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Some Results on Annihilating-ideal Graphs

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Abstract. The annihilating-ideal graph of a commutative ring R, denoted by $\mathbb{AG}(R)$, is a graph whose vertex set consists of all non-zero annihilating ideals and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. Here we show that if R is a reduced ring and the independence number of $\mathbb{AG}(R)$ is finite, then the edge chromatic number of $\mathbb{AG}(R)$ equals its maximum degree and this number equals $2^{|\mathrm{Min}(R)|-1} - 1$; also, it is proved that the independence number of $\mathbb{AG}(R)$ equals $2^{|\mathrm{Min}(R)|-1}$, where $\mathrm{Min}(R)$ denotes the set of minimal prime ideals of R. Then we give some criteria for a graph to be isomorphic with an annihilating-ideal graph of a ring. For example, it is shown that every bipartite annihilating-ideal graph is a complete bipartite graph with at most two horns. Among other results, it is shown that a finite graph $\mathbb{AG}(R)$ is not Eulerian, and that it is Hamiltonian if and only if R contains no Gorenstain ring as its direct summand.

1 Introduction

Throughout this paper, all graphs are assumed to be undirected simple graphs. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *neighborhood*, *closed-neighborhood* and *degree* of a vertex v of G are denoted by $N_G(v)$, $N_G[v]$, and $d_G(v)$, respectively. The subscript G is usually dropped when there is no confusion. Also, the *minimum degree* and the *maximum degree* of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *girth* of a graph G, denoted by $\mathrm{girth}(G)$, is the length of a shortest cycle contained in G. If G does not contain any cycle, its girth is defined to be infinity. The *distance* between two vertices u and v of a graph is denoted by $d(u,v)$. The *diameter* of a connected graph G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of the vertices of G. An *induced subgraph* of G on $X \subseteq V(G)$, denoted by $G[X]$, is the subgraph with the vertex set $V(G[X]) = X$ and the edge set $E(G[X]) = \{\{u,v\} \in E(G) \mid u,v \in X\}$. A vertex v in a connected graph G is called a *cut* vertex if $G \setminus \{v\} = G[V(G) \setminus \{v\}]$ is a disconnected graph. A *bipartite graph* is a graph whose vertex set can be divided into two disjoint parts X and Y such that both of the induced subgraphs $G[X]$ and $G[Y]$ have no edges. Moreover, a *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then it is said to be a *star graph*. A vertex of a bipartite graph G with parts X and Y, is called an *end* vertex if $\deg(v) = 1$; also, it is called a *full* vertex if either $N(v) = X$ or $N(v) = Y$. For a graph G, the *independence number* of G and the *edge chromatic number* of G are denoted by $\alpha(G)$ and $\chi'(G)$, respectively. For more details about the terminology of graphs used here, see [17].

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There are many papers on assigning a graph to the set of ideals of a ring [1–3, 5, 7, 9–11, 14, 15]. Let R be a commutative ring with identity. We call an ideal I of R, an *annihilating-ideal* if there exists a non-zero ideal J of R such that $IJ = (0)$. We use the notations $\mathbb{I}(R)$ and $\mathbb{A}(R)$, for the set of non-zero ideals of R and the set of annihilating-ideals of R, respectively. Also, by $\operatorname{Min}(R)$, we denote the set of all minimal prime ideals of R. By the *annihilating-ideal graph* of R, $\mathbb{AG}(R)$, we mean the graph with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. The concept of the annihilating-ideal graph of a commutative ring was first introduced in [5]. We say that a graph G is an annihilating-ideal graph if $G \cong \mathbb{AG}(R)$ for some ring R. In Section 2, it is shown that for every reduced ring R,

$$\chi'(\mathbb{AG}(R)) = \Delta(\mathbb{AG}(R)) = \alpha(\mathbb{AG}(R)) - 1 = 2^{|\operatorname{Min}(R)|-1} - 1.$$

Moreover, we find a sufficient condition under which $\mathbb{AG}(R)$ belongs to Class 1, for every Artinian ring R. In Section 3 we give some criteria for a graph to be an annihilating-ideal graph. For example, we prove that any bipartite annihilating-ideal graph is a complete bipartite graph with at most two horns. Section 4 is devoted to investigating the cycles in annihilating-ideal graphs. Finally, we show that a finite annihilating-ideal graph $\mathbb{AG}(R)$ is not Eulerian, and this graph is Hamiltonian if and only if R contains no Gorenstein ring as its direct summand.

2 The Independence Number and the Edge Chromatic Number

In this section we use the maximal intersecting families to obtain a lower bound for the independence number of $\mathbb{AG}(R)$.

Proposition 2.1 *If $\alpha(\mathbb{AG}(R)) < \infty$, then every element of R is either a zero-divisor or a unit. Moreover, if R is Noetherian, then R has finitely many maximal ideals.*

Proof Suppose to the contrary that R contains an element, say x, which is neither a zero-divisor nor a unit. Then it is clear that $\{(zx^n) \mid n \in \mathbb{N}\}$, in which z is a zero-divisor, is an infinite independent set of $\mathbb{AG}(R)$, a contradiction. Moreover, if R is Noetherian, then [16, Corollary 9.36] and the fact that the set of associated prime ideals of a Noetherian ring is a finite non-empty set, imply that R has finitely many maximal ideals. ∎

We note that the converse of the previous proposition is not true. To see this, let $R \cong S \times T$, where S and T are Artinian local rings and $|\mathbb{I}(T)| = \infty$. Then

$$\{S \times I \mid I \text{ is a nontrivial ideal of } T\}$$

is an infinite independent set of $\mathbb{AG}(R)$. So $\alpha(\mathbb{AG}(R)) = \infty$.

Let R be a decomposable ring such that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where every R_i is a ring. Then we use the following notation:

$$\mathcal{S}(R) = \big\{ (0) \neq I = I_1 \times I_2 \times \cdots \times I_n \lhd R \,|\, \forall\, 1 \leq k \leq n : \ I_k \in \{(0), R_k\} \big\}.$$

Also, we denote the induced subgraph of $\mathbb{AG}(R)$ on $\mathcal{S}(R)$ by $G_{\mathcal{S}}(R)$.

Lemma 2.2 *If $R \cong R_1 \times R_2 \times \cdots \times R_n$ is a ring, then $\alpha(G_{\mathcal{S}}(R)) = 2^{n-1}$.*

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Proof For every ideal $I = I_1 \times I_2 \times \cdots \times I_n$, let $\Delta_I = \{k \mid 1 \leq k \leq n \text{ and } I_k = R_k\}$. Then two distinct vertices I and J in $G_{\mathcal{S}}(R)$ are not adjacent if and only if $\Delta_I \cap \Delta_J \neq \varnothing$. So there is a one-to-one correspondence between the independent sets of $G_{\mathcal{S}}(R)$ and the set of families of pairwise intersecting subsets of $[n] = \{1, 2, \ldots, n\}$. Assume that $\mathcal{F}$ is a maximum intersecting family of the subsets of $[n]$. Setting $\mathcal{A} = \{I \in \mathcal{S}(R) \mid \Delta_I \in \mathcal{F}\}$, we deduce that $\mathcal{A}$ is an independent set of $G_{\mathcal{S}}(R)$ with maximum size. This implies that $\alpha(G_{\mathcal{S}}(R)) = |\mathcal{A}| = |\mathcal{F}|$. So [13, Lemma 2.1] completes the proof. ■

Using [4, Theorem 8.7] and Lemma 2.2, we have the following corollary.

Corollary 2.3 *Let R be an Artinian ring with n maximal ideals. Then $\alpha(\mathbb{AG}(R)) \geq 2^{n-1}$; moreover, the equality holds if and only if R is reduced.*

Lemma 2.4 ([12, Proposition 1.5]) *Let R be a ring and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a finite set of distinct minimal prime ideals of R. Let $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. Then $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$.*

Proposition 2.5 *If $|\mathrm{Min}(R)| \geq n$, then $\alpha(\mathbb{AG}(R)) \geq 2^{n-1}$.*

Proof Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ be a subset of $\mathrm{Min}(R)$ and $S = R \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i$. By Lemma 2.4, there exists a ring isomorphism $R_S \cong R_{\mathfrak{p}_1} \times \cdots \times R_{\mathfrak{p}_n}$. On the other hand, if I_S, J_S are two non-adjacent vertices of $\mathbb{AG}(R_S)$, then it is not difficult to check that I, J are two non-adjacent vertices of $\mathbb{AG}(R)$. Thus $\alpha(\mathbb{AG}(R)) \geq \alpha(\mathbb{AG}(R_S))$ and so by Lemma 2.2, we deduce that $\alpha(\mathbb{AG}(R)) \geq 2^{n-1}$. ■

From the previous proposition, we have the following immediate corollary which shows that the finiteness of $\alpha(\mathbb{AG}(R))$ implies the finiteness of the set of minimal prime ideals of R.

Corollary 2.6 *If R contains infinitely many minimal prime ideals, then the independence number of $\mathbb{AG}(R)$ is infinite.*

Vizing's Theorem [18, p. 16] states that if G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. A graph G belongs to Class 1 if $\chi'(G) = \Delta(G)$ and it belongs to Class 2 if $\chi'(G) = \Delta(G) + 1$.

Now we determine sufficient conditions under which an annihilating-ideal graph belongs to Class 1. First of all, we recall the following lemma.

Lemma 2.7 ([6, Corollary 5.4]) *Let G be a simple graph. Suppose that for every vertex u of maximum degree, there exists an edge $\{u, v\}$ such that $\Delta(G) - d(v) + 2$ is more than the number of vertices with maximum degree in G. Then $\chi'(G) = \Delta(G)$.*

Recall that for every local ring $(R, \mathfrak{m})$, $\mathrm{Ann}(\mathfrak{m})$ is a vector space on the field $\frac{R}{\mathfrak{m}}$. The dimension of this vector space, denoted by $r(R)$, is called the *type* of R. By [1, Theorem 3], $\mathfrak{m}^2 = (0)$ if and only if $\mathbb{AG}(R)$ is a complete graph and in this case $\chi'(\mathbb{AG}(R)) = \Delta(\mathbb{AG}(R))$ if and only if R has an even number of non-trivial ideals.

Theorem 2.8 *Let $(R, \mathfrak{m})$ be a local ring with t proper ideals. If $\mathfrak{m}^2 \neq (0)$ and $r(R) = r < \log_2(t+2) - 1$, then $\mathbb{AG}(R)$ belongs to Class 1.*

Proof Since $\dim_{\frac{R}{\mathfrak{m}}} \operatorname{Ann}(\mathfrak{m}) = r$, there are $2^r - 1$ non-zero ideals (subspaces) which are contained in $\operatorname{Ann}(\mathfrak{m})$. Also, it is clear that every ideal $I \subseteq \operatorname{Ann}(\mathfrak{m})$ is adjacent to every other vertex of $\mathbb{AG}(R)$. So a vertex I of $\mathbb{AG}(R)$ has maximum degree if and only if $I \subseteq \operatorname{Ann}(\mathfrak{m})$. For every such vertex, we have

$$\Delta(\mathbb{AG}(R)) - d(\mathfrak{m}) + 2 = t - 2 - (2^r - 1) + 2 > 2^{r+1} - 2 - 2^r + 1 = 2^r - 1.$$

Therefore, the assertion follows from Lemma 2.7. ∎

Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a ring, where $(R_i, \mathfrak{m}_i)$ is an Artinian local ring of type r_i and $|\mathbb{I}(R_i)| = t_i$ for every $1 \le i \le n$. With no loss of generality, we can assume that $t_1 = t_2 = \cdots = t_k > t_{k+1} \ge \cdots \ge t_n$, for some positive integer $k \le n$. It is not hard to check that I is a vertex of maximum degree if and only if

$$I = (0) \times \cdots \times (0) \times \operatorname{Ann}(\mathfrak{m}_j) \times (0) \times \cdots \times (0)$$

for some $1 \le j \le k$ and in this case $M_j = R_1 \times \cdots \times R_{j-1} \times \mathfrak{m}_j \times R_{j+1} \times \cdots \times R_n$ is an adjacent vertex to I. Thus $\Delta(\mathbb{AG}(R)) - d(M_j) + 2 = t_1 \prod_{i=2}^n (t_i + 1) - 2^{r_j} + 1$, where 2^{r_j} is the number of ideals contained in $\operatorname{Ann}(\mathfrak{m}_j)$. Moreover, the number of vertices with maximum degree is $\sum_{i=1}^k (2^{r_i} - 1)$. Now setting $s = \max\{r_1, r_2, \ldots, r_k\}$, Lemma 2.7 implies that the sufficient condition for $\mathbb{AG}(R)$ to be of Class 1 is that

$$t_1 \prod_{i=2}^n (t_i + 1) > \sum_{i=1}^k 2^{r_i} + 2^s - k - 1.$$

Therefore, we have proved the following result.

Proposition 2.9 *Let $R \cong R_1 \times R_2 \times \cdots \times R_n$ be a ring, where every R_i is an Artinian local ring of type r_i and $|\mathbb{I}(R_i)| = t_i$, such that $t_1 = t_2 = \cdots = t_k > t_{k+1} \ge \cdots \ge t_n$ and $r_1 \ge r_2 \ge \cdots \ge r_k$. If $t_1 \prod_{i=2}^n (t_i + 1) > \sum_{i=1}^k 2^{r_i} + 2^{r_1} - k - 1$, then $\mathbb{AG}(R)$ belongs to Class* 1*.*

An Artinian local ring $(R, \mathfrak{m})$ is called *Gorenstein* if R has type one. Also, it is said that an Artinian ring R is Gorenstein if $R_{\mathfrak{m}}$ is a Gorenstein ring for every maximal ideal $\mathfrak{m}$ of R. From the previous proposition, we have the following immediate corollary.

Corollary 2.10 *If R is an Artinian Gorenstein ring, then $\chi'(\mathbb{AG}(R)) = \Delta(\mathbb{AG}(R))$.*

It is clear that any finite direct product of fields is a Gorenstein ring. So, we have the following corollary.

Corollary 2.11 *Let $R \cong F_1 \times F_2 \times \cdots \times F_n$, where every F_i is a field. Then $\chi'(\mathbb{AG}(R)) = \Delta(\mathbb{AG}(R)) = 2^{n-1} - 1$.*

We finish this section with the following theorem.

Theorem 2.12 *If R is a reduced ring and $\alpha(\mathbb{AG}(R)) < \infty$, then*

$$\chi'(\mathbb{AG}(R)) = \Delta(\mathbb{AG}(R)) = \alpha(\mathbb{AG}(R)) = 2^{|\operatorname{Min}(R)|-1} - 1.$$

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Proof Since $\alpha(\mathbb{AG}(R)) < \infty$, Proposition 2.1 implies that every element of R is either a zero-divisor or a unit. Moreover, Corollary 2.6 implies that R contains only finitely many minimal prime ideals. Let $\operatorname{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$. Then by Lemma 2.4, $R_S \cong R_{\mathfrak{p}_1} \times R_{\mathfrak{p}_2} \times \cdots \times R_{\mathfrak{p}_n}$, where $S = \bigcup_{i=1}^n \mathfrak{p}_i$. Let I and J be two non-trivial ideals of R. If $I_S = J_S$, then for every $x \in I$, there are elements $s \in S$ and $y \in J$ such that $sx = y$ which implies that $x = s^{-1}y \in J$. Thus $I \subseteq J$. A similar proof shows that $J \subseteq I$. Therefore, there is a one-to-one correspondence between the ideals of R and the ideals of R_S. Next we show that $IJ = (0)$ if and only if $I_S J_S = (0)$. If $IJ = (0)$, then it is clear that $I_S J_S = (0)$. Now suppose that $I_S J_S = (0)$ and choose $x \in I$ and $y \in J$. Then there exists an element $t \in S$ such that $txy = 0$. By [8, Corollary 2.4], t is not a zero-divisor and so $xy = 0$. Thus $IJ = (0)$ if and only if $I_S J_S = (0)$. The above argument shows that $\mathbb{AG}(R) \cong \mathbb{AG}(R_S)$. By [12, Proposition 1.1], we can assume that $R_S \cong F_1 \times F_2 \times \cdots \times F_n$, where every F_i is a field. Therefore, Lemma 2.2 and Corollary 2.11 complete the proof. ∎

3 Some Criteria for Graphs to be Annihilating-ideal Graphs

First we determine when a bipartite graph is an annihilating-ideal graph. For this we need the following two lemmas.

Lemma 3.1 *Assume that R is a ring such that $\mathbb{AG}(R)$ is a bipartite graph with parts X, Y. If $I \in X$, $J \in Y$, and $IJ \neq (0)$, then either $d(I) = 1$ or $d(J) = 1$.*

Proof Let $I \in X$ and $J \in Y$ be two non-adjacent vertices of $\mathbb{AG}(R)$. Then it is clear that $N(I) \cup N(J) \subseteq N[IJ]$. If $IJ \in X$, then $N(J) = N(J) \setminus Y \subseteq N[IJ] \setminus Y = \{IJ\}$ and thus $d(J) = 1$. If $IJ \in Y$, then a similar proof shows that $d(I) = 1$. ∎

As an immediate consequence, we obtain the following result.

Corollary 3.2 *If G is a bipartite annihilating-ideal graph containing no end vertices, then G is a complete bipartite graph.*

Now we recall the following theorem from [5, Theorem 2.1].

Theorem 3.3 *For every ring R the annihilating-ideal graph $\mathbb{AG}(R)$ is connected and* $\operatorname{diam}(\mathbb{AG}(R)) \leq 3$. *Moreover, if $\mathbb{AG}(R)$ contains a cycle, then* $\operatorname{girth}(\mathbb{AG}(R)) \leq 4$.

Let G be a bipartite graph with parts X, Y. We define a vertex $v \in V(G)$ to be *full* if either $N(v) = X$ or $N(v) = Y$.

Lemma 3.4 *Let $\mathbb{AG}(R)$ be a bipartite annihilating-ideal graph with parts X, Y. If there exists a vertex $I \in X$ such that $d(I) = 1$, then the unique element in $N(I)$ is a full vertex.*

Proof Suppose to the contrary that $J \in N(I)$ is not a full vertex. Then there is a non-trivial ideal $K \in X$ such that $KJ \neq (0)$ and the shortest one of the possible paths linking I and K is as shown in Figure 1

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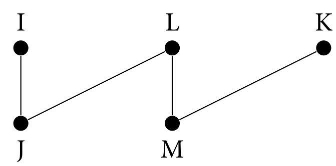

Figure 1.

for some non-trivial ideals L, M of R. Thus $d(I,K) \geq 4$, which is impossible by Theorem 3.3. ∎

Recall that a *two-star graph* is a graph G consisting of two star graphs with a bridge connecting the two sub-centers x and y. A *horn* in a graph consists of some end vertices all adjacent to a common vertex.

Theorem 3.5 *Every bipartite, annihilating-ideal graph is a complete bipartite graph with at most two horns.*

Proof Let $G \cong \mathbb{AG}(R)$ be a bipartite, annihilating-ideal graph for some ring R. If G is a star graph, then there is nothing to prove. So we can assume that $\mathbb{AG}(R)$ is a bipartite graph with parts X, Y, and $|X| > 1$, $|Y| > 1$. We claim that there exist $I \in X$ and $J \in Y$ such that I and J are full vertices. If G is complete bipartite, then there is nothing to prove. So, we assume that G is not a complete bipartite graph and there exist non-trivial ideals $K \in X$ and $L \in Y$ such that K is not adjacent to L. So by Lemma 3.1 and with no loss of generality, we can assume that $d(K) = 1$. Let $J \in N(K)$. Then $J \in Y$ is a full vertex by Lemma 3.4. Now assume that each vertex in X is not full. Then $N(\mathfrak{a}) \subsetneq Y$ for any $\mathfrak{a} \in X$. If $d(\mathfrak{a}) = 1$ for any $\mathfrak{a} \in X$, then $N(\mathfrak{a}) = \{J\}$, for any $\mathfrak{a} \in X$, and thus each vertex in $Y \setminus \{J\}$ is an isolated vertex, which contradicts Theorem 3.3. Hence there exists $\mathfrak{a} \in X$ with $2 \leq d(\mathfrak{a})$ and $N(\mathfrak{a}) \subsetneq Y$. Fix a vertex $J_1 \in Y \setminus N(\mathfrak{a})$. Then J_1 is not adjacent to $\mathfrak{a}$ and so $d(J_1) = 1$ by Lemma 3.1. It follows from Lemma 3.4 that the unique element in $N(J_1)$ is full, a contradiction again. So the claim is proved. Set

$$A = \{\mathfrak{p} \in V(G) \mid \mathfrak{p} \text{ is adjacent to } I \text{ and } d(\mathfrak{p}) > 1\},$$
$$U = \{\mathfrak{p} \in V(G) \mid \mathfrak{p} \text{ is adjacent to } I \text{ and } d(\mathfrak{p}) = 1\},$$
$$B = \{\mathfrak{q} \in V(G) \mid \mathfrak{q} \text{ is adjacent to } J \text{ and } d(\mathfrak{q}) > 1\},$$
$$V = \{\mathfrak{q} \in V(G) \mid \mathfrak{q} \text{ is adjacent to } J \text{ and } d(\mathfrak{q}) = 1\}.$$

Then $V(G) = A \cup B \cup U \cup V \cup \{I, J\}$ and the graph G is of the type in Figure 2

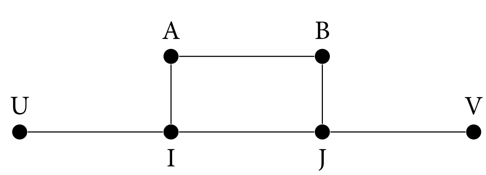

Figure 2.

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where A, B, U, and V are pairwise disjoint subsets of $V(G)$. To complete the proof, we show that the induced subgraph on $A \cup B$ is a complete bipartite graph. Suppose to the contrary, $\mathfrak{p} \in A$ and $\mathfrak{q} \in B$ are two nonadjacent vertices. Then Lemma 3.1 implies that either $d(\mathfrak{p}) = 1$ or $d(\mathfrak{q}) = 1$, a contradiction. Hence the induced subgraph on $A \cup B$ is a complete bipartite graph, as desired. ∎

From the previous theorem, we have the following immediate corollary.

Corollary 3.6 *Let G be an annihilating-ideal graph. If G is a tree, then G is either a star or a two-star graph.*

Proof From Theorem 3.5, we deduce that G is of the type in Figure 2. Since G is a tree, then either $A = \varnothing$ or $B = \varnothing$. So the assertion follows. ∎

Recall that a graph G is a *refinement of a star graph* if it contains a vertex, say v, such that $N[v] = V(G)$.

From [5, Theorem 2.2], we know that an annihilating-ideal graph $\mathbb{AG}(R)$ is a refinement of a star graph if and only if either $Z(R)$ is an annihilator ideal or $R \cong F \times D$, where F is a field and D is an integral domain. In the following theorem we investigate the existence of end vertices in an annihilating-ideal graph that is not a refinement of a star graph.

Theorem 3.7 *Let G be an annihilating-ideal graph which is not a refinement of a star graph. Then G has an end vertex if and only if G has a cut vertex.*

Proof Let $G \cong \mathbb{AG}(R)$ for some ring R. If G has a vertex I of degree one, then $\mathrm{Ann}(I)$ is the unique vertex, adjacent to I. Thus $\mathrm{Ann}(I)$ is a cut vertex of G. Conversely, assume that G has a cut vertex, say K. Then there exist two disjoint subsets X, Y such that $V(G) \setminus \{K\} = X \cup Y$ and $IJ \neq (0)$, for every ideal $I \in X$ and $J \in Y$. Suppose to the contrary, G contains no end vertex. Now from Theorem 3.3, we deduce that for every two ideals $I \in X$ and $J \in Y$, there exist ideals $L \in X$ and $M \in Y$ such that $IL = (0)$ and $JM = (0)$. So, $N(L) \cup N(M) \subseteq N[LM]$. Thus $LM \in N[I] \cap N[J] \subseteq \{K\}$, and so $LM = K$. Therefore, $KI = KJ = (0)$, for every ideal $I \in X$ and every ideal $J \in Y$. This implies that G is a refinement of a star graph, a contradiction. ∎

Example 3.8 The graphs in Figures 3 and 4 are not annihilating-ideal graphs, where U consists of finitely or infinitely many end vertices.

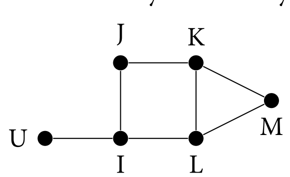

Figure 3.

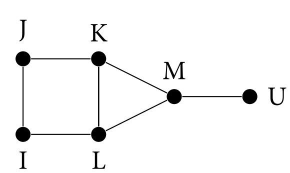

Figure 4.

To see this, find $N[IM]$.

Let Δ, δ denote the maximum and minimum degrees of a graph, respectively.

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Lemma 3.9 *Let G be an annihilating-ideal graph and I be a vertex of G with maximum degree. If $J \notin N(I)$, then $N(J) \subseteq N(I)$.*

Proof Let $G \cong \mathbb{AG}(R)$, where R is a ring, and choose $J \notin N(I)$. Then $N(I) \cup N(J) \subseteq N[IJ] \setminus \{I\}$. If $I \in N(IJ)$, then $|N(I) \cup N(J)| = \Delta$, and so $N(J) \subseteq N(I)$. Now, suppose $IJ \notin N(I)$. Then $N(I) \subseteq N(IJ)$. Since $|N(IJ)| \leq |N(I)| = \Delta$, we have $N(I) = N(IJ)$, and so $IJ \notin N(J)$. Hence $N(J) \subseteq N(IJ) = N(I)$. ∎

In the next result we give a necessary condition on the minimum and maximum degree of the vertices of a graph G to be an annihilating-ideal graph.

Theorem 3.10 *If $\lceil \delta(|V(G)| - \Delta - 1)/\Delta) \rceil + 1 > \Delta$, then G is not an annihilating-ideal graph.*

Proof Let $\lceil \delta(|V(G)| - \Delta - 1)/\Delta) \rceil + 1 > \Delta$ and suppose to the contrary, $G \cong \mathbb{AG}(R)$ for some ring R. Choose a vertex I with maximum degree and let $N(I) = \{J_1, \ldots, J_\Delta\}$. Also, assume that $V(G) \setminus N[I] = \{K_1, \ldots, K_{n-\Delta-1}\}$, where $n = |V(G)|$. By Lemma 3.9, $N(K_i) \subseteq N(I)$ for every $1 \leq i \leq n - \Delta - 1$. On the other hand, each K_i must be adjacent to at least δ vertices. So the pigeonhole principle implies that there must exist K_i with degree $d(K_i) \geq \lceil \delta(n - \Delta - 1)/\Delta) \rceil + 1 > \Delta$, a contradiction. ∎

For any graph G, we denote the set of all vertices with maximum degree, by I_Δ.

Corollary 3.11 *If G is an annihilating-ideal graph, then $G[I_\Delta]$ is either connected or the graph of isolated vertices.*

Proof Suppose $G \cong \mathbb{AG}(R)$ for some ring R, and assume that $G[I_\Delta]$ is not connected. Then there exist two vertices I and J in the same connected component such that $IJ = (0)$. Clearly, there exists a vertex, say K, such that K is not adjacent to both I and J. Thus from Lemma 3.9 we have $N(K) = N(I)$ and $N(K) = N(J)$. Therefore, $N(I) = N(J)$, a contradiction. ∎

4 Cycles in Annihilating-ideal Graphs

In this section, it is shown that the *core* of any annihilating-ideal graph is a union of triangles and rectangles. Recall that the core of a graph G is the subgraph induced on all vertices of cycles of G. Also, we prove that a finite annihilating-ideal graph is Hamiltonian if and only if the ring contains no Gorenstein ring as its direct summand. First, we need the following lemma.

Lemma 4.1 *If $I - J - K$ is a path in $\mathbb{AG}(R)$, then either $N(M) \subseteq \overline{N(J)}$ for every non-adjacent vertex M to J, or $I - J - K$ is contained in a cycle of length ≤ 4.*

Proof Let $I - J - K$ be a path in $\mathbb{AG}(R)$. If there exists a vertex $L \neq J$ such that $L \in N(I) \cap N(K)$, then $I - J - K - L - I$ is a cycle of length 4. So, assume that $N(I) \cap N(K) = \{J\}$. If M is a non-adjacent vertex with J, then $N(J) \cup N(M) \subseteq \overline{N(JM)}$. Thus $JM \in \overline{N(I)} \cap \overline{N(K)}$. If $JM \in N(I) \cap N(K)$, then $JM = J$ and this

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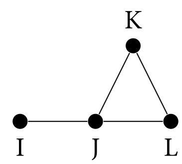
implies that $N(M) \subseteq \overline{N(J)}$. So, with no loss of generality, we can assume that $JM = I$. Thus $K \in N(I)$ and hence $I - J - K - I$ is a triangle in $\mathbb{AG}(R)$. ■

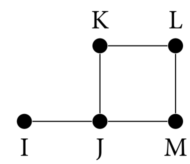
In [5], the authors proved that if $\mathbb{AG}(R)$ contains a cycle, then its girth does not exceed 4 . The next theorem is a vast strengthening of this result.

Theorem 4.2 *The core $\mathcal{K}$ of $\mathbb{AG}(R)$ is a union of triangles and rectangles. Moreover, any vertex of $\mathbb{AG}(R)$ is either a vertex of the core $\mathcal{K}$ of $\mathbb{AG}(R)$ or else is an end vertex of $\mathbb{AG}(R)$.*

Proof If $I \in \mathcal{K}$, then I is part of a cycle $I - J - K - L - \cdots - I$. If L is adjacent to I, then I is in a rectangle. Thus we can assume that L and I are not adjacent vertices. So Lemma 4.1 implies that $K \in N(L) \subseteq \overline{N(I)}$. Therefore, I is in a triangle. This proves the first statement. For the second statement, we can assume $|V(\mathbb{AG}(R))| \geq 3$. Suppose to the contrary, I is a vertex of $\mathbb{AG}(R)$ such that $I \notin \mathcal{K}$ and I is not an end vertex. Choose $J \in \mathcal{K}$. Then Lemma 4.1 implies that J lies in either a triangle or a rectangle. By Theorem [5, Theorem 2.1] $d(I,J) \leq 3$. To get a contradiction, we consider the following cases.

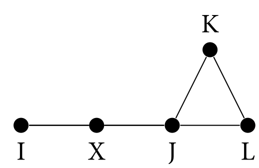
Case 1: $d(I,J) = 1$. In this case, $\mathbb{AG}(R)$ contains one of the following subgraphs:

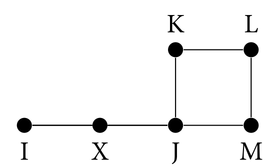




Since $I \notin \mathcal{K}$ and I is not an end vertex, there exists a vertex $P \in N(I) \setminus \{J, K, L, M\}$. So Lemma 4.1 implies that either $P - I - J$ lies in a cycle with length ≤ 4 or $K \in N(L) \subseteq N(I)$. Thus $I \in \mathcal{K}$, a contradiction.

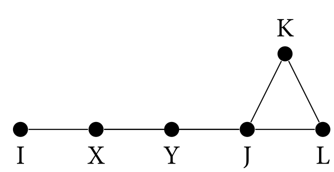
Case 2: $d(I,J) = 2$. In this case, $\mathbb{AG}(R)$ contains one of the following subgraphs:

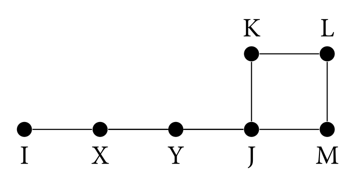




Again from Lemma 4.1, we deduce that either $I - X - J$ lies in a cycle of length ≤ 4 or $K \in N(L) \subseteq N(I)$, a contradiction.

Case 3: $d(I,J) = 3$. In this case, $\mathbb{AG}(R)$ contains one of the following subgraphs:





Therefore, we have $\operatorname{diam}(\mathbb{AG}(R)) \geq d(I,K) \geq 4$, a contradiction. ■

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Theorem 4.3 *Let R be a ring with finite annihilating-ideal graph. Then $\mathbb{AG}(R)$ is Hamiltonian if and only if R contains no Gorenstein ring as its direct summand.*

Proof Since $\mathbb{AG}(R)$ is a finite graph, [5, Theorem 1.4] and [4, Theorem 8.7] imply that $R \cong R_1 \times R_2 \times \cdots \times R_n$, where every $(R_i, \mathfrak{m}_i)$ is an Artinian local ring. If R contains a Gorenstein ring as its direct summand, then with no loss of generality, we can assume that R_1 is a Gorenstein local ring. Thus the type of R_1 is 1, and so $\text{Ann}(\mathfrak{m}_1) \times (0) \times \cdots \times (0)$ is the only vertex adjacent with $\mathfrak{m}_1 \times R_2 \times \cdots \times R_n$. Hence $\mathbb{AG}(R)$ is not Hamiltonian. Now assume that R contains no Gorenstein ring as its direct summand. Then every R_i is not a Gorenstein ring. Indeed, every vertex of $\mathbb{AG}(R)$ is contained in $M_i = R_1 \times \cdots \times \mathfrak{m}_i \times \cdots \times R_n$ for some $1 \leq i \leq n$. Since R_i is not Gorenstein, $\text{Ann}(M_i) = \text{Ann}(\mathfrak{m}_i)$ contains more than one non-trivial ideal. Thus $d(M_i) > 1$ for every maximal ideal M_i of R, and hence $\mathbb{AG}(R)$ contains no end vertex. Therefore, Theorem 4.2 implies that $\mathbb{AG}(R)$ is Hamiltonian. ■

The following corollary is obtained immediately from the previous theorem.

Corollary 4.4 *If $\mathbb{AG}(R)$ is a finite graph and R contains a field as its direct summand, then $\mathbb{AG}(R)$ is not a Hamiltonian graph.*

Finally, we finish this paper with the following result.

Proposition 4.5 *Finite annihilating-ideal graphs are not Eulerian.*

Proof Assume that R is a ring such that $\mathbb{AG}(R)$ is a finite graph. Then by [5, Theorem 1.4], R contains only finitely many ideals. Thus [4, Theorem 8.7] implies that $R \cong R_1 \times R_2 \times \cdots \times R_n$ for some positive integer n, where every R_i is an Artinian local ring. Let $\mathfrak{m}$ be the unique maximal ideal of R_1. Then the degree of the vertex $M = \mathfrak{m} \times R_2 \times \cdots \times R_n$ in $\mathbb{AG}(R)$ equals the number of non-zero ideals contained in $\text{Ann}(\mathfrak{m}_1) \times (0) \times \cdots \times (0)$. So $d(M) = 2^r - 1$, where r is the type of the local ring R_1. Therefore, by [17, Theorem 1.2.26], $\mathbb{AG}(R)$ is not Eulerian. ■

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