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A NOTE ON THE LÉVY-KHINCHIN REPRESENTATION OF NEGATIVE DEFINITE FUNCTIONS ON HILBERT SPACES

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Dedicated to Robert Edwards in recognition of 25 years' distinguished contribution to mathematics in Australia, on the occasion of his retirement

Abstract

We study negative definite functions on a Hilbert space $\mathcal{H}$ and use their properties to give a proof of the Lévy-Khinchin formula for an infinitely divisible probability distribution on $\mathcal{H}$.

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1. Introduction

An important branch of probability theory where Fourier analysis plays a major role is the theory of convergence of suitably normalized sums of independent random variables. The study of the asymptotic behaviour of such sums leads to infinitely divisible distributions and to their Lévy-Khinchin representations. These have originally been studied on Euclidean spaces (see Courrège [4]); further research was then directed towards other structures, in particular there exists now a well-established theory on locally compact groups and algebraic semigroups (see Berg and Forst [3], Berg, Christensen and Ressel [2], and Heyer [10] for an account in these frameworks).

In the last decades, through the influence of quantum physics, in particular quantum field theory, and stochastic processes the infinite dimensional case has become more and more important, as the characteristics of large scale systems are most clearly understood in the framework of an infinite dimensional model. Hilbert space has traditionally been considered as an infinite dimensional extension of Euclidean space appropriate for the solution to many problems (not the only one of course). It was therefore natural to extend the Lévy-Khinchin formula to this situation. To my knowledge, the first proof of the Lévy-Khinchin formula on Hilbert space is due to S. R. S. Varadhan [15] (see Parthasarathy [12]). His proof is based on shift compactness and the concentration function. There is another more proabilistic proof by Gihman and Skorohod [6]. The purpose of the present note is to clarify the rôle of negative definiteness of the generating function of an infinitely divisible distribution in the Hilbert space context and to give a proof of the formula that uses its properties. We shall not deal with more general situations here, in particular we shall not consider infinitely divisible distributions on Banach spaces (see however Araujo [1] and Linde [19] for an account).

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2. S-topology, positive definite, and negative definite functions

2.1. In what follows, the symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of natural numbers, of real numbers and of complex numbers, respectively; $\mathcal{H}$ denotes a real, separable Hilbert space with inner product (x, y) and norm $\|x\| = \sqrt{(x,x)}$. The space $\mathcal{H}$ will be considered as a topological vector space in three different ways: with the norm topology, the weak topology, and the S-topology (also called Hilbert-Schmidt topology). A typical neighbourhood of $0 \in \mathcal{H}$ for the third-mentioned topology is the set

$$V_S := \{y \in \mathcal{H} / \|\sqrt{S}y\| \leq 1\} = \{y \in \mathcal{H} / (y, Sy) \leq 1\},$$

where S is a positive selfadjoint, nuclear operator on $\mathcal{H}$; that is, a positive selfadjoint, compact, linear operator with a finite trace. (For details about compact operators see Edwards [5], Chapter 9.) Such an operator is also called an S-operator. S is an S-operator if and only if $\sqrt{S}$ is a positive selfadjoint Hilbert-Schmidt operator. We will freely use terms like S-continuous, S-convergent, S-neighbourhood etc.

2.2 PROPOSITION. *On an infinite dimensional Hilbert space the S-topology lies strictly between the strong and the weak topologies.*

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PROOF. The relationship between the strong and the S-topology is clear. Now let U be a weak neighbourhood of $0 \in \mathcal{H}$ of the form

$$U = \{y \in \mathcal{H} / |(y, z_1)| \leq 1, \ldots, |(y, z_n)| \leq 1\} \qquad (z_1, \ldots, z_n \in \mathcal{H}),$$

let E be the subspace of $\mathcal{H}$ spanned by $z_1, \ldots, z_n$, and let P_E be the projection $\mathcal{H} \to E$. Since E is finite dimensional, there exists $\delta > 0$ such that

$$\{y \in E / \|y\| \leq \delta\} \subseteq U.$$

Then for all $y \in \mathcal{H}$ such that $(y, P_E y) \leq \delta^2$ we have $\|P_E y\| \leq \delta$ and hence $P_E y \in U$. Therefore

$$|(y, z_k)| = |(P_E y, z_k)| \leq 1$$

for all k, that is, $y \in U$. $S := P_E/\delta^2$ is an S-operator such that $V_S \subseteq U$. On the other hand, if S is an S-operator such that $(y, Sy) > 0$ for all $y \in \mathcal{H} \setminus \{0\}$ then $\ker S = \{0\}$ and the set $\{y \in \mathcal{H} / (y, Sy) \leq 1\}$ does not contain a weak neighbourhood.

2.3 REMARK. The S-topology and the weak topology coincide on bounded sets. Indeed, if S is an S-operator then on any bounded set $B \subseteq \mathcal{H}$ the function

$$\begin{aligned} B &\to \mathbf{R} \\ y &\to (y, Sy) \end{aligned}$$

is weakly continuous (this is even true for compact operators S). It follows that S-continuous functions are weakly sequentially continuous.

2.4. The symbol $\mathcal{B}$ will stand for the σ-algebra of Borel subsets of $\mathcal{H}$ and $\mathcal{M}_+(\mathcal{B})$ is the convex cone of bounded, positive measures on $\mathcal{B}$. By $\mathcal{M}_+^1(\mathcal{B})$ we mean the convex subset of probability measures (distributions) on $\mathcal{B}$. The Dirac probability at the point $x \in \mathcal{H}$ will be denoted by δ_x. The space $\mathcal{M}_+(\mathcal{B})$ (and $\mathcal{M}_+^1(\mathcal{B})$) will be endowed with the topology of weak convergence, that is, the coarsest topology such that all functions

$$\begin{aligned} \mathcal{M}_+(\mathcal{B}) &\to \mathbf{C} \\ \mu &\to \int_{\mathcal{H}} f \, d\mu \end{aligned}$$

are continuous, where f runs through the space $\mathbf{C}_b(\mathcal{H})$ of complex-valued, bounded, norm-continuous functions on $\mathcal{H}$. With this topology, $\mathcal{M}_+(\mathcal{B})$ becomes a separable, completely metrizable topological space (cf. Parthasarathy II.6 [12]).

The *convolution* of two measures $\mu, \nu \in \mathcal{M}_+(\mathcal{B})$ is the Borel measure $\mu * \nu$ defined in the usual way:

$$\mu * \nu(B) := \int \mu(B - x)\nu(dx).$$

Convolution makes $\mathcal{M}_+(\mathcal{B})$ a metric, commutative semigroup; in particular convolution is a jointly weakly continuous mapping $\mathcal{M}_+(\mathcal{B}) \times \mathcal{M}_+(\mathcal{B}) \to \mathcal{M}_+(\mathcal{B})$ (see Linde [11]). The set $\mathcal{M}_+^1(\mathcal{B})$ is a weakly closed subsemigroup of $\mathcal{M}_+(\mathcal{B})$ and compactness in the space $\mathcal{M}_+(\mathcal{B})$ is well understood by Prokhorov's now classical theorem [13].

The *Fourier transform* of the measure $\mu \in \mathcal{M}_+^1(\mathcal{H})$ is the function $\hat{\mu}\colon \mathcal{H} \to \mathbb{C}$ defined by

$$\hat{\mu}(y) := \int_{\mathcal{H}} e^{i(x,y)} \mu(dx) \qquad (y \in \mathcal{H}).$$

As in the locally compact case, this Fourier transform is a useful tool for the characterization of measures (uniqueness theorem), for establishing criteria of weak compactness in $\mathcal{M}_+(\mathcal{B})$, and for the construction of measures (Minlos-Sazonov theorem).

2.5. We will need the following criterion for relative compactness in $\mathcal{M}_+(\mathcal{B})$ (see Gihman-Skorohod [6]). A subset $M \subseteq \mathcal{M}_+(\mathcal{B})$ is weakly relatively compact, if and only if the set of its Fourier transforms satisfies the two conditions

(i) $\sup_{\mu \in M} \hat{\mu}(0) < \infty$;

(ii) there exists a family $(S_{\varepsilon,\mu})$ $(\varepsilon > 0, \mu \in M)$ of S-operators such that

(α) for some orthonormal basis (e_k) of $\mathcal{H}$.

$$\lim_m \sup_{\mu \in M} \sum_{k \geq m} (e_k, S_{\varepsilon,\mu} e_k) = 0$$

for all $\varepsilon > 0$, and

(β) $\hat{\mu}(0) - \operatorname{Re} \hat{\mu}(y) \leq \varepsilon$, whenever $(y, S_{\varepsilon,\mu} y) \leq 1$.

2.6. A function $\varphi\colon \mathcal{H} \to \mathbb{C}$ is called *positive definite*, if for all choices $y_1, \ldots, y_n \in \mathcal{H}$ the matrix

$$(\varphi(y_k - y_l))_{k,l}$$

is positive Hermitean. The important theorem of Minlos-Sazonov reads: *A function $\varphi\colon \mathcal{H} \to \mathbb{C}$ is the Fourier transform $\hat{\mu}$ of some measure $\mu \in \mathcal{M}_+(\mathcal{B})$ if and only if*

(i) *φ is positive definite and*

(ii) *φ is S-continuous (or $\operatorname{Re} \varphi$ is S-continuous at $0 \in \mathcal{H}$).*

A function $\psi\colon \mathcal{H} \to \mathbb{C}$ is called *negative definite* if for all choices of $y_1, \ldots, y_n \in \mathcal{H}$ the matrix

$$(\psi(y_k) + \overline{\psi(y_l)} - \psi(y_k - y_l))_{k,l}$$

is positive Hermitean. The real part $\operatorname{Re} \psi$ of a negative definite function is also negative definite and hence $\operatorname{Re} \psi \geq 0$. For further properties of negative definite functions see Berg and Forst [3]. The two most important facts about negative definite functions are their algebraic relationship with positive definite functions

(Schoenberg's theorem, see Berg and Forst [3]) and subadditivity of $\sqrt{|\psi|}$, which implies the quadratic growth property, crucial in the construction of a Lévy measure:

2.7 PROPOSITION. *Let $\psi\colon \mathcal{H} \to \mathbb{C}$ be a negative definite function, S-continuous at the origin and vanishing there. Then, given $\varepsilon > 0$, there exists an S-operator S_ε such that*

$$|\psi(y)| \le \varepsilon + (y, S_\varepsilon y) \qquad (y \in \mathcal{H}).$$

PROOF. By subadditivity of $\sqrt{|\psi|}$ one obtains

$$|\psi(2y)| \le 4|\psi(y)| \qquad (y \in \mathcal{H}).$$

Choose an S-operator such that $|\psi(y)| \le \varepsilon$ whenever $(y, Sy) \le 1$. It is now possible to apply Parthasarathy [12], p. 172 to obtain the desired estimate, but for the reader's convenience we include here the (short) argument. Let $y \in \mathcal{H}$ be fixed. If $(y, Sy) \le 1$, any S_ε will do. Otherwise, choose $n = n(y) \in \mathcal{N}$ in such a way that $\frac{1}{4} \le (2^{-n}y, S(2^{-n}y)) \le 1$. We estimate

$$|\psi(y)| = |\psi(2^n 2^{-n} y)| \le 4^n |\psi(2^{-n}y)| \le 4^n \varepsilon \le 4\varepsilon(y, Sy),$$

and we may put $S_\varepsilon := 4\varepsilon S$.

3. Generating functions of infinitely divisible distributions and their characterization and representation

3.1. A convolution semigroup of probability measures is a family $(\pi_t)_{t>0}$ of probability distributions such that

$$\pi_s * \pi_t = \pi_{s+t}$$

for all $s, t > 0$. (π_t) is called *weakly continuous* if $(\pi_t) \underset{t\to\infty}{\longrightarrow} \delta_0$ weakly. The mapping $t \to \pi_t$ is then continuous. A measure $\pi \in \mathcal{M}_+^1(\mathcal{B})$ is called *infinitely divisible* if for any $n \in \mathbb{N}$ there exists a measure $\sigma_n \in \mathcal{M}_+^1(\mathcal{B})$ such that $\sigma_n^n = \pi$ (nth root). Each member π_t of a semigroup of probability measures is clearly infinitely divisible. The Fourier transform of an infinitely divisible measure π vanishes nowhere on $\mathcal{H}$; see Guichardet [7], p. 57. By a standard argument of function theory there exists a unique function $\psi\colon \mathcal{H} \to \mathbb{C}$ such that

(i) $\psi(0) = 0$,
(ii) ψ is continuous,
(iii) $\hat{\pi} = e^{-\psi}$,

(see Linde [11]). The function ψ is often called the *generating function* of the infinitely divisible probability measure π. (This terminology seems to appear for the first time in Hazod [8] and Heyer [9] in connection with locally compact groups.) The nth root σ_n is infinitely divisible and, by uniqueness of ψ, its generating function is $-\psi/n$; in particular σ_n is unique. There is the following characterization of generating functions.

3.2 THEOREM. (a) *A function* $\psi\colon \mathcal{H} \to \mathbf{C}$ *generates an infinitely divisible probability distribution if and only if it has the following three properties:*

(i) $\psi(0) = 0$,

(ii) ψ *is* S*-continuous* (*or* $\operatorname{Re}\psi$ *is* S*-continuous at* $0 \in \mathcal{H}$),

(iii) ψ *is negative definite.*

(b) *In this case there exists a unique weakly continuous semigroup of probability measures* π_t *such that*

$$\hat{\pi}_t = e^{-t\psi}.$$

PROOF. If the function ψ generates an infinitely divisible probability distribution π, then (i) is part of the definition of ψ; moreover, this function is S-continuous since $\hat{\pi}$ is. As all the functions $e^{-\psi/n}$ are transforms of measures and therefore positive definite assertion (iii) is satisfied by Schoenberg's theorem. On the other hand, if the function ψ satisfies (i)–(iii), then the functions $\varphi_t := e^{-t\psi}$ are positive definite (again by Schoenberg's theorem) and S-continuous, so that there exists a family of measures $\pi_t \in \mathcal{M}^1_+(\mathcal{B})$ such that $\varphi_t = \hat{\pi}_t$ (2.6). The semigroup property now follows from the equality $\varphi_s\varphi_t = \varphi_{s+t}$ and 2.5 in combination with uniqueness of the Fourier transform may be applied to show weak continuity.

3.3 REMARK. A. V. Skorohod [14], p. 11 claims that characteristic functions are weakly continuous. This claim is true in the sequential sense only. If S is a nondegenerate S-operator (that is, all eigenvalues are strictly positive) then S is not weakly continuous (see the proof of 2.2). Hence the characteristic function of the centered Gauss measure with covariance S is not weakly continuous.

3.4. A finite dimensional subspace $V \subseteq \mathcal{H}$ of dimension n will be identified with $\mathcal{R}^n$ via the isomorphism

$$\mathbf{R}^n \to V \qquad (x_1, \dots, x_n) \to \sum_{k=1}^{n} x_k e_k$$

where $(e_1, \dots, e_n)$ is an orthonormal basis of V with respect to the inner product of $\mathcal{H}$. It is then clear what Lebesgue measure on V means and integrals $\int_V dz$ make sense since these notions are independent of the choice of the orthonormal basis. Note that

$$\int_V e^{-\|z\|^2}\, dz = \pi^{n/2}.$$

Define the auxiliary function $g\colon \mathcal{H} \to \mathcal{R}$ by

$$g(x) := e^{-\|x\|^2/4}.$$

The function $(1-g)$ will serve as a density function. Any function that is bounded, positive outside the origin, and behaves like $c\|x\|^2$ near the origin would be possible here, but the chosen function is the handiest for the present purposes. A classical function is $\|x\|^2/(1+\|x\|^2)$. We will need the following

3.5 LEMMA. *Let S be an S-operator. Then for all finite dimensional subspaces of $V \subseteq \mathcal{H}$ of dimension n we have*

$$\pi^{-n/2}\int_V e^{-\|z\|^2}(z, Sz)\, dz \le \tfrac{1}{4}\operatorname{trace} S.$$

PROOF. Let $S_V := P_V S P_V$. Then S_V is a positive selfadjoint operator on V such that $\operatorname{trace} S_V \le \operatorname{trace} S$. Let $(e_1, \ldots, e_n)$ be an eigenbasis of S_V with eigenvalues $\lambda_1, \ldots, \lambda_n$ and let z_k be the kth component of a vector z with respect to this basis. We may compute

$$\begin{aligned}\pi^{-n/2}\int_V e^{-\|z\|^2}(z, Sz)\, d2 &= \pi^{-n/2}\int_V e^{-\|z\|^2}(z, P_V S P_V z)\, dz\\ &= \sum_{k=1}^{n}\lambda_k \pi^{-n/2}\int_{\mathbf{R}^n} e^{-\|z\|^2} z_k^2\, dz.\end{aligned}$$

The integral has the value $\frac{1}{4}\pi^{n/2}$, whence the desired estimate.

3.6 REMARK. To my knowledge, the first proof of the Lévy-Khinchin representation in the case of a Hilbert space was given by Varadhan [15] (see Parthasarathy [12]). His proof is analytic; it is based on shift compactness and the concentration function. Other more probabilistic constructions of the Lévy measure are now also available (see Gihman and Skorohod [6]). I shall give a proof here that exploits the properties of the generating function stated in Theorem 3.2. The core of any proof of the Lévy-Khinchin formula is the construction of the Lévy measure. This can be done by letting $t \to 0$ in the net $(1-g)\pi_t/t$. In the infinite dimensional case also the approximation by finite dimensional subspaces plays a role. Contrary to other proofs of the Lévy-Khinchin formula, we shall first let the parameter t go to zero and then go to the dimensional limit; this approach seems to give better insight into the difference between the finite and the infinite dimensional cases.

3.7 THEOREM (Lévy-Khinchin-Varadhan). (a) *A function $\psi\colon \mathcal{H} \to \mathbf{C}$ generates an infinitely divisible probability distribution if and only if it can be written*

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as

$$\psi(y) = i(a,y) + (y,Sy) - \int_{\mathcal{H}\setminus\{0\}} [e^{-i(x,y)} - 1 + ig(x)(x,y)] \frac{\mu(dx)}{1-g(x)} \qquad (y \in \mathcal{H}).$$

Here $a \in \mathcal{H}$, S *is an* S*-operator and* $\mu \in \mathcal{M}_+(\mathcal{H})$.

(b) *The representation* (a, S, μ) *of* ψ *is unique.*

PROOF. Concerning uniqueness of the representation we refer the reader to Gihman and Skorohod [6], p. 395. In order to prove part (a) let us first suppose that $\psi\colon \mathcal{H} \to \mathbb{C}$ generates the convolution semigroup (π_t) of probability measures. Fix an ascending system $\mathcal{V}$ of finite dimensional subspaces of $\mathcal{H}$ such that $\bigcup \mathcal{V}$ is dense in $\mathcal{H}$. For any subspace $V \in \mathcal{V}$ we write

$$\pi_{t,V} := P_V(\pi_t)$$

for the projection of π_t onto V. The family $(\pi_{t,V})_{t>0}$ is a convolution semigroup of probabilities on $\mathcal{H}$ for each V. Since the Fourier transform of the measure

$$(1-g)\frac{\pi_{t,V}}{t} \in \mathcal{M}_+(V)$$

is the function

$$y \to \pi^{-n/2} \int_V e^{-\|z\|^2} \frac{e^{-t\psi(y)} - e^{-t\psi(y-z)}}{t}\, dz \qquad (y \in V),$$

Lévy's continuity theorem applies to show that there exists a measure $\mu_V \in \mathcal{M}_+(V)$ such that

$$(1-g)\frac{\pi_{t,V}}{t} \underset{t\to 0}{\to} \mu_V \quad \text{weakly}$$

and

$$\hat{\mu}_V(y) = \pi^{-n/2} \int_V e^{-\|z\|^2} (\psi(y-z) - (\psi(y))\, dz \qquad (y \in V).$$

We consider μ_V also as a measure on $\mathcal{B}$ and apply 2.5 to show that the net $(\mu_V)_{V\in\mathcal{V}}$ is relatively compact in $\mathcal{M}_+(\mathcal{B})$. First let S_1 be as in Proposition 2.7 for $\varepsilon = 1$. We have

$$\begin{aligned}\hat{\mu}_V(0) &= \pi^{-n/2} \int_V e^{-\|z\|^2} \psi(z)\, dz \\ &\le \pi^{-n/2} \int_V e^{-\|z\|^2} (1 + (z, S_1 z))\, dz. \\ &\le 1 + \tfrac{1}{4} \operatorname{trace} S_1 \\ &< \infty\end{aligned}$$

for all V by Lemma 3.5; this is 2.5(i). Now note that $\operatorname{Re}\psi$ too is negative definite. Applying again Proposition 2.7, but this time to the function $\operatorname{Re}\psi$, we obtain an S-operator S_ε such that

$$\operatorname{Re}\psi(y) \le \varepsilon + \varepsilon(y, S_\varepsilon y) \tag{1}$$

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for all $y \in \mathcal{H}$. Since the function $\sqrt{\operatorname{Re}\psi}$ is subadditive and since $\operatorname{Re}\psi$ is symmetric (see Berg and Forst [3]) we have

$$\sqrt{\operatorname{Re}\psi(y)} - \sqrt{\operatorname{Re}\psi(z)} \le \sqrt{\operatorname{Re}\psi(y-z)}$$

and likewise

$$\sqrt{\operatorname{Re}\psi(z)} - \sqrt{\operatorname{Re}\psi(y)} \le \sqrt{\operatorname{Re}\psi(y-z)}.$$

that is,

$$|\sqrt{\operatorname{Re}\psi(z)} - \sqrt{\operatorname{Re}\psi(y)}| \le \sqrt{\operatorname{Re}\psi(y-z)},$$

whence

$$\operatorname{Re}\psi(y) + \operatorname{Re}\psi(z) - \operatorname{Re}\psi(y-z) \le 2\sqrt{\operatorname{Re}\psi(y)}\sqrt{\operatorname{Re}\psi(z)} \tag{2}$$

for all $y, z \in \mathcal{H}$. We may now use (2), (1), and 3.5 to compute for $y \in \mathcal{H}$

$$\begin{aligned}
&\hat{\mu}_V(0) - \operatorname{Re}\hat{\mu}_V(y)\\
&\quad= \pi^{-n/2}\int_V e^{-\|z\|^2}[\operatorname{Re}\psi(z) + \operatorname{Re}\psi(P_V y) - \operatorname{Re}\psi(P_V y - z)]\,dz\\
&\quad\le 2\pi^{-n/2}\sqrt{\operatorname{Re}\psi(P_V y)}\int_V e^{-\|z\|^2}\sqrt{\operatorname{Re}\psi(z)}\,dz\\
&\quad= 2\pi^{-n/2}\sqrt{\operatorname{Re}\psi(P_V y)}\left(\int_{\substack{V\\ \operatorname{Re}\psi<1}} + \int_{\substack{V\\ \operatorname{Re}\psi\ge 1}}\right) e^{-\|z\|^2}\sqrt{\operatorname{Re}\psi(z)}\,dz\\
&\quad\le 2\sqrt{\operatorname{Re}\psi(P_V y)}\left(1 + \pi^{-n/2}\int_V e^{-\|z\|^2}\operatorname{Re}'\psi(z)\,dz\right)\\
&\quad\le 2\sqrt{\varepsilon}\sqrt{1 + (P_V y, S_\varepsilon P_V y)}\left(1 + \pi^{-n/2}\int_V e^{-\|z\|^2}(1 + (z, S_1 z))\,dz\right)\\
&\quad\le 2\sqrt{\varepsilon}\sqrt{1 + (P_V y, S_\varepsilon P_V y)}(2 + \tfrac{1}{4}\operatorname{trace} S_1).
\end{aligned}$$

The conditions α and β of 2.5 can now be verified with $S_{\varepsilon,V} := P_V S_\varepsilon P_V$. This finishes the proof of relative compactness of the net (μ_V). As in the proof of the classical Lévy-Khinchin formula we now write for $y \in V$

$$\begin{aligned}
\frac{\hat{\pi}_{t,V}(y) - 1}{t} &= \int_{\mathcal{H}} [e^{-i(x,y)} - 1]\frac{\pi_{t,V}(dx)}{t} - i(a_{t,V}, y) - (y, S_{t,V} y)\\
&\quad + \int_{\mathcal{H}} \frac{[e^{-i(x,y)} - 1 + g(x)(i(x,y) + \frac{1}{2}(x,y)^2]}{1 - g(x)}(1 - g(x))\frac{\pi_{t,V}(dx)}{t}
\end{aligned}$$

with a vector $a_{t,V} \in V$ and a positive selfadjoint operator $S_{t,V}$ on V. As $t \to 0$, the left-hand side converges to $-\psi(y)$ and by weak convergence the integral converges to

$$\int_{\mathcal{H}} [e^{-i(x,y)} - 1 + g(x)(i(x,y) + \tfrac{1}{2}(x,y)^2)]\frac{\mu_V(dx)}{1 - g(x)}.$$

Hence $ia_{t,V} \to ia_V$ and $S_{t,V} \to S_V$, the former being purely imaginary and the latter real. Extending to $y \in \mathcal{H}$ we have obtained the (essentially classical) formula

$$\begin{aligned}\psi(P_V y) &= i(a_V, y) + (y, P_V S_V P_V y)\\ &\quad - \int_{\mathcal{H}} [e^{-i(x,y)} - 1 + g(x)(i(x,y) + \tfrac{1}{2}(x,y)^2)] \frac{\mu_V(dx)}{1 - g(x)}.\end{aligned}$$

By relative compactness we may assume that the net $(\mu_V)_V$ converges weakly to a Borel measure $\mu \in \mathcal{M}_+(\mathcal{B})$. Then the net $\psi(P_V y)$ converges to $\psi(y)$ and the integral converges, the integrand being continuous and bounded. It follows that the nets $i(a_V, y)$ and $(y, P_V S_V P_V y)$ both converge along $\mathcal{V}$. By the uniform boundedness principle there exists a vector $a \in \mathcal{H}$ such that

$$i(a_V, y) \to i(a, y) \qquad (y \in \mathcal{H}).$$

A similar argument can be applied to the quadratic terms $(y, P_V S_V P_V y)$. By polarization we first obtain pointwise convergence of the symmetric bilinear forms $(y, z) \to (y, P_V S_V P_V z)$. Applying the uniform boundedness principle to the net $(P_V S_V P_V z)_V$ we see that for any $z \in \mathcal{H}$ there exists a vector $S'z \in \mathcal{H}$ such that

$$(y, P_V S_V P_V z) \to (y, S'z) \qquad (y \in \mathcal{H}).$$

Moreover, $S'\colon \mathcal{H} \to \mathcal{H}$ is a symmetric linear map. We obtain the representation

$$\begin{aligned}\psi(y) &= i(a, y) + (y, S'y)\\ &\quad - \int_{\mathcal{H}\setminus\{0\}} [e^{-i(x,y)} - 1 + g(x)(i(x,y) + \tfrac{1}{2}(x,y)^2)] \frac{\mu(dx)}{1 - g(x)}.\end{aligned}$$

Since

$$y \to \int_{\mathcal{H}\setminus\{0\}} g(x)(x,y)^2 \frac{\mu(dx)}{1 - g(x)}$$

is a continuous quadratic form, the above representation can also be written as

$$\psi(y) = i(a, y) + (y, Sy) - \int_{\mathcal{H}\setminus\{0\}} [e^{-i(x,y)} - 1 + ig(x)(x,y)] \frac{\mu(dx)}{1 - g(x)}$$

with a new symmetric linear map $S\colon \mathcal{H} \to \mathcal{H}$. The relation

$$(y, Sy) = \lim_{k\to\infty} \frac{\operatorname{Re}\psi(ky)}{k^2}$$

shows that S is positive. In order to finish this part of the proof it remains to show that S is nuclear. Indeed, from the representation we see $0 \le (y, Sy) \le \operatorname{Re}\psi(y)$, hence, by S-continuity of ψ and by the equality $\psi(0) = 0$, the mapping $y \to (y, Sy)$ is S-continuous at $0 \in \mathcal{H}$. Consequently there exists an S-operator S_1 such that

$$(y, Sy) \le 1 \quad \text{whenever } (y, S_1 y) \le 1.$$

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Then by bilinearity and positivity it is clear that $(y, Sy) \leq (y, S_1 y)$ for all $y \in \mathcal{H}$; hence S is nuclear. Finally note that the functions

$$y \to -i(a, y), \qquad y \to (y, Sy),$$

and

$$y \to \int_{\mathcal{H}\setminus\{0\}} [e^{-i(x,y)} - 1 + ig(x)(x, y)] \frac{\mu(dx)}{1 - g(x)}$$

satisfy the conditions (i)–(iii) of Theorem 3.2. This is shown as in the classical case (see Courrège [4]).

4. Examples

We study here some semigroups of probabilities on $\mathbf{R}^{\mathbf{N}}$ that are supported by $l^2 = \{x \in \mathbf{R}^{\mathbf{N}} / \|x\|^2 := \sum x_k^2 < \infty\}$.

4.1 *The translation semigroup.* Let $a = (a_k)_{k \in \mathbf{N}}$ be a sequence of real numbers. Of course, the product measures $\pi_t = \bigotimes_{k \in \mathbf{N}} \delta_{ta_k}$, $(t > 0)$ are concentrated on l^2 if and only if $a \in l^2$. (π_t) is then a convolution semigroup on l^2 and its generating function is

$$\psi(y) := -i(a, y).$$

4.2 *The Gaussian semigroup.* Let

$$\nu_u(dx) := \frac{1}{\sqrt{2\pi u}} e^{-x^2/2u}\, dx$$

be the centered normal distribution on $\mathbf{R}$ with variance $u > 0$ and let $v = (v_k)$ be a sequence of real numbers > 0. We consider the product probability measure $\pi_t = \bigotimes_{k \in \mathbf{N}} \nu_{tv_k}$, $(t > 0)$ on $\mathbf{R}^{\mathbf{N}}$. It is well known that the measures π_t are concentrated on l^2 if and only if $\sum_{k \in \mathbf{N}} v_k < \infty$, that is, if and only if the matrix

$$S := \begin{pmatrix} v_0 & & & 0 \\ & v_1 & & \\ & & v_2 & \\ 0 & & & \ddots \end{pmatrix}$$

induces a nuclear operator on l^2, that is, $(v_k) \in l^1$. Indeed, denoting for $\beta > 0$ by $h_\beta \colon \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ the function defined by

$$h_\beta(x) := e^{-\beta\|x\|^2},$$

we have $h_\beta \uparrow 1_{l^2}$ as $\beta \downarrow 0$. We compute

$$\int h_\beta \, d\pi_1 = \prod_k \frac{1}{\sqrt{2\pi v_k}} \int_{\mathbf{R}} e^{-\beta x_k^2 - x_k^2/2v_k} \, dx_k$$
$$= \left(\prod_k \frac{1}{1+2\beta v_k} \right)^{1/2}.$$

This quantity tends to 1 as $\beta \downarrow 0$ if and only if $\sum v_k < \infty$. Since

$$\hat{\pi}_t = e^{-\frac{t}{2} \sum v_k y_k^2},$$

the generating function is given by $\psi_v(y) = \frac{1}{2}(y, Sy)$.

4.3 *The Cauchy semigroup.* Let

$$\gamma_\alpha(dx) := \frac{\alpha}{\pi(x^2+\alpha^2)} \, dx$$

be the centered Cauchy distribution on $\mathbf{R}$ with parameter $\alpha > 0$. Let $\rho = (\rho_k)$ be a sequence of real numbers > 0 and consider the product measure $\pi_t := \bigotimes_k \gamma_{t\rho_k}$ on $\mathbf{R}^{\mathbf{N}}$. Using here the function $k_\beta \colon \mathbf{R}^{\mathbf{N}} \to \mathbf{R}$ defined by

$$k_\beta(x) := \prod_k \frac{1}{1+\beta x_k^2} \qquad (\beta > 0)$$

we obtain $\int k_\beta \, d\pi_1 = \prod_k 1/(1+\sqrt{\beta}\rho_k)$. Letting $\beta \downarrow 0$ we see that π_1 is supported by l^2 if and only if $\sum \rho_k < \infty$. Since $\hat{\gamma}_\alpha(u) = e^{-\alpha|u|}, (u \in \mathbf{R})$ the generating function ψ_ρ is given by $\psi_\rho(y) = (\rho, |y|)$, $(y \in l^2)$, where we have put $|y| = (|y_0|, |y_1|, \dots)$. Although the function ψ_ρ is well defined for $\rho \in l^2$, it is S-continuous if and only if $\rho \in l^1$. Indeed, if $\rho \in l^1$ then

$$(\rho, |y|)^2 = \left(\sum \sqrt{\rho_k} \sqrt{\rho_k} |y_k| \right)^2$$
$$\leq \left(\sum \rho_k \right) \left(\sum \rho_k y_k^2 \right)$$
$$= (y, Sy)$$

where

$$S = \left(\sum \rho_k \right) \begin{pmatrix} \rho_0 & & 0 \\ & \rho_1 & \\ 0 & & \ddots \end{pmatrix}.$$

Conversely, if ψ is S-continuous then it generates an infinitely divisible probability distribution π on l^2. The induced measure in $\mathbf{R}^{\mathbf{N}}$ by the injection $l^2 \to \mathbf{R}^{\mathbf{N}}$ must be a product of Cauchy distributions and therefore, by what was shown above, $\sum \rho_k < \infty$.

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