x)$.

Since $h_1\,(Abc)\,b_\alpha \equiv 0$ and $h_1\,(Abc)$ is symmetrical in the symbols of the second and third quadrics the only possible factor is $(B\gamma x)$.

$h_1\,(Abc)\,(B_\gamma)\,(A,\,C)\,a_\gamma\,(b_x\,c)$* is the only covariant of this type.

Three factors of type $(A\beta x)$

These factors may be, (1) $(A_\beta C)\,(B_\gamma)$, (2) $(C_\alpha B_\gamma)$.

The types of covariants are :—

(1) $h_1\,(Abc)\,(A_\beta C)\,(B\gamma)\,a_\gamma\,(b_x\,c)$.*

(2) $h_1\,(Abc)\,(C_\alpha B_\gamma)\,(A\ B)\,a_\gamma\,(b_x\,c)$.*

Five factors of type $(A\beta x)$.

The only possible factors are $(A_\gamma B_\alpha C_\beta)$ and any covariant of this type involves the factor $(C_\alpha B)\,(C,\,B)$.

Type (i) . $h_1\,N$. N may have no factors, two or four of type $(A\beta x)$.

No factors of type $(A\beta x)$.

The types of covariants are :—

$h_1\,(B\,C)\,a_x,\quad h_1\,(BA)\,(AC)\,a_x.$

Two factors of type $(A\beta x)$.

These factors may be, (1) $(A_\beta C)$, (2) $({}_\beta C_\alpha)$, (3) $(C_\beta)\,(B_\alpha)$, (4) $(C_\beta)\,(B_\gamma)$, (5) $(C_\alpha B)$.

The types of covariants are :—

(1) $h_1\,(A_\beta C)\,(A\,B)\,a_x$.*

(4) $h_1\,(C_\beta)\,(B_\gamma)\,a_x\,({}_\beta a_\gamma)$.*

(5) $h_1\,(C_\alpha B)\,a_x$.

Four factors of type $(A\beta x)$.

These factors may be, (1) $({}_\gamma B_\alpha C_\beta)$, (2) $(A_\gamma B_\alpha C)$, (3) $({}_\alpha B_\gamma A)\,(C\beta)$ (4) $({}_\alpha C_\beta)\,(A_\gamma B)$, (5) $(B_\gamma A_\beta C)$.

The only type of covariant is :—

(5) $h_1\,(B_\gamma A_\beta C)\,a_x$ *

D. $k=(ABCxx)$.

$k^2=(ABCxx)^2$ is the only covariant in this group.

§ (10). *K_4 Group.*

From *loc. cit.* 489, we see that possible covariants in this group are of types (*a*) $F_4''\,(Bac)\,(Abc)\,(a_\gamma b)\,(c,\gamma)$ and its dual.

(*b*) $F_4''\,(Bac)\,a_\gamma\,(A,)$ and its dual.

(*c*) $F_4''\,h_3\,(\gamma)$.

(*d*) $F_4''\,(A, B, c, \gamma)$.

F_4'' may occur with the following F_1 and F_2 factors :—

$c_x,\, c_\alpha,\, c_\beta,\, b_\gamma,\, a_\gamma,\, (A\gamma),\, (B\gamma),\, (BC),\, (CA),\, (AB)$.

Type (*a*). $F_4''\,(Bac)\,(Abc)\,(a_\gamma b)\,(c,\gamma)$. The unpaired γ can only appear as a_γ or b_γ and in either case the covariant will have a factor. A similar proof applies to the dual case and accordingly there are no covariants of this type.

Type (*b*). $F_4''\,(Bac)\,a_\gamma\,(A,)$. The unpaired (A) necessitates a factor (A_γ) or $(A, B)\,(B_\gamma)$ and in either case a γ is left unpaired. It may be paired off by means of the factor a_γ or b_γ, the former of which involves the factor a_γ^2 and the latter leaves an unpaired b which is impossible. A similar proof applies to the dual case and therefore there are no covariants of this type.

Types (*c*) *and* (*d*). Both types leave an unpaired γ and as in type (*b*) this is impossible.

§(11). *List of Identities used in the Reduction of the Covariants.*

I. $a_\beta (A\gamma x) \equiv a_\gamma (A\beta x) - a_x (A\beta\gamma)$.

II. $F_4 b_x \equiv h_1 b_\alpha + (Cab)(B\alpha x)$.

III. $(A\beta\gamma)(B\alpha x) \equiv (BA)(\alpha\beta\gamma x) + (B\alpha\gamma)(A\beta x)$.

IV. $F_4 b_\gamma \equiv (Cab)(B\alpha\gamma) + (BC) b_\alpha a_\gamma$.

V. $F_4 (A\beta x) \equiv (AB)(C\beta\alpha) a_x + (AB)(C\alpha x) a_\beta - (AC)(B\alpha x) a_\beta$.

VI. $F_4 (B\gamma x) \equiv h_1 (B\alpha\gamma) + (B\alpha\gamma)(BC) a_x$.

VII. $F_4 (Abc) \equiv (Cba)(AB) c_\alpha + (Bca)(AC) b_\alpha$.

VIII. $F_4 (A\beta\gamma) \equiv (C\beta\alpha)(AB) a_\gamma + (B\gamma\alpha)(AC) a_\beta$.

IX. $k b_\alpha \equiv (B\alpha x) h_2 + (AB)(C\alpha) b_x$.

X. $h_1 c_\beta \equiv (Bac)(C\beta x) + (BC) c_x a_\beta$.

XI. $h_1 (A\gamma\beta) \equiv (AB)(C\beta x) a_\gamma + (AC)(B\gamma x) a_\beta - (BC)(A\beta x) a_\gamma$.

XII. $k(\alpha\beta\gamma x) \equiv (B\gamma x)(C\alpha x)(A\beta x) + (B\alpha x)(C\beta x)(A\gamma x)$.

§ (12). *Typical Reductions.*

To reduce—$(Bac)(Cab)({}_\gamma B_\alpha)(B, C)({}_\gamma a_x c) b_\alpha = E$. .

$E \equiv [F_4 b_x - h_1 b_\alpha](Bac)(B\gamma)(B, C)({}_\gamma a_x c) b_\alpha$ by II.

$\equiv F_4 (Bac)(B\gamma)(B, C)({}_\gamma a_x c)({}_\alpha b_x) \bmod b_\alpha^2$.

$\equiv [F_4 (B, C)({}_\alpha b_x a)][(Bac)(B\gamma) a_\gamma c_x]$.

$\equiv$ product of two covariants.

$\equiv 0$.

To reduce—$(Bac)(C\beta x)(BC) c_x a_\beta$.

$h_1 c_\beta \equiv (Bac)(C\beta x) + (BC) c_x a_\beta$ by X.

Squaring both sides we have

$h_1^2 c_\beta^2 \equiv (Bac)^2 (C\beta x)^2 + (BC)^2 c_x^2 a_\beta^2 + 2(Bac)(C\beta x)(BC) c_x a_\beta$

i.e. $0 \equiv (Bac)(BC)(C\beta x) c_x a_\beta$.

Some covariants were reduced by splitting a capital letter up into its two components, *e.g.*

To reduce—$(A\beta x)(B\gamma x)(AC)(CB)({}_\beta a_x b_\gamma) = E$.

$E = 4\, a_{1\beta} a_{2x} b_{1\gamma} b_{2x} (a_1 a_2 C)(C b_1 b_2)({}_\beta a_x b_\gamma)$.

$\equiv 4\, a_{1\beta} a_{2x} a_{1\gamma} b_{2x} (b_1 a_2 C)(C b_1 b_2)({}_\beta a_x b_\gamma)$

$+ 4\, a_{1\beta} a_{2x} a_{2\gamma} b_{2x} (a_1 b_1 C)(C b_1 b_2)({}_\beta a_x b_\gamma) \bmod c_\gamma^2$

$\equiv 4X + 4Y$.

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Where $X = ({}_{\gamma} a_{1\beta} a_x b_{\gamma}) \; a_{2c} b_{2x} (b_1 \, a_2 \, C) \; (C \, b_1 \, b_2)$

$= $ product of two covariants.

$\equiv 0.$

$Y = ({}_{\gamma} a_{2x} b_{\gamma}) \; a_{1\beta} b_{2x} (a_1 \, b_1 \, C) \; (C \, b_1 \, b_2) \; ({}_{\beta} a_x).$

$= $ product of two covariants.

$\equiv 0.$

Therefore $E \equiv 0$.

The reduction of $(Bca) \; (A\beta) \; (AC) \; (CB) \; c_{\beta} a_x$ is interesting. It is deduced by means of a differential operator from

$$(Bca) \; (A\beta) \; (AB) \; c_{\beta} \; a_x$$

in the same way that Professor Turnbull proved that the invariant $(Bca) \; (A\beta\gamma) \; (AC) \; (CB) \; c_{\beta} a_{\gamma}$ was reducible.* This was the only case in which this type of reduction was possible, as usually a covariant involving $(AC) \; (CB)$ is easier to reduce than the corresponding one involving the single factor $(A \, B)$.

§ (13). From the list of covariants given the complete list of contravariants—concomitants involving solely the variable u—can at once be written down by the principle of duality, *i.e.* by changing Italic letters to Greek and *vice versa* and by writing u for x.

For example the covariant $(A\beta x)^2$ at once yields the contravariant $(Abu)^2$ and the covariant

$(A\beta x)(B\gamma x)(AB)({}_{\beta} c \, {}_{a} b_{\gamma})$ the contravariant $(Abu)(Bcu)(AB)(b_{\gamma} \, a_{\beta} \, c)$.

§ 14. It is worth noting that all the covariants are of even order and that there is no irreducible covariant of higher order than the sixth. There is only one sextic $- h_1 \, h_2 \, h_3 \, a_x \; b_x \; c_x$ which is symmetrical and of degree three in each of the coefficients of the quadrics. In some cases there appear to be two or more covariants of the same order and of the same degree in the coefficients. The simplest example is the following—the covariants h_1^2, $h_1(BC)a_x$ are both of the second order and of degree (1, 2, 2) in the coefficients of the quadrics. All efforts to prove these two equivalent have failed as in the other cases.

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* *Proc. Lond. Math. Soc.*, 2.23, 423–7 (1924).