A PAIR OF DUAL INTEGRAL EQUATIONS OCCURRING IN DIFFRACTION THEORY

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1. Dual integral equations of the form

$$\int_k^\infty u^{-\mu-\nu}(u^2-k^2)^\alpha \psi(u) J_\mu(xu)du = f(x), \quad 0 \leqq x \leqq 1, \quad \ldots\ldots\ldots\ldots(1)$$

$$\int_0^\infty \psi(u) J_\nu(xu)du = g(x), \quad x > 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $f(x)$ and $g(x)$ are given functions, $\psi(x)$ is unknown, $k \geqq 0$, μ, ν and α are real constants, have applications to diffraction theory and also to dynamical problems in elasticity. The special cases $\nu = -\mu$, $\alpha = 0$ and $\nu = \mu = 0$, $0 < \alpha^2 < 1$ were treated by Ahiezer (**1**). More recently, equations equivalent to the above were solved by Peters (**2**) who adapted a method used earlier by Gordon (**3**) for treating the (extensively studied) case $\mu = \nu$, $k = 0$.

We present here a method of solution which has affinities with the " elementary " method introduced by Sneddon (**4**) for the case $\mu = \nu = k = 0$ and developed by Copson (**5**) for the case $\mu = \nu$, $k = 0$. The essence of the method is the reduction of the dual integral equations to a single integral equation: the integral equations which arise in the study of equations (1) and (2) (see Lemmas 2 and 3 below) have not apparently been treated before and seem to be of interest in their own right; they generalise the integral equations discussed by Copson (**6**) and Jones (**7**).

The analysis throughout is formal: orders of integration, and of integration and differentiation, are inverted freely. The solutions obtained may always be verified by substitution into the original dual integral equations.

2. We first state some preliminary results, in the form of three lemmas.

Lemma 1. (Sonine's " second " integral, (**8**), p. 415).

$$\int_0^\infty J_\mu(bt) \frac{J_\nu\{a\sqrt{(t^2+z^2)}\}}{(t^2+z^2)^{\frac{1}{2}\nu}} t^{\mu+1}dt$$

$$= \begin{cases} 0, & 0 < a < b, \\ \dfrac{b^\mu}{a^\nu}\left\{\dfrac{\sqrt{(a^2-b^2)}}{z}\right\}^{\nu-\mu-1} J_{\nu-\mu-1}\{z\sqrt{(a^2-b^2)}\}, & 0 < b < a, \end{cases}$$

for all real or complex z, provided $Re(\nu) > Re(\mu) > -1$.

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Lemma 2. The solution of the integral equation

$$\int_a^x \psi(\rho)(x^2-\rho^2)^{\frac{1}{2}\beta} J_\beta\{k\sqrt{(x^2-\rho^2)}\}d\rho = h(x), \quad \text{...............(3)}$$

where a is a finite constant, is

$$\psi(x) = k\frac{d}{dx}\int_a^x \rho h(\rho)(x^2-\rho^2)^{-\frac{1}{2}(\beta+1)} I_{-(\beta+1)}\{k\sqrt{(x^2-\rho^2)}\}d\rho. \quad \text{......(4)}$$

More symmetrically, the solution of the integral equation

$$\int_a^x \psi(\rho)\left\{\frac{\sqrt{(x^2-\rho^2)}}{k}\right\}^\beta J_\beta\{k\sqrt{(x^2-\rho^2)}\}d\rho = h(x) \quad \text{..................(5)}$$

is

$$\psi(x) = \frac{d}{dx}\int_a^x \rho h(\rho)\left\{\frac{\sqrt{(x^2-\rho^2)}}{k}\right\}^{-(\beta+1)} I_{-(\beta+1)}\{k\sqrt{(x^2-\rho^2)}\}d\rho. \quad \text{......(6)}$$

If J_β is replaced by I_β in (3) or (5) then $I_{-(\beta+1)}$ must be replaced by $J_{-(\beta+1)}$ in (4) or (6).

To prove this result we put $x^2-a^2 = X$, $\rho^2-a^2 = Y$, $\psi(\rho)/\rho = \Psi(Y)$, $2h(x) = H(x)$ in (3) which gives

$$\int_0^X \Psi(Y)(X-Y)^{\frac{1}{2}\beta} J_\beta\{k\sqrt{(X-Y)}\}dY = H(X), \quad \text{..................(7)}$$

an integral equation of convolution type. Thus, taking the Laplace transform of both sides of (7) and using the convolution theorem (see (**9**)), we find, denoting the Laplace transform of $f(x)$ by $\mathscr{L}\{f\}$ or $\bar{f}$, that

$$\bar{\Psi}\mathscr{L}\{X^{\frac{1}{2}\beta}J_\beta(kX^{\frac{1}{2}})\} = \bar{H}. \quad \text{..............................(8)}$$

But (see (**10**), p. 178)

$$\mathscr{L}\{X^{\frac{1}{2}\beta}J_\beta(kX^{\frac{1}{2}})\} = (k/2)^\beta \exp\{-k^2/(4p)\}/p^{\beta+1} \quad \text{...............(9)}$$

and so (8) yields

$$\bar{\Psi}/p = (2/k)^\beta p^\beta \exp\{k^2/(4p)\}\bar{H}. \quad \text{...........................(10)}$$

Now, using (9) again, we may write

$$p^\beta \exp\{k^2/(4p)\} = \mathscr{L}\{(ik/2)^{\beta+1}X^{-\frac{1}{2}(\beta+1)}J_{-(\beta+1)}(ikX^{\frac{1}{2}})\}$$

which, since (see (**8**), p. 77)

$$I_\nu(x) = i^{-\nu}J_\nu(ix), \quad I_\nu(ix) = i^\nu J_\nu(x), \quad \text{.......................(11)}$$

allows (10) to be written in the form

$$\bar{\Psi}/p = (k/2)\mathscr{L}\{X^{-\frac{1}{2}(\beta+1)}I_{-(\beta+1)}(kX^{\frac{1}{2}})\}\bar{H}.$$

Hence, by the convolution theorem and the well-known property (see (**10**), p. 168) $\bar{f}/p = \mathscr{L}\left\{\int_0^x f(t)dt\right\}$, we find

$$\Psi(X) = \frac{k}{2}\frac{d}{dX}\int_0^X (X-Y)^{-\frac{1}{2}(\beta+1)} I_{-(\beta+1)}\{k\sqrt{(X-Y)}\}H(Y)dY. \quad \text{.........(12)}$$

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Reverting to the original variables we immediately obtain (4). The pair (5) and (6) follow at once from (3) and (4) and the effect of replacing J_β by I_β may be seen on replacing k by ik and using (11). Note that, by an elementary change of variables, we see that, if the lower limit of integration in (7) is replaced by a finite constant A, then the same must be done in the solution (12).

Lemma 3. The solution of the integral equation

$$\int_x^\infty \psi(\rho)(\rho^2-x^2)^{\frac{1}{2}\beta} I_\beta\{k\sqrt{(\rho^2-x^2)}\}d\rho = m(x) \quad \text{.................(13)}$$

is

$$\psi(x) = -k\frac{d}{dx}\int_x^\infty \rho m(\rho)(\rho^2-x^2)^{-\frac{1}{2}(\beta+1)} J_{-(\beta+1)}\{k\sqrt{(\rho^2-x^2)}\}d\rho. \quad \text{...(14)}$$

More symmetrically, the solution of the integral equation

$$\int_x^\infty \psi(\rho)\left\{\frac{\sqrt{(\rho^2-x^2)}}{k}\right\}^\beta I_\beta\{k\sqrt{(\rho^2-x^2)}\}d\rho = m(x) \quad \text{..............(15)}$$

is

$$\psi(x) = -\frac{d}{dx}\int_x^\infty \rho m(\rho)\left\{\frac{\sqrt{(\rho^2-x^2)}}{k}\right\}^{-(\beta+1)} J_{-(\beta+1)}\{k\sqrt{(\rho^2-x^2)}\}d\rho \quad \text{...(15')}$$

As in Lemma 2, if I_β is replaced by J_β in (13) or (15) then $J_{-(\beta+1)}$ must be replaced by $I_{-(\beta+1)}$ in (14) or (15'). The formulae remain valid if the upper limit of integration is a finite constant b throughout.

To prove this result we proceed as in the proof of Lemma 2, putting $x^2 = X$, $\rho^2 = Y$, $\psi(\rho)/\rho = \Psi(Y)$, $2m(x) = M(X)$ in (13), and we find

$$\int_X^\infty \Psi(Y)(Y-X)^{\frac{1}{2}\beta} I_\beta\{k\sqrt{(Y-X)}\}dY = M(X). \quad \text{..............(16)}$$

To solve this equation we first consider the equation

$$\int_X^B \Psi_B(Y)(Y-X)^{\frac{1}{2}\beta} I_\beta\{k\sqrt{(Y-X)}\}dY = M(X). \quad \text{..............(17)}$$

The substitutions $Y = -u$, $X = -t$, $B = -a$, $\Psi_B(Y) = F(u)$, $M(X) = N(t)$ yield

$$\int_a^t F(u)(t-u)^{\frac{1}{2}\beta} I_\beta\{k\sqrt{(t-u)}\}du = N(t)$$

or, by (11),

$$\int_a^t F(u)(t-u)^{\frac{1}{2}\beta} J_\beta\{ik\sqrt{(t-u)}\}du = i^\beta N(t).$$

Hence by (7) and (12) (with the lower limit of integration replaced by a—cf. the end of the proof of Lemma 2), we find

$$F(u) = \frac{ik}{2}\frac{d}{du}\int_a^u (u-t)^{-\frac{1}{2}(\beta+1)} I_{-(\beta+1)}\{ik\sqrt{(u-t)}\} i^\beta N(t)dt$$

which gives

$$\Psi_B(X) = -\frac{k}{2}\frac{d}{dX}\int_X^B M(Y)(Y-X)^{-\frac{1}{2}(\beta+1)}J_{-(\beta+1)}\{k\sqrt{(Y-X)}\}dY. \quad \text{...(18)}$$

Thus, letting B tend to infinity in (17) and (18) we obtain, as the solution of (16),

$$\Psi(X) = -\frac{k}{2}\frac{d}{dX}\int_X^\infty M(Y)(Y-X)^{-\frac{1}{2}(\beta+1)}J_{-(\beta+1)}\{k\sqrt{(Y-X)}\}dY \quad \text{...(19)}$$

which immediately yields (14) when we revert to the original variables. The pair of equations (17) and (18) give the solution in the case when the upper limit of integration is a finite constant and the alternative pair of equations (15), (15′) follow at once from (13) and (14).

3. We now turn to the solution of the dual integral equations (1) and (2). Since these equations are linear in $\psi(x)$ we may write

$$\psi(x) = \psi_1(x)+\psi_2(x), \quad \text{....(20)}$$

where $\psi_1(x)$ denotes the solution in the case $g \equiv 0$, $\psi_2(x)$ the solution in the case $f \equiv 0$.

By Lemma 1, a particular solution of the equation

$$\int_0^\infty \psi_1(u)J_\nu(xu)du = 0, \quad x>1, \quad \text{....(21)}$$

is the function

$$\psi_1(x) = x^{\nu+1}(x^2-k^2)^{-\frac{1}{2}\lambda}J_\lambda\{\rho\sqrt{(x^2-k^2)}\},$$

provided $\rho<1$, $\lambda>\nu>-1$. Thus we are led, by superposition, to the function

$$\psi_1(x) = x^{\nu+1}(x^2-k^2)^{-\frac{1}{2}\lambda}\int_0^1 \phi_1(\rho)J_\lambda\{\rho\sqrt{(x^2-k^2)}\}d\rho, \quad \text{....(22)}$$

with $\lambda>\nu>-1$ and $\phi_1(\rho)$ a function to be determined; if we substitute (22) into (21), invert the order of integrations and use Lemma 1, we see that the function (22) automatically satisfies equation (2) (with $g \equiv 0$).

To determine the function $\phi_1(\rho)$ we substitute (22) into (1), put $u^2-k^2 = v^2$ and invert the order of integrations:

$$\int_0^1 \phi_1(\rho)\left[\int_0^\infty (v^2+k^2)^{-\frac{1}{2}\mu}v^{1+2\alpha-\lambda}J_\mu\{x\sqrt{(v^2+k^2)}\}J_\lambda(\rho v)dv\right]d\rho = f(x),$$

$$0\leqq x\leqq 1 \quad \text{....(23)}$$

Thus, if we choose $2\alpha-\lambda = \lambda$, i.e. $\lambda = \alpha$, we can apply Lemma 1 to the inner integral in (23) which, after some simplification, yields the integral equation

$$\int_0^x \rho^\alpha\phi_1(\rho)(x^2-\rho^2)^{\frac{1}{2}(\mu-\alpha-1)}J_{\mu-\alpha-1}\{k\sqrt{(x^2-\rho^2)}\}d\rho = k^{\mu-\alpha-1}x^\mu f(x), \quad \text{...(24)}$$

provided $\mu>\alpha>\nu>-1$.

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But now, by Lemma 2, equation (24) has solution

$$\phi_1(x) = k^{\mu-\alpha}x^{-\alpha}\frac{d}{dx}\int_0^x \rho^{\mu+1}f(\rho)(x^2-\rho^2)^{\frac{1}{2}(\alpha-\mu)}I_{\alpha-\mu}\{k\sqrt{(x^2-\rho^2)}\}d\rho, \quad \text{.........(25)}$$

whence, by (22),

$$\psi_1(x) = k^{\mu-\alpha}x^{\nu+1}(x^2-k^2)^{-\frac{1}{2}\alpha}\int_0^1 \rho^{-\alpha}J_\alpha\{\rho\sqrt{(x^2-k^2)}\}$$

$$\cdot\left[\frac{d}{d\rho}\int_0^\rho u^{\mu+1}f(u)(\rho^2-u^2)^{\frac{1}{2}(\alpha-\mu)}I_{\alpha-\mu}\{k\sqrt{(\rho^2-u^2)}\}du\right]d\rho. \quad \text{.........(26)}$$

To determine $\psi_2(x)$ we rewrite (1) (with $f \equiv 0$) in the form

$$\int_0^\infty (v^2+k^2)^{-\frac{1}{2}(\mu+\nu+1)}v^{2\alpha+1}\psi_2\{\sqrt{(v^2+k^2)}\}J_\mu\{x\sqrt{(v^2+k^2)}\}dv = 0, \quad \text{......(27)}$$

with $0 \leqq x \leqq 1$, $v^2 = u^2-k^2$. Thus, using Lemma 1 again, we see that a particular solution of (27) is the function

$$\psi_2\{\sqrt{(v^2+k^2)}\} = (v^2+k^2)^{\frac{1}{2}+\frac{1}{2}\nu}v^{\lambda-2\alpha}J_\lambda(\rho v), \quad \rho > x,\ \mu > \lambda > -1,$$

i.e.

$$\psi_2(u) = u^{1+\nu}(u^2-k^2)^{\frac{1}{2}\lambda-\alpha}J_\lambda\{\rho\sqrt{(u^2-k^2)}\}, \quad \rho > x,\ \mu > \lambda > -1.$$

Therefore, if we take

$$\psi_2(x) = x^{1+\nu}(x^2-k^2)^{\frac{1}{2}\lambda-\alpha}\int_1^\infty \phi_2(\rho)J_\lambda\{\rho\sqrt{(x^2-k^2)}\}d\rho, \quad \mu > \lambda > -1, \text{...(28)}$$

then equation (1) (with $f \equiv 0$) will be automatically satisfied and, on substituting (28) into (2) and inverting the order of integrations, we find

$$\int_1^\infty \phi_2(\rho)\left[\int_0^\infty u^{1+\nu}(u^2-k^2)^{\frac{1}{2}\lambda-\alpha}J_\nu(xu)J_\lambda\{\rho\sqrt{(u^2-k^2)}\}du\right]d\rho = g(x),$$

$$x > 1. \quad \text{.........(29)}$$

To apply Lemma 1 to the inner integral in (29) we must take $\frac{1}{2}\lambda-\alpha = -\frac{1}{2}\lambda$, i.e. $\lambda = \alpha$, and, after some simplification, this yields the integral equation

$$\int_x^\infty \phi_2(\rho)\rho^{-\alpha}(\rho^2-x^2)^{\frac{1}{2}(\alpha-\nu-1)}I_{\alpha-\nu-1}\{k\sqrt{(\rho^2-x^2)}\}d\rho = k^{\alpha-\nu-1}x^{-\nu}g(x), \text{......(30)}$$

provided $\mu > \alpha > \nu > -1$.

But now, by Lemma 3, the solution of equation (30) is

$$\phi_2(x) = -k^{\alpha-\nu}x^\alpha\frac{d}{dx}\int_x^\infty \rho^{1-\nu}g(\rho)(\rho^2-x^2)^{\frac{1}{2}(\nu-\alpha)}J_{\nu-\alpha}\{k\sqrt{(\rho^2-x^2)}\}d\rho, \quad \text{......(31)}$$

whence, by (28)

$$\psi_2(x) = -k^{\alpha-\nu}x^{1+\nu}(x^2-k^2)^{-\frac{1}{2}\alpha}\int_1^\infty \rho^\alpha J_\alpha\{\rho\sqrt{(x^2-k^2)}\}$$

$$\cdot\left[\frac{d}{d\rho}\int_\rho^\infty u^{1-\nu}g(u)(u^2-\rho^2)^{\frac{1}{2}(\nu-\alpha)}J_{\nu-\alpha}\{k\sqrt{(u^2-\rho^2)}\}du\right]d\rho. \text{.........(32)}$$

Thus the solution of equations (1) and (2) is given by (20), (26) and (32), provided $\mu>\alpha>\nu>-1$.

The solution may be written in various forms. Thus, integrating by parts in (26), we have

$$\psi_1(x) = k^{\mu-\alpha}x^{\nu+1}(x^2-k^2)^{-\frac{1}{2}\alpha}\Big[[\rho^{-\alpha}J_\alpha\{\rho\surd(x^2-k^2)\}F(\rho)]_0^1 + \int_0^1 \surd(x^2-k^2)\rho^{-\alpha}J_{\alpha+1}\{\rho\surd(x^2-k^2)\}F(\rho)d\rho\Big]$$

$$= k^{\mu-\alpha}x^{\nu+1}(x^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\surd(x^2-k^2)\}F(1) + k^{\mu-\alpha}x^{\nu+1}(x^2-k^2)^{\frac{1}{2}(1-\alpha)}\int_0^1 \rho^{-\alpha}J_{\alpha+1}\{\rho\surd(x^2-k^2)\}F(\rho)d\rho, \quad \text{.........(33)}$$

where

$$F(\rho) = \int_0^\rho u^{\mu+1}f(u)(\rho^2-u^2)^{\frac{1}{2}(\alpha-\mu)}I_{\alpha-\mu}\{k\surd(\rho^2-u^2)\}du \quad \text{...........(34)}$$

and we have used the result ((8), p. 45)

$$\frac{d}{dx}\{x^{-\nu}J_\nu(xy)\} = -yx^{-\nu}J_{\nu+1}(xy). \quad \text{.....................(35)}$$

Similarly, integrating (32) by parts we find

$$\psi_2(x) = k^{\alpha-\nu}x^{1+\nu}(x^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\surd(x^2-k^2)\}G(1) + k^{\alpha-\nu}x^{1+\nu}(x^2-k^2)^{\frac{1}{2}(1-\alpha)}\int_1^\infty \rho^\alpha J_{\alpha-1}\{\rho\surd(x^2-k^2)\}G(\rho)d\rho, \quad \text{......(36)}$$

where

$$G(\rho) = \int_\rho^\infty u^{1-\nu}g(u)(u^2-\rho^2)^{\frac{1}{2}(\nu-\alpha)}J_{\nu-\alpha}\{k\surd(u^2-\rho^2)\}du, \quad \text{.........(37)}$$

and we have used the result ((8), p. 45)

$$\frac{d}{dx}\{x^\nu J_\nu(xy)\} = yx^\nu J_{\nu-1}(xy), \quad \text{.......................(38)}$$

the asymptotic behaviour of $J_\nu(x)$ for large x and the assumption that $g(x)$ is such as to make $G(\rho) = o(\rho^{\frac{1}{2}-\alpha})$ as $\rho\to\infty$.

4. It is worth noting that the quantities that are of interest in applications are not only the solution $\psi(x)$ of equations (1) and (2) but also the left-hand sides of these equations, for the ranges $x>1$ and $0\leqq x\leqq 1$, respectively, i.e. the functions

$$f_1(x) = \int_k^\infty u^{-\mu-\nu}(u^2-k^2)^\alpha\psi(u)J_\mu(xu)du, \quad x>1, \quad \text{...........(39)}$$

$$g_1(x) = \int_0^\infty \psi(u)J_\nu(xu)du, \quad 0\leqq x\leqq 1. \quad \text{....................(40)}$$

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There are several ways of evaluating these, perhaps the most direct being to substitute (20), (22) and (28) into the right-hand sides of (39) and (40). This yields, after inversion of the order of integrations,

$$f_1(x) = \int_0^1 \phi_1(\rho)\left[\int_k^\infty u^{-\mu-\nu}(u^2-k^2)^\alpha J_\mu(xu) \right.$$
$$\left. .\,u^{\nu+1}(u^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\rho\sqrt{(u^2-k^2)}\}du\right]d\rho$$
$$+\int_1^\infty \phi_2(\rho)\left[\int_k^\infty u^{-\mu-\nu}(u^2-k^2)^\alpha J_\mu(xu)\right.$$
$$\left. .\,u^{\nu+1}(u^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\rho\sqrt{(u^2-k^2)}\}du\right]d\rho$$
$$= x^{-\mu}\int_0^1 \rho^{-\alpha}\phi_1(\rho)\left\{\frac{\sqrt{(x^2-\rho^2)}}{k}\right\}^{\mu-\alpha-1} J_{\mu-\alpha-1}\{k\sqrt{(x^2-\rho^2)}\}d\rho$$
$$+x^{-\mu}\int_1^x \rho^{\alpha}\phi_2(\rho)\left\{\frac{\sqrt{(x^2-\rho^2)}}{k}\right\}^{\mu-\alpha-1} J_{\mu-\alpha-1}\{k\sqrt{(x^2-\rho^2)}\}d\rho, \quad \ldots(41)$$

where, after putting $v^2 = u^2-k^2$ in the inner integrals, we have used Lemma 1 and the fact that $x>1$. Similarly,

$$g_1(x) = \int_0^1 \phi_1(\rho)\left[\int_0^\infty u^{\nu+1}(u^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\rho\sqrt{(u^2-k^2)}\}J_\nu(xu)du\right]d\rho$$
$$+\int_1^\infty \phi_2(\rho)\left[\int_0^\infty u^{\nu+1}(u^2-k^2)^{-\frac{1}{2}\alpha}J_\alpha\{\rho\sqrt{(u^2-k^2)}\}J_\nu(xu)du\right]d\rho$$
$$= x^\nu\int_x^1 \rho^{-\alpha}\phi_1(\rho)\left\{\frac{\sqrt{(\rho^2-x^2)}}{k}\right\}^{\alpha-\nu-1} I_{\alpha-\nu-1}\{k\sqrt{(\rho^2-x^2)}\}d\rho$$
$$+x^\nu\int_1^\infty \rho^{-\alpha}\phi_2(\rho)\left\{\frac{\sqrt{(\rho^2-x^2)}}{k}\right\}^{\alpha-\nu-1} I_{\alpha-\nu-1}\{k\sqrt{(\rho^2-x^2)}\}d\rho, \quad \ldots\ldots(42)$$

where we have used Lemma 1, the fact that $0\leqq x\leqq 1$ and equation (11).

Thus, if we now substitute (25) and (31) into (41) and (42), we obtain

$$f_1(x) = kx^{-\mu}\int_0^1 (x^2-\rho^2)^{\frac{1}{2}(\mu-\alpha-1)}J_{\mu-\alpha-1}\{k\sqrt{(x^2-\rho^2)}\}\frac{dF}{d\rho}d\rho$$
$$-k^{1+2\alpha-\mu-\nu}x^{-\mu}\int_1^\infty \rho^{2\alpha}(x^2-\rho^2)^{\frac{1}{2}(\mu-\alpha-1)}J_{\mu-\alpha-1}\{k\sqrt{(x^2-\rho^2)}\}\frac{dG}{d\rho}d\rho \quad \ldots\ldots\ldots(43)$$

and

$$g_1(x) = k^{1+\mu+\nu-2\alpha}x^\nu\int_x^1 \rho^{-2\alpha}(\rho^2-x^2)^{\frac{1}{2}(\alpha-\nu-1)}I_{\alpha-\nu-1}\{k\sqrt{(\rho^2-x^2)}\}\frac{dF}{d\rho}d\rho$$
$$-kx^\nu\int_1^\infty (\rho^2-x^2)^{\frac{1}{2}(\alpha-\nu-1)}I_{\alpha-\nu-1}\{k\sqrt{(\rho^2-x^2)}\}\frac{dG}{d\rho}d\rho, \quad \ldots\ldots(44)$$

where $F(\rho)$, $G(\rho)$ are given by (34) and (37), respectively.

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5. Finally we observe that the conditions $\mu > \alpha > \nu > -1$ imposed above can be circumvented in various ways. One of these is to alter the order of the Bessel functions occurring in (1) and (2) before starting the solution.

Thus, if we multiply (1) by x^{μ} and differentiate both sides of the resulting equation with respect to x, using (38), we obtain the equations

$$\int_k^\infty u^{1-\mu-\nu}(u^2-k^2)^\alpha \psi(u) J_{\mu-1}(xu)\,du = x^{-\mu}\frac{d}{dx}\{x^\mu f(x)\}, \quad 0 \leqq x \leqq 1, \quad \text{.........(45)}$$

$$\int_0^\infty \psi(u) J_\nu(xu)\,du = g(x), \quad x > 1, \quad \text{........................(46)}$$

which may be solved as above provided $\mu-1>\alpha>\nu>-1$. If now we multiply both sides of (46) by $x^{-\nu}$ and repeat the above procedure (using (35)), we find

$$\int_k^\infty u^{-\{(\mu-1)+(\nu+1)\}}(u^2-k^2)^\alpha \phi(u) J_{\mu-1}(xu)\,du = x^{-\mu}\frac{d}{dx}\{x^\mu f(x)\}, \quad 0 \leqq x \leqq 1, \quad (47)$$

$$\int_0^\infty \phi(u) J_{\nu+1}(xu)\,du = -x^\nu \frac{d}{dx}\{x^{-\nu} g(x)\}, \quad x > 1, \quad \text{...........(48)}$$

where $\phi(x) = x\psi(x)$, which may be solved as above provided $\mu-1>\alpha>\nu+1 > -1$.

Similarly, if we multiply (1) by $x^{-\mu}$, (2) by x^ν and repeat the above procedure, we find

$$\int_k^\infty u^{-\{(\mu+1)+(\nu-1)\}}(u^2-k^2)^\alpha \phi(u) J_{\mu+1}(xu)\,du = -x^{\mu}\frac{d}{dx}\{x^{-\mu} f(x)\}, \quad 0 \leqq x \leqq 1, \quad \text{.........(49)}$$

$$\int_0^\infty \phi(u) J_{\nu-1}(xu)\,du = x^{-\nu} \frac{d}{dx}\{x^{\nu} g(x)\}, \quad x > 1, \quad \text{..............(50)}$$

where $\phi(x) = x\psi(x)$, which may be solved as above provided $\mu+1>\alpha>\nu-1 > -1$.

An alternative is to integrate instead of differentiating. Thus, if we multiply (1) by $x^{\mu+1}$ and integrate with respect to x, from 0 to x, using (38), the behaviour of $J_\nu(x)$ for small x and the fact that $\mu > -1$, we obtain the equations

$$\int_k^\infty u^{-\mu-\nu-1}(u^2-k^2)^\alpha \psi(u) J_{\mu+1}(xu)\,du = x^{-\mu-1}\int_0^x u^{\mu+1} f(u)\,du, \quad 0 \leqq x \leqq 1, \quad (51)$$

$$\int_0^\infty \psi(u) J_\nu(xu)\,du = g(x), \quad x > 1, \quad \text{.......................(52)}$$

which may be solved as above provided $\mu+1>\alpha>\nu>-1$, $\mu+1>0$. Finally, if we now multiply (52) by $x^{-\nu}$ and differentiate with respect to x, we obtain

$$\int_k^\infty u^{-\{(\mu+1)+(\nu+1)\}}(u^2-k^2)^\alpha \phi(u) J_{\mu+1}(xu)\,du = x^{-\mu-1}\int_0^x u^{\mu+1} f(u)\,du, \quad 0 \leqq x \leqq 1, \text{........(53)}$$

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$$\int_0^\infty \phi(u)J_{\nu+1}(xu)du = -x^\nu \frac{d}{dx}\{x^{-\nu}g(x)\}, \quad x>1, \quad \text{..............(54)}$$

where $\phi(x) = x\psi(x)$, which may be solved as above provided $\mu+1>\alpha>\nu+1>-1$, $\mu+1>0$. These processes may be repeated and combined as often as necessary. Other examples of the use of such methods are given by Peters (**2**) and Noble (**11**) for the case $k = 0$.

A more satisfactory method of extending the range of values of the parameters, as well as some discussion of the conditions to be imposed on the functions $f(x)$ and $g(x)$ in equations (1) and (2) for the various procedures to be valid, will be given in a subsequent paper.

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