8
Quantum Electrodynamics in 1+1 Dimensions

8.1 The Abelian Higgs Model

Instantons imply drastic changes in the spectrum of theories with essentially Abelian gauge invariance in $1+1$ and $2+1$ dimensions. We say essentially Abelian, since we include in this class theories which are spontaneously broken to a residual $U(1)$ invariance. In $1+1$ dimensions we consider the theory defined by the Lagrangian density [61],

$$\mathcal{L} = (D_\mu\phi)^* (D^\mu\phi) - \frac{\lambda}{4}(\phi^*\phi)^2 - \frac{\mu^2}{2}\phi^*\phi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}, \tag{8.1}$$

where

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + iA_\mu\phi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{8.2}$$

We take $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$, but we have replaced $A_\mu \to \frac{1}{e}A_\mu$. The Lagrangian is invariant under a local gauge transformation which has a natural multiplication law corresponding to the group $U(1)$

$$\begin{aligned} \phi &\to e^{i\Lambda(x,t)}\phi = g(x,t)\phi \quad g(x,t)\in U(1) \\ A_\mu &\to e^{i\Lambda(x,t)}(A_\mu - i\partial_\mu)e^{-i\Lambda(x,t)} = A_\mu - \partial_\mu\Lambda. \end{aligned} \tag{8.3}$$

Then

$$\begin{aligned} D_\mu\phi \to{}& \partial_\mu\left(e^{i\Lambda(x,t)}\phi\right) + i(A_\mu - \partial_\mu\Lambda(x,t))e^{i\Lambda(x,t)}\phi \\ &= e^{i\Lambda(x,t)}\partial_\mu\phi + e^{i\Lambda(x,t)}i\partial_\mu\Lambda(x,t)\phi + e^{i\Lambda(x,t)}iA_\mu\phi - e^{i\Lambda(x,t)}i\partial_\mu\Lambda(x,t)\phi \\ &= e^{i\Lambda(x,t)}D_\mu\phi. \end{aligned} \tag{8.4}$$

We impose that $\lim_{|x|\to\infty} g(x,t) = 1$. This gives an effective topological compactification of the space since the gauge transformation at spatial infinity must be the same in all directions.

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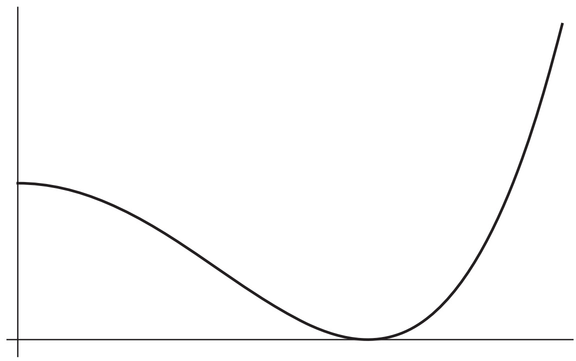

Figure 8.1. The symmetric breaking potential $U(|\phi|)$

There are two cases to consider depending on the sign of μ^2. For $\mu^2 > 0$ the quadratic part of the Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - \frac{\mu^2}{2}\phi^*\phi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \tag{8.5}$$

with corresponding equations of motion

$$\begin{aligned} \partial_\mu \partial^\mu \phi + \mu^2 \phi &= 0 \\ \partial_\mu F^{\mu\nu} &= 0, \end{aligned} \tag{8.6}$$

which describe a free, massive, scalar particle and a massless vector field, the free, electromagnetic field. The conserved current corresponding to gauge invariance is

$$j^\mu = \phi^* \partial^\mu \phi - (\partial^\mu \phi^*)\,\phi. \tag{8.7}$$

External charges that are well separated experience the usual Coulomb force. This is true in any dimension except in $1+1$ dimensions, the case that we are considering first. Here, the Coulomb force is independent of the separation and it costs an infinite amount of energy to separate two charges to infinity. We say that charges are confined. Furthermore, there is no photon. There is no transverse direction for the polarization states of the photon. The spectrum consists of bound states of particle–anti-particle pairs, which are stable. They cannot disintegrate since there is no photon.

For the other case with $\mu^2 < 0$, the potential (as depicted in Figure 8.1) is of the symmetry breaking type

$$U(|\phi|) = \frac{\lambda}{4}|\phi|^4 - \frac{|\mu^2|}{2}|\phi|^2 + C, \tag{8.8}$$

where the C is adjusted so that the potential vanishes at the minimum. The minimum is at $|\phi|^2 = \frac{|\mu^2|}{\lambda}$. We fix the gauge so that $\Im m(\phi) = 0$, and we write

$$\phi = \frac{|\mu|}{\sqrt{\lambda}} + \eta \tag{8.9}$$

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with $\eta \in \mathbf{R}$. Then we get the Lagrangian density

$$\mathcal{L} = (\partial_\mu - iA_\mu)\left(\frac{|\mu|}{\sqrt{\lambda}} + \eta\right)(\partial^\mu + iA^\mu)\left(\frac{|\mu|}{\sqrt{\lambda}} + \eta\right) - \frac{\lambda}{4}\left(\frac{|\mu|}{\sqrt{\lambda}} + \eta\right)^4 - \frac{\mu^2}{2}\left(\frac{|\mu|}{\sqrt{\lambda}} + \eta\right)^2 - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}, \tag{8.10}$$

which yields the quadratic part

$$\mathcal{L}_0 = \partial_\mu \eta \partial^\mu \eta + \frac{\mu^2}{\lambda}A_\mu A^\mu + \mu^2\eta^2 - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}. \tag{8.11}$$

This now corresponds to a scalar particle with a mass of $\frac{\mu}{\sqrt{2}}$ and a vector particle of mass $\frac{\mu}{\sqrt{\lambda}}e$. Then the expectation is that the potential between particles should drop off exponentially with the usual Yukawa factor

$$e^{-\frac{r}{M}} \tag{8.12}$$

with $M = \frac{|\mu|}{\sqrt{2}}$ or $M = \frac{|\mu|}{\sqrt{\lambda}}e$. We will find, surprisingly, that this is again not true in $1+1$ dimensions. Instantons change the force between the particles and actually imply confinement. The only difference between the cases $\mu^2 > 0$ and $\mu^2 < 0$ is that the force is exponentially smaller (in $\hbar$) for the case $\mu^2 < 0$; however, it is still independent of separation.

8.2 The Euclidean Theory and Finite Action

To see this result, we must analyse the Euclidean theory. Here the Lagrangian density is

$$\mathcal{L} = \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) + \frac{\lambda}{4}\left(\phi^*\phi - a^2\right)^2 \tag{8.13}$$

adding a constant, where $a = \frac{|\mu|}{\sqrt{\lambda}}$. These are three positive terms. For a configuration of finite Euclidean action, each term must give a finite contribution when integrated over $\mathbf{R}^2$. This implies that $\phi^*\phi \to a^2$, $D_\mu\phi \to 0$ and $F_{\mu\nu} \to 0$ faster than $\frac{1}{r}$.

$$\begin{aligned}
\phi^*\phi \to a^2 &\Rightarrow \lim_{r\to\infty} \phi = g(\theta)a \\
F_{\mu\nu} \to 0 &\Rightarrow iA_\mu \to \tilde{g}\partial_\mu(\tilde{g})^{-1} + o\left(\frac{1}{r^2}\right) = i\partial_\mu\Lambda \\
&\qquad \text{for} \quad \tilde{g} = e^{i\Lambda} \\
D_\mu\phi \to 0 &\Rightarrow \partial_\mu g(\theta)a + \tilde{g}\partial_\mu(\tilde{g})^{-1}g(\theta)a \\
&= \left(-g(\theta)\partial_\mu g^{-1}(\theta) + \tilde{g}\partial_\mu(\tilde{g})^{-1}\right)g(\theta)a = 0.
\end{aligned} \tag{8.14}$$

This is satisfied if $g(\theta) = \tilde{g}$. Thus the configurations with finite Euclidean action are characterized by $g(\theta)$. $g(\theta)$ defines a mapping of the circle at infinity parametrized by θ into the group $U(1)$, which is just the unit circle as in Figure 8.2.

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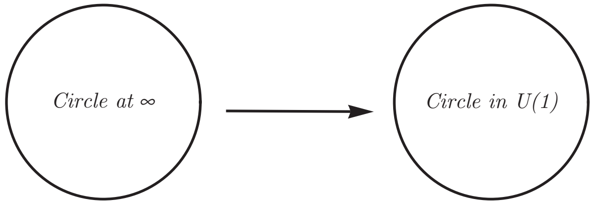

Figure 8.2. Mapping the spatial circle at ∞ to the circle in $U(1)$

8.2.1 Topological Homotopy Classes

The space of such maps separates into homotopically inequivalent classes. These classes are characterized by the winding number of the map. A map from one class cannot be continuously deformed to any other map from another class. It is intuitively obvious that there are an infinite number of classes each corresponding to a winding number. We can take

$$\begin{aligned} g^0(\theta) &= 1 \qquad n=0 \\ g^1(\theta) &= e^{i\theta} \qquad n=1 \\ &\cdot \\ &\cdot \\ &\cdot \\ g^\nu(\theta) &= e^{i\nu\theta} \qquad n=\nu. \end{aligned} \tag{8.15}$$

Given $g(\theta)$ we can extract ν by the formula

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta g(\theta)\frac{d}{d\theta}g^{-1}(\theta). \tag{8.16}$$

If $g(\theta) = e^{i\nu\theta}$, $\frac{d}{d\theta}g^{-1}(\theta) = -i\nu g^{-1}(\theta)$ thus

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} g(\theta)\left(-i\nu\right)g^{-1}(\theta) = \frac{\nu}{2\pi}2\pi = \nu. \tag{8.17}$$

If we make an arbitrary, infinitesimal change in $g(\theta)$,

$$g(\theta) \to e^{i\epsilon\Lambda(\theta)}g(\theta) = g(\theta) + i\epsilon\Lambda(\theta)g(\theta) \tag{8.18}$$

with $\Lambda(\theta)$ of compact support in $[0, 2\pi)$,

$$\begin{aligned} \delta g(\theta) &= i\Lambda(\theta)g(\theta) \\ \delta\left(g(\theta)\frac{d}{d\theta}g^{-1}(\theta)\right) &= i\Lambda(\theta)g(\theta)\frac{d}{d\theta}g^{-1}(\theta) + g(\theta)\frac{d}{d\theta}\left(-i\Lambda(\theta)g^{-1}(\theta)\right) \\ &= i\Lambda(\theta)g(\theta)\frac{d}{d\theta}g^{-1}(\theta) - i\Lambda(\theta)g(\theta)\frac{d}{d\theta}g^{-1}(\theta) - i\frac{d}{d\theta}\Lambda(\theta) \\ &= -i\frac{d}{d\theta}\Lambda(\theta). \end{aligned} \tag{8.19}$$

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Thus

$$\delta\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta\delta\left(g(\theta)\frac{d}{d\theta}g^{-1}(\theta)\right) = \frac{i}{2\pi}(-i)\int_0^{2\pi} d\theta\frac{d}{d\theta}\Lambda(\theta)$$
$$= \frac{1}{2\pi}\left(\Lambda(2\pi)-\Lambda(0)\right) = 0. \tag{8.20}$$

Thus for each class, ν is an invariant under arbitrary continuous deformation. Furthermore, if $g(\theta) = g_{\nu_1}(\theta)g_{\nu_2}(\theta)$, then

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta g_{\nu_1}(\theta)g_{\nu_2}(\theta)\frac{d}{d\theta}\left(g_{\nu_2}^{-1}(\theta)g_{\nu_1}^{-1}(\theta)\right)$$
$$= \frac{i}{2\pi}\int_0^{2\pi} d\theta g_{\nu_1}(\theta)\left(g_{\nu_2}(\theta)\frac{d}{d\theta}g_{\nu_2}^{-1}(\theta)\right)g_{\nu_1}^{-1}(\theta) + g_{\nu_1}(\theta)\frac{d}{d\theta}g_{\nu_1}^{-1}(\theta)$$
$$= \nu_1 + \nu_2. \tag{8.21}$$

Finally, using $iA_\mu = g\partial_\mu g^{-1} + o\left(\frac{1}{r^2}\right)$

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta g(\theta)\frac{d}{d\theta}g^{-1}(\theta) = \frac{i}{2\pi}\int_0^{2\pi} d\theta r i\hat{r}_\mu\epsilon_{\mu\nu}A_\nu = -\frac{1}{2\pi}\oint_{r=\infty} dx_\mu A_\mu$$
$$= -\frac{1}{2\pi}\int d^2x\partial_\mu\epsilon_{\mu\nu}A_\nu = -\frac{1}{4\pi}\int d^2x\epsilon_{\mu\nu}F_{\mu\nu} = -\left(\frac{\Phi}{2\pi}\right), \tag{8.22}$$

giving that the flux is quantized in units of 2π. In each homotopy class, the configuration of minimum action must be stationary and hence satisfy the Euclidean equations of motion. Because the solutions with different ν cannot be obtained from each other by continuous deformation, there should be an infinite action barrier between each class.

8.2.2 Nielsen–Olesen Vortices

The solutions for each ν are known to exist and are called the Nielsen–Olesen vortices [96]. They are described by two radial functions, for $\nu = 1$

$$A_\mu = \epsilon_{\mu\nu}r_\nu\frac{\Phi(r)}{2\pi r^2}$$
$$\phi(r) = e^{i\theta}f(r). \tag{8.23}$$

This form implies the equations

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r)\right) + \frac{1}{r^2}\left(1-\frac{\Phi(r)}{2\pi}\right)^2 f(r) - \mu^2 f(r) + \lambda f^3(r) = 0 \tag{8.24}$$

and

$$-\frac{1}{e^2}\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\frac{\Phi(r)}{2\pi}\right) + \frac{f^2(r)}{r}\left(\frac{\Phi(r)}{2\pi}-1\right) = 0. \tag{8.25}$$

A solution exists, as depicted in Figure 8.3, with the behaviour for the magnetic field $B(r)$

$$B(r) = \frac{1}{2\pi r}\frac{d}{dr}\frac{\Phi(r)}{2\pi} \to Ce^{-erf(r)}$$

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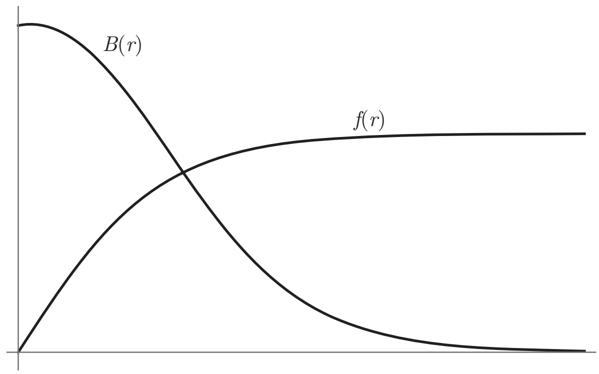

Figure 8.3. The form of the function $f(r)$ and the magnetic field $B(r)$

$$\begin{aligned} f(r) &\to a \\ \Phi(r) &\to \Phi, \end{aligned} \tag{8.26}$$

where $\Phi(r)$ has the interpretation of being equal to the magnetic flux inside the radius r while Φ is the total magnetic flux in the soliton, which is quantized in units of 2π. The magnetic field is concentrated around the origin and both fields approach the vacuum configuration exponentially fast with a non-trivial winding number.

This solution mediates tunnelling between inequivalent classical vacua, which correspond to classical configurations with zero energy. The energy is given by (for $A_0 = 0$)

$$\mathcal{E} = \int dx \frac{1}{2e^2} \left(\partial_0 A_1\right)^2 + \left(\partial_0 \phi\right)^* \left(\partial_0 \phi\right) + \left(D_1 \phi\right)^* \left(D_1 \phi\right) + \lambda \left(\phi^* \phi - a^2\right)^2. \tag{8.27}$$

The simplest zero-energy configuration is $\phi = a$, $A_\mu = 0$. There exists, however, the freedom to transform this solution by a local gauge transformation that depends only on space, which keeps the gauge condition $A_0 = 0$ invariant,

$$\phi \to g(x) a \qquad A_1 \to -ig(x)\partial_1 g^{-1}(x). \tag{8.28}$$

We impose the additional condition, the $\lim_{x\to\infty} g(x) \to 1$, which is consistent with our desire to consider a theory with arbitrary local excitations but asymptotically no excitations. Then we get the effective compactification of the spatial hypersurface. Topologically it is now just a circle and $g(x)$ again maps the circle that is space onto the circle in $U(1)$. These maps are characterized by winding numbers. Thus the classical vacua

$$\begin{aligned} \phi &= g_\nu(x) a \\ A_1 &= -ig_\nu(x) \frac{d}{dx} g_\nu^{-1}(x) \end{aligned} \tag{8.29}$$

indexed by $\nu \in \mathbb{Z}$ are homotopically inequivalent. We cannot deform one into another while staying at $\mathcal{E} = 0$. However, the energy barrier between them is

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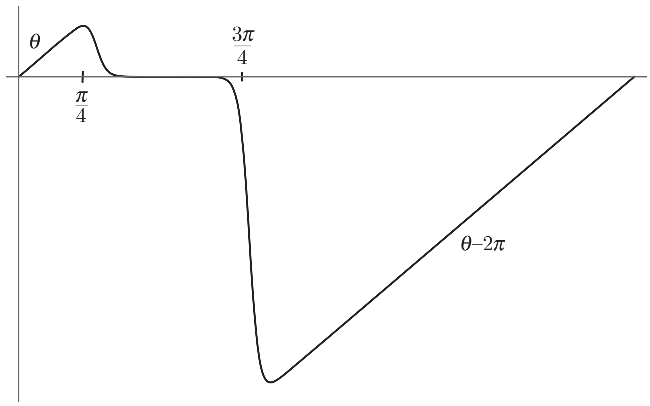

Figure 8.4. The form of the gauge transformation $\Lambda(\theta)$

not infinite. The Nielsen–Olesen vortices interpolate between these vacua. To see this we must transform the Nielsen–Olesen vortex into a form suitable for the description in terms of many vacua, *i.e.* to the gauge $A_0 = 0$.

We first perform a gauge transformation $g(\theta) = e^{-i\Lambda(\theta)}$, which has the limit at spacetime infinity given by

$$\Lambda(\theta) = \begin{cases} \theta & \text{for } \theta \in (0, \pi/4) \\ \pi/4 \to 0 & \text{for } \theta \in (\pi/4, \pi/4 + \epsilon) \\ 0 & \text{for } \theta \in (\pi/4 + \epsilon, 3\pi/4 - \epsilon) \\ 0 \to -5\pi/4 & \text{for } \theta \in (3\pi/4 - \epsilon, 3\pi/4) \\ (\theta - 2\pi) & \text{for } \theta \in (3\pi/4, 2\pi) \end{cases} \tag{8.30}$$

as drawn in Figure 8.4. The corresponding $g(\theta)$ is topologically trivial; we can simply deform the two saw-tooth humps to zero (non-trivial winding number requires a $\Lambda(\theta)$ discontinuous by $2n\pi$ between its value at $\theta = 0$ and $\theta = 2\pi$). Therefore, the gauge transformation can be continued everywhere inside the spacetime and define a gauge transformation at all points. This gauge transformation (note this is an inverse transformation, $\Lambda \to -\Lambda$) takes

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda(\theta) \tag{8.31}$$

and it is easy to see that this vanishes asymptotically, except where $\Lambda(\theta) = 0$, *i.e.* for $\theta \in (\pi/4 + \epsilon, 3\pi/4 - \epsilon)$. Thus $\tilde{A}_\mu \to 0$ for $t \in [-\infty, T]$, T finite, exponentially fast as $|x| \to \infty$, as depicted in Figure 8.5.

Now we further perform the gauge transformation to put $\tilde{A}_0 = 0$ everywhere; this is easily implemented by the choice

$$\Lambda(x,t) = \int_{-\infty}^{t} dt' \tilde{A}_0(x,t')$$

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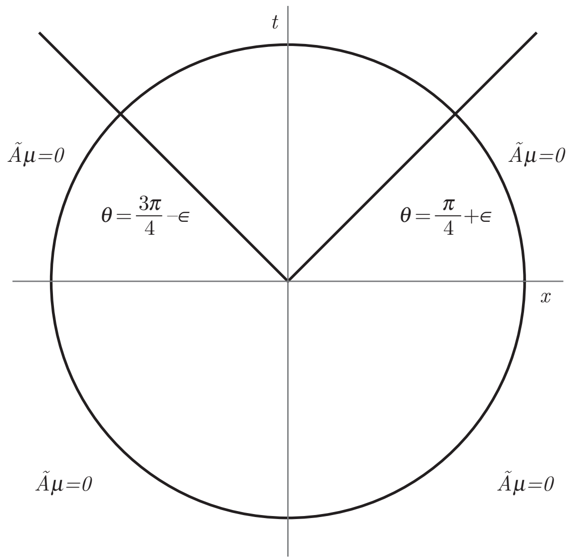

Figure 8.5. The regions of spacetime where the gauge field vanishes

$$A_1(x,t) \to \tilde{\tilde{A}}_1(x,t) = \tilde{A}_1(x,t) - \partial_x \Lambda(x,t) = \tilde{A}_1(x,t) - \partial_x \int_{-\infty}^{t} dt' \tilde{A}_0(x,t'). \tag{8.32}$$

This makes $\tilde{\tilde{A}}_0 = 0$ everywhere, but maintains $\tilde{\tilde{A}}_1 \to 0$ for $t \in [-\infty, T]$, T finite, exponentially fast as $|x| \to \infty$, since both $\tilde{A}_1 \to 0$ and $\tilde{A}_0 \to 0$ exponentially fast at spatial infinity in the region $t \in [-\infty, T]$, due to the first gauge transformation. Thus as $t \to -\infty$, $\tilde{\tilde{A}}_1 = 0$, but as $t \to +\infty$ we have,

$$\tilde{\tilde{A}}_1 \to g(x)\partial_x g^{-1}(x) \tag{8.33}$$

with

$$g(x) = e^{i\int_{-\infty}^{\infty} dt' \tilde{A}_0(x,t')} g(\theta) e^{i\theta}, \tag{8.34}$$

where $g(\theta) = e^{-i\Lambda(\theta)}$ is our first gauge transformation and $e^{i\int_{-\infty}^{\infty} dt' \tilde{A}_0(x,t')}$ is the second gauge transformation that put $\tilde{\tilde{A}}_0 = 0$. The final factor $e^{i\theta}$ is the asymptotic gauge transformation of the Nielsen–Olesen vortex. The first two factors are topologically trivial gauge transformations: in each case the exponent can be continuously switched to zero, thus the winding number of the gauge transformation $e^{i\theta}$, which is 1, is unchanged. However, the two trivial factors manage to bunch all of the non-trivial winding of $e^{i\theta}$ into the spatial line x at $t = \infty$.

Thus we have put the Nielsen–Olesen vortex in a gauge where it interpolates from the vacuum configuration $g(x) = 1$ at $t = -\infty$ to the non-trivially transformed vacuum configuration $g(x) = e^{i\int_{-\infty}^{\infty} dt' \tilde{A}_0(x,t')} g(\theta) e^{i\theta}$. The situation

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is exactly analogous to the problem of a periodic potential on a line. The classical vacua form a denumerable infinity of local minima indexed by the winding number n. There is a finite energy barrier between each one, and the Nielsen–Olesen vortex is the instanton that mediates the tunnelling between them.

8.3 Tunnelling Transitions

We can calculate the matrix element

$$\langle \nu = n | \, e^{\frac{-\hat{H}T}{\hbar}} \, | \nu = 0 \rangle = \mathcal{N} \int_{\nu[\phi_{in}]=0}^{\nu[\phi_f]=n} \mathcal{D}(A_1, \phi^*, \phi) \, e^{-\frac{S_\phi^E}{\hbar}} \tag{8.35}$$

in the semi-classical approximation. The functional integral is simply identified with the integral over all finite action field configurations with $\nu = n$. The continuation from Euclidean space automatically projects on the vacuum in this sector. The critical point of the action contains the vortex with $\nu = n$; however, this is not the most important configuration. The most important configurations correspond to n_+ vortices with $\nu = 1$ and n_- vortices with $\nu = -1$, widely separated, such that $n_+ - n_- = n$. The action for such a configuration is very close to $(n_+ + n_-) S^E(\nu = 1)$. The entropy factor, counting the degeneracy of the configuration, is

$$\frac{(TL)^{n_+ + n_-}}{n_+! n_-!}. \tag{8.36}$$

In comparison, for a single vortex with $\nu = n$, the action is presumably smaller, but the entropy factor is just TL, since there is only one object. Thus the dilute multi-instanton configurations are arbitrarily more important as $TL \to \infty$. Then

$$\begin{aligned} \langle \nu = n | \, e^{\frac{-\hat{H}T}{\hbar}} \, | \nu = 0 \rangle = \mathcal{N} \det{}_0^{-\frac{1}{2}} \sum_{n_+=0}^{n_+=\infty} \sum_{n_-=0}^{n_-=\infty} \frac{1}{n_+! n_-!} \left(TL e^{-\frac{S_0^E}{\hbar}} K \right)^{n_+ + n_-} \\ \times \delta_{n_+ - n_-, n}, \end{aligned} \tag{8.37}$$

where K^{-2} (so that it appears in the formula as just K) is given by the ratio of the determinant prime corresponding to the quadratic part of the Lagrangian in the presence of one vortex, divided by the determinant of the free quadratic part (written as $\det_0$), and the Jacobian factors from the usual change of variables that take into account zero modes. The prefactor is set equal to one by choosing the normalization $\mathcal{N}$. Now using the formula

$$\delta_{a,b} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta(a-b)} \tag{8.38}$$

we get

$$\langle \nu = n | \, e^{\frac{-\hat{H}T}{\hbar}} \, | \nu = 0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta n} \sum_{n_+, n_- = 0}^{\infty} \frac{e^{in_+\theta} e^{-in_-\theta}}{n_+! n_-!} \left(TL e^{-\frac{S_0^E}{\hbar}} K \right)^{n_+ + n_-}$$

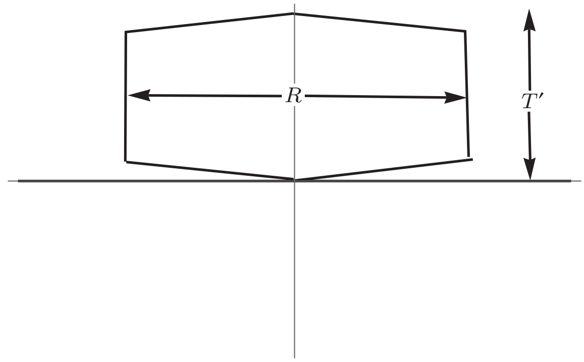

Figure 8.6. Creation of a pair of charges at the origin, separation by R, held for time T' and then annihilated

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta n} e^{\left(TLe^{-\frac{S_0^E}{\hbar}} K\left(e^{i\theta}+e^{-i\theta}\right)\right)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta n} e^{\left(2TLe^{-\frac{S_0^E}{\hbar}} K\cos\theta\right)} \\
&= \int_0^{2\pi} d\theta e^{\frac{-\mathcal{E}(\theta)T}{\hbar}} \langle \nu = n|\theta\rangle\langle\theta|\nu = 0\rangle .
\end{aligned}
\tag{8.39}
$$

Thus we find the infinite set of classical vacua rearrange themselves to form a band of states parametrized by θ with energy density

$$\frac{\mathcal{E}(\theta)}{L} = -2Ke^{-\frac{S_0^E}{\hbar}}\cos\theta \tag{8.40}$$

and the matrix element

$$\langle \nu = n|\theta\rangle = \frac{e^{in\theta}}{\sqrt{2\pi}}. \tag{8.41}$$

8.4 The Wilson Loop

This rearrangement of the vacua has important consequences for the force between charges. Consider the creation of an external charged particle and anti-particle pair. We create them at the origin, separate them by a large distance R, hold them at this separation for a long time T', and then we let them come together and annihilate, as depicted in Figure 8.6. A particle of charge q, in an electromagnetic field, experiences an additional change to its wave function by the factor

$$e^{-i\frac{q}{e}\int dx_\mu A_\mu}. \tag{8.42}$$

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Consider external charges governed by a dynamics with a Lagrangian

$$L = \frac{1}{2}\dot{x}_i^2 + q\dot{x}_i A_i - qA_0 - V(x_i). \tag{8.43}$$

The equation of motion is

$$\ddot{x}_i + q\vec{A}_i(x_l) - q\dot{x}_j\partial_i\vec{A}_j(x_l) + q\partial_i A_0(x_l) + \partial_i V(x_l) = 0, \tag{8.44}$$

which can be rewritten

$$\ddot{x}_i - q\dot{x}_j\epsilon_{jik}B_k(x_l) = -\partial_i V(x_l) + q\vec{E}_i(x_l), \tag{8.45}$$

where $E_i(x_l) = \partial_t\vec{A}_i(x_l) + \partial_i A_0(x_l)$ is the electric field and $B_i(x_l) = \epsilon_{jik}\partial_j\vec{A}_k(x_l)$ is the magnetic field. Thus the action in the functional integral for the particle is augmented by the term

$$\begin{aligned} e^{-i\frac{S^0}{\hbar}} &\to e^{-i\frac{S^0}{\hbar}} e^{-i\frac{q}{e}\int dt\left(\dot{x}_i\vec{A}_i(x_l) - qA_0(x_l)\right)} \\ &= e^{-i\frac{S^0}{\hbar}} e^{-i\frac{q}{e}\int dx^\mu A_\mu}. \end{aligned} \tag{8.46}$$

For an anti-particle the additional factor is, of course,

$$e^{i\frac{q}{e}\int dx^\mu A_\mu}. \tag{8.47}$$

Thus for our trajectory, the additional factor becomes a closed integral in the exponent,

$$e^{-i\frac{q}{e}\oint dx^\mu A_\mu}. \tag{8.48}$$

We perform the functional integral over the gauge and scalar fields treating our particles as external, with their dynamics controlled by $V(x_l)$. However, the wave functions of the particles will change by this additional factor, which we must take into account. When we integrate over A_μ, ϕ, ϕ^* we obtain the matrix element of the operator (in Euclidean space)

$$W = e^{-\frac{q}{e}\oint dx_\mu A_\mu}. \tag{8.49}$$

This is called the Wilson loop operator. The matrix element of the operator behaves approximately as

$$W \sim e^{-E(R)T'\left(\frac{q}{e}\right)}. \tag{8.50}$$

If $E(R) \sim CR$ for some constant C, the interaction between the charges is said to be confining, and the expectation value of the Wilson loop operator will behave like

$$\langle W\rangle \sim e^{-CA\left(\frac{q}{e}\right)}, \tag{8.51}$$

where A is the area of the loop. This is the celebrated criterion of area law behaviour of the Wilson loop for confining interactions. If, on the other hand, the $E(R) \sim D$ for some constant D, we get

$$\langle W\rangle \sim e^{-DP\left(\frac{q}{e}\right)}, \tag{8.52}$$

where P is the perimeter of the loop. Such behaviour of the Wilson loop does not imply confining interactions.

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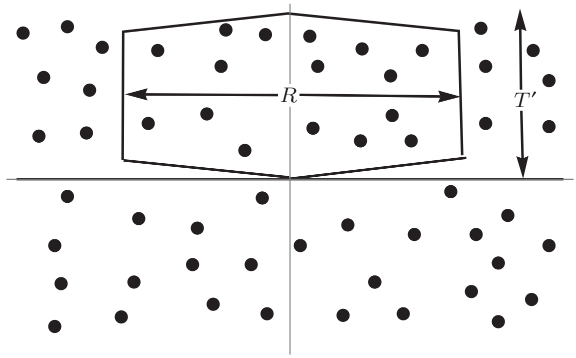

Figure 8.7. A dilute gas of instantons and anti-instantons surround the Wilson loop

8.4.1 Expectation Value of the Wilson Loop Operator

Thus we wish to evaluate

$$\langle\theta|\,W\,|\theta\rangle = \frac{\int \mathcal{D}\,(A_\mu,\phi,\phi^*)\,e^{-\frac{S^E}{\hbar}}\,e^{i\nu\theta}W}{\int \mathcal{D}\,(A_\mu,\phi,\phi^*)\,e^{-\frac{S^E}{\hbar}}\,e^{i\nu\theta}} \tag{8.53}$$

in the semi-classical approximation. For the numerator we divide the summation over the positions of the vortices (instantons) and the anti-vortices (anti-instantons) into those inside the loop and those outside the loop, as depicted in Figure 8.7. We drop the contribution from vortices situated on or near the boundary; these form a negligible part of the set of all configurations, if the size of the loop is much larger than the size of the vortices.

The integrand splits neatly into a part from outside and a part from inside the Wilson loop

$$\begin{aligned} S &= S^{outside} + S^{inside} \\ \nu &= \nu^{outside} + \nu^{inside} \end{aligned} \tag{8.54}$$

however,

$$W = e^{2\pi i\frac{q}{e}\nu^{inside}}. \tag{8.55}$$

Inside the volume available is RT', while outside the volume available is $LT - RT'$, for each vortex. We sum independently over the vortices and the anti-vortices, inside and outside the loop, with no constraint on their numbers. The contribution inside has $\theta \to \theta + \frac{2\pi q}{e}$, thus we get

$$\begin{aligned} \langle\theta|\,W\,|\theta\rangle &= e^{\left(2Ke^{-\frac{S_0^E}{\hbar}}\left(\left(LT-RT'\right)\cos\theta+RT'\cos\left(\theta+\frac{2\pi q}{e}\right)-LT\cos\theta\right)\right)} \\ &= e^{\left(2Ke^{-\frac{S_0^E}{\hbar}}RT'\left(-\cos\theta+\cos\left(\theta+\frac{2\pi q}{e}\right)\right)\right)}. \end{aligned} \tag{8.56}$$

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Then comparing with Equation (8.50) we get

$$E(R) = 2R\left(\cos(\theta) - \cos\left(\theta + \frac{2\pi q}{e}\right)\right) K e^{-\frac{S_0^E}{\hbar}} \tag{8.57}$$

and hence

$$E(R) \sim R \tag{8.58}$$

implying confinement.

We can also calculate

$$\begin{aligned}
\langle\theta|\,\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}\,|\theta\rangle &= \frac{1}{2LT}\int d^2x\,\langle\theta|\,\epsilon_{\mu\nu}F_{\mu\nu}\,|\theta\rangle \\
&= -\frac{4\pi}{2LT}\frac{\int \mathcal{D}(A_\mu,\phi,\phi^*)\,\nu e^{-\frac{S_0^E}{\hbar}}e^{i\nu\theta}}{\int \mathcal{D}(A_\mu,\phi,\phi^*)\,e^{-\frac{S_0^E}{\hbar}}e^{i\nu\theta}} \\
&= \frac{4\pi}{2LT}i\frac{d}{d\theta}\ln\left(\int \mathcal{D}(A_\mu,\phi,\phi^*)\,e^{-\frac{S_0^E}{\hbar}}e^{i\nu\theta}\right) \\
&= \frac{4\pi}{2LT}i\frac{d}{d\theta}\left(2Ke^{-\frac{S_0^E}{\hbar}}LT\cos(\theta)\right) \\
&= -i4\pi Ke^{-\frac{S_0^E}{\hbar}}\sin(\theta).
\end{aligned} \tag{8.59}$$

For small θ from Equation (8.40), removing a constant, we have

$$\frac{\mathcal{E}(\theta)}{L} = Ke^{-\frac{S_0^E}{\hbar}}\theta^2. \tag{8.60}$$

Also,

$$\begin{aligned}
\langle\theta|\,F_{12}\,|\theta\rangle &= -i4\pi Ke^{-\frac{S_0^E}{\hbar}}\theta \\
E(R) &= 2R\left(\theta^2 - \left(\theta + \frac{2\pi q}{e}\right)^2\right)Ke^{-\frac{S_0^E}{\hbar}}.
\end{aligned} \tag{8.61}$$

This lends itself to the following interpretation. In the θ vacuum, there exists an electric field that is proportional to θ with a corresponding energy density proportional to θ^2. The external charges induce an electric field between them, proportional to their charges. The energy changes by the separation of the charges multiplied by the energy density, which in this case is clearly

$$\left(\theta + \frac{2\pi q}{e}\right)^2. \tag{8.62}$$

There exist non-linear effects that convert these to periodic functions in $\frac{q}{e}$. This is because the theory contains particles of charge e. For $q > e$, a charged particle–anti-particle pair can be created, which then can migrate to the oppositely charged external charges, lowering their charge and hence the induced electric field. Thus q's are equivalent modulo e.

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Our analysis, although encouraging, cannot work in higher dimensions. In $1+1$ dimensions, the flux of each instanton inside the loop must totally pass through the loop, independent of its position inside the loop. In $3+1$ dimensions the instantons are not like flux tubes, they are $O(4)$-symmetric objects. Instead of the Wilson loop, we would require some analogous "Wilson three-dimensional hypersurface" to reach the same conclusion. Confinement must, however, imply the area law for the usual Wilson loop, in any dimensions. Thus we do not expect instantons to be responsible for confinement in higher dimensions. We can, as we shall see in Chapter 9, circumvent this problem in $2+1$ dimensions by introducing a mild non-Abelian nature.

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