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A SUM OF RECIPROCALS OF LEAST COMMON MULTIPLES

BY

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The purpose of this note is to prove the following theorem conjectured by P. Erdös.

THEOREM. *Let $a_0, a_1, \ldots, a_k$ be integers satisfying $1 \le a_0 < a_1 < \cdots < a_k$, and let $[a_{i-1}, a_i]$ denote the least common multiple of a_{i-1} and a_i. Then*

$$\frac{1}{[a_0, a_1]} + \frac{1}{[a_1, a_2]} + \cdots + \frac{1}{[a_{k-1}, a_k]} \le 1 - \frac{1}{2^k}, \tag{1}$$

with equality occurring if and only if $a_i = 2^i$ for $1 \le i \le k$.

Proof. For $i = 1, 2, \ldots, k$, let $c_i = [a_{i-1}, a_i]$, and let

$$s_i = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_i}.$$

Then $c_i = u_i a_{i-1} = v_i a_i$ where $u_i > v_i \ge 1$. Hence

$$\frac{1}{c_i} \le \frac{1}{a_i}, \tag{2}$$

and, since $c_i^{-1} \le (u_i - v_i) c_i^{-1}$,

$$\frac{1}{c_i} \le \frac{1}{a_{i-1}} - \frac{1}{a_i}. \tag{3}$$

It follows from (3) that

$$s_i \le \frac{1}{a_0} - \frac{1}{a_i}. \tag{4}$$

To establish (1) we consider three cases which exhaust all possible conditions on the integers $a_0, a_1, \ldots, a_k$.

CASE 1. $a_k \le 2^k$. Then, by (4),

$$s_k \le 1 - \frac{1}{a_k} \le 1 - \frac{1}{2^k}.$$

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CASE 2. $a_i > 2^i$ for $1 \le i \le k$. Then, by (2),

$$s_k \le \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} < \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.$$

CASE 3. $a_j \le 2^j$ for some positive integer $j < k$, and $a_i > 2^i$ for $j+1 \le i \le k$. Then, by (2) and (4),

$$s_k = s_j + \frac{1}{c_{j+1}} + \cdots + \frac{1}{c_k} < 1 - \frac{1}{2^j} + \frac{1}{2^{j+1}} + \cdots + \frac{1}{2^k} = 1 - \frac{1}{2^k}.$$

Thus (1) holds in all three cases. Further, it is immediate that equality occurs in (1) when $a_i = 2^i$ for $1 \le i \le k$.

Suppose next that

$$s_k = 1 - \frac{1}{2^k}.$$

Then, by (4), we have $1 - 2^{-k} \le 1 - a_k^{-1}$ so that $a_k \ge 2^k$; and we cannot have $a_k > 2^k$ for Case 2 and Case 3 show that this would lead to $s_k < 1 - 2^{-k}$. Hence

$$a_k = 2^k.$$

If $k = 1$ there is nothing further to prove. For $k > 1$, we have, by (1) with $k-1$ in place of k, and (2), that

$$1 - \frac{1}{2^{k-1}} \ge s_{k-1} = s_k - \frac{1}{c_k} \ge 1 - \frac{1}{2^k} - \frac{1}{2^k} = 1 - \frac{1}{2^{k-1}}.$$

Hence

$$s_{k-1} = 1 - \frac{1}{2^{k-1}},$$

and repetition yields the desired conclusion that

$$a_i = 2^i \quad \text{for } 1 \le i \le k.$$

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