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COVERING GROUPS WITH SUBGROUPS

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A group is *covered* by a collection of subgroups if it is the union of the collection. The intersection of an irredundant cover of n subgroups is known to have index bounded by a function of n, though in general the precise bound is not known. Here we confirm a claim of Tompkinson that the correct bound is 16 when n is 5. The proof depends on determining all the 'minimal' groups with an irredundant cover of five maximal subgroups.

1. INTRODUCTION

A *covering* or *cover* of a group G is a collection of subgroups of G whose union is G. We use the term n-cover for a cover with n members. The cover is *irredundant* if no proper sub-collection is also a cover. Neumann [**5**] obtained a uniform bound for the index of the intersection of an irredundant n-cover; see Tompkinson [**7**] for an improved bound. We shall write $f(n)$ for the largest index $|G : D|$ over all groups G with an irredundant n-cover with intersection D. An immediate consequence is that such a group G has a permutation representation of degree at most $f(n)$, with kernel $\mathrm{core}_G(D)$. In particular $G/\mathrm{core}_G(D)$ is a finite group with an irredundant n-cover whose intersection is core-free.

The groups with an irredundant core-free intersection covering are known precisely when $n = 3$ (Scorza [**6**]) and when $n = 4$ (Greco [**4**, p.58]): see Propositions 2.3 and 2.4 below. Partial results are known for $n = 5$: Greco [**3**] lists all groups with an irredundant 5-cover in which all pairwise intersections are the same; and Tompkinson [**7**] claims that $f(5) = 16$.

The aim of the present article is to fill in some of the missing detail when $n = 5$. We are concerned with irredundant, core-free intersection 5-covers in which all five subgroups of the cover are maximal. A cover in which all subgroups are maximal we shall call *maximal*.

THEOREM 1.1. *Let G be a group with a maximal irredundant cover of five subgroups with core-free intersection D. Then either*

(a) *$D = 1$ and G is elementary Abelian of order 16; or*
(b) *$D = 1$ and $G \cong \mathrm{Alt}_4$; or*
(c) *$|D| = 3$, $|G| = 48$ and G embeds in $\mathrm{Alt}_4 \times \mathrm{Alt}_4$.*

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Theorem 1.2. $f(5) = 16$.

2. Preliminary results

The following results will be needed below. Where no proof is given it is either very easy or a reference is given.

Lemma 2.1. *Let $\{A_i\colon 1 \leqslant i \leqslant m\}$ be a (maximal) irredundant covering of a group G with intersection D. If N is a normal subgroup of G contained in D then $\{A_i/N\colon 1 \leqslant i \leqslant m\}$ is a (maximal) irredundant cover of G/N.*

Lemma 2.2. (See [1, Lemma 2.2]) *Let $\mathcal{A} = \{A_i\colon 1 \leqslant i \leqslant m\}$ be an irredundant covering of a group G whose intersection is D.*

(a) *If p is a prime, x a p-element of G and $|\{i\colon x \in A_i\}| = n$ then either $x \in D$ or $p \leqslant m - n$.*

(b) $\bigcap_{j \neq i} A_j = D \quad (1 \leqslant i \leqslant m)$.

(c) *If $\bigcap_{i \in S} A_i = D$ whenever $|S| = n$ then $\left|\bigcap_{i \in T} A_i\colon D\right| \leqslant m - n + 1$ whenever $|T| = n - 1$.*

(d) *If $\mathcal{A}$ is maximal and U is an Abelian minimal normal subgroup of G then, if $|\{i\colon U \leqslant A_i\}| = n$, either $U \subseteq D$, or $|U| \leqslant m - n$.*

Proposition 2.3. (Scorza [6]) *Let $\{A_i : 1 \leqslant i \leqslant 3\}$ be an irredundant cover with core-free intersection D of a group G. Then $D = 1$ and $G \cong C_2 \times C_2$.*

Proposition 2.4. (Greco [4]) *Let $\{A_i\colon 1 \leqslant i \leqslant 4\}$ be an irredundant cover with core-free intersection D of a group G. If the cover is maximal then* either

(a) *$D = 1$ and $G \cong \mathrm{Sym}_3$ or $G \cong C_3 \times C_3$*; or

(b) *$|D| = 2$, $|G| = 18$ and G embeds into $\mathrm{Sym}_3 \times \mathrm{Sym}_3$.*

If the cover is not maximal then either

(c) *$D = 1$ and $G \cong D_8$, or $G \cong C_4 \times C_2$, or $G \cong C_2 \times C_2 \times C_2$*; or

(d) *$|D| = 2$ and $G \cong D_8 \times C_2$.*

Lemma 2.5. *Let G be a group with a maximal irredundant 5-cover with core-free intersection D.*

(a) *G is a 2-group if and only if $D = 1$ and G is elementary of order 16.*

(b) *G is not a 3-group.*

Proof: Let $G = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5$ be a maximal irredundant cover for a p-group G, with core-free intersection D. Now $\Phi(G) \subseteq D$ so $D \trianglelefteq G$, therefore $D = 1$, and G is elementary Abelian. By Lemma 2.2(b), (c), $|M_i \cap M_j \cap M_k| \leqslant 2$ whenever i, j, k are distinct. When $p = 2$, therefore, $|G| \leqslant 16$. Also $|G| \geqslant 8$ since otherwise G

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does not have five maximal subgroups. However $|G| = 8$ is impossible. For, if $|G| = 8$ and $|M_1 \cap M_2 \cap M_3| = 2$ then $G = M_1 \cup M_2 \cup M_3$, contradicting the irredundance of the cover; and if $M_1 \cap M_2 \cap M_3 = 1$ then $|M_1 \cup M_2 \cup M_3| = 7$, so G is covered by four of the M_i, again a contradiction. Conversely if $\langle a, b, c, d\rangle$ is elementary of order 16, then $\langle a, b, c\rangle$, $\langle a, b, d\rangle$, $\langle a, c, d\rangle$, $\langle b, c, d\rangle$, $\langle ab, bc, cd\rangle$ provide a maximal irredundant core-free intersection cover.

When $p = 3$ we conclude that $M_i \cap M_j \cap M_k = 1$ for all distinct i, j, k. $|G| > 9$ since an elementary Abelian group of order 9 has only four maximal subgroups; in particular, no pairwise intersection is trivial. Hence $|M_i \cap M_j| = 3$ $(i \neq j)$. By the inclusion-exclusion principle $|G| = 5.9 - 10.3 + 10.1 - 5.1 + 1 = 21$, which is not a power of 3, a contradiction. ∎

LEMMA 2.6. *Let F be finite field with q elements. Suppose that*

$$F^2 = S_1 \cup S_2 \cup \ldots \cup S_m \tag{2.1}$$

where S_i is a translate of a one dimensional subspace U_i $(1 \leqslant i \leqslant m)$. Then $m \geqslant q$ and

(a) *if $m = q$, $U_1 = U_i$ $(1 \leqslant i \leqslant q)$;*
(b) *if $m = q + 1$ and the union (2.1) is irredundant, then the subspaces U_i are distinct and, for some $r \in F$, $S_i = U_i + r$ $(1 \leqslant i \leqslant q + 1)$;*

and

(c) *if $m = q + 2$ and the union (2.1) is irredundant then the subspaces U_i $(1 \leqslant i \leqslant q + 2)$ do not cover F^2.*

PROOF: Firstly note that $mq \geqslant q^2$ so $m \geqslant q$. Now observe that F^2 can be thought of as an affine plane in which the lines are the translates of one-dimensional vector subspaces. The result then has an easy, and presumably well known, geometrical proof. We give a sketch.

(a) In this case the space is covered by the q lines S_i, each containing exactly q points. Hence these lines are parallel and one of them passes through the origin.

(b) We are to prove that $q+1$ lines have a common point if their union is irredundant and equal to F^2. There are at most q mutually parallel lines, so S_1 and S_2 say, meet at a point P. Let $A = S_1 \cup S_2$. Every line S_i $(3 \leqslant i \leqslant q+1)$ meets A in at least one point. Since $\left|F^2 \setminus A\right| = q^2 - (2q - 1) = (q-1)^2$ no line S_i $(3 \leqslant i \leqslant q + 1)$ meets A in more than one point. If $q = 2$ then S_3 is incident with P since neither S_1 nor S_2 is redundant. Hence we may suppose that $q > 2$. Now no two S_i $(3 \leqslant i \leqslant q + 1)$ meet outside A. Suppose $P \in S_i$ but $P \notin S_j$ for some i, j satisfying $3 \leqslant i, j \leqslant q+1$. Then S_j is parallel to just one of S_1, S_2, say to S_1, and also parallel to just one of S_2, S_i therefore to S_i, a contradiction since S_1 and S_i are not parallel. That is, if three of

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the lines S_i $(1 \leqslant i \leqslant q+1)$ pass through P then all do, and we are done. Suppose that none of S_i $(3 \leqslant i \leqslant q+1)$ is incident with P. Then all S_j $(3 \leqslant j \leqslant d)$ are parallel to S_1 and all S_k $(d+1 \leqslant k \leqslant q+1)$ are parallel to S_2, for some d satisfying $3 \leqslant d < q+1$, or else the union (2.1) is redundant. It follows that $|S_i \cap S_j| = 1$ if $i \in \{1, 3, \ldots, d\}$ and $j \in \{2, d+1, \ldots, q+1\}$, and is zero otherwise; in particular all three-fold intersections are empty. Hence, counting points, $q^2 = q(q+1) - (d-1)(q-d+2)$ whence $q = (d-1)(q-d+2)$. However, both right-side factors are greater than 1, and hence have a prime common factor which therefore divides both q and $q+1$, a contradiction.

(c) In this case $q > 2$. Two of the lines, say S_1 and S_2, are parallel. It is enough to show that there is another pair of parallels. If there is not, all the lines S_i $(3 \leqslant i \leqslant q+2)$ are incident in pairs, and each is incident with each of S_1 and S_2. Since the complement of $S_1 \cup S_2$ has cardinality $q^2 - 2q = q(q-2)$, it follows that $S_i \cap S_j \subseteq S_1 \cup S_2$ $(3 \leqslant i < j \leqslant q+2)$. If all these intersections are the same, say lying in S_1, then counting shows that S_2 is redundant. Hence $S_1 \cap S_h \neq S_1 \cap S_k$ for some h, $k \in \{3, \ldots, q+2\}$. Then $S_h \cap S_k$ is incident with S_2, and there is some S_t $(3 \leqslant t \leqslant q+2)$ for which $S_2 \cap S_t \neq S_2 \cap S_h$. But then one of $S_h \cap S_t$ or $S_k \cap S_t$ is not incident with S_1, a contradiction. ∎

LEMMA 2.7. *Let G be a group with the following structure: $O_3(G)$ is elementary Abelian of index 2 in G, and G has trivial centre. There does not exist a maximal irredundant 5-cover of G.*

PROOF: Let us suppose that the result is false, and that G is a minimal counterexample. Note that $|G| > 6$ since Sym_3 is not a counterexample.

Let

$$G = M_1 \cup M_2 \cup M_3 \cup M_4 \cup M_5 \tag{2.2}$$

be a maximal irredundant cover of G with core-free intersection D. Then $|M_i| > 2$ $(1 \leqslant i \leqslant 5)$. Therefore either

(a) for some i, $M_i = V := O_3(G)$ and $|G : M_j| = 3$ $(j \neq i)$; or
(b) $|G : M_j| = 3$ for all j.

Now $D \cap V = 1$ by Lemma 2.1 since $D \cap V \trianglelefteq G$. Let a be an involution of G. Since $\langle a \rangle$ is a Sylow 2-subgroup of G every 2-element of G is conjugate to a. Define

$$S_i := \{x \in V : a^x \in M_i\}, \quad 1 \leqslant i \leqslant 5.$$

Either $S_i = \emptyset$ or S_i is a coset of $X_i := V \cap M_i$ in V, and there is at most one of the first type. For all $x \in V$, there is an i for which $a^x \in M_i$ so

$$V = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5. \tag{2.3}$$

From Lemma 2.2(c) the intersection of every triple of the subgroups X_i $(1 \leqslant i \leqslant 5)$ is trivial. In the case (a) suppose that $M_5 = V$, so that the pairwise intersections $X_i \cap X_j$ $(1 \leqslant i < j \leqslant 4)$ are all trivial. In particular $|V| = 9$. Also $S_5 = \emptyset$ and

$$V = S_1 \cup S_2 \cup S_3 \cup S_4.$$

In this union all the S_i are essential since if, say, S_1 were omissible, then M_1 would be omissible in (2.2). However Lemma 2.6 now shows that the subgroups X_i $(1 \leqslant i \leqslant 4)$ are distinct. They therefore cover V making M_5 redundant, a contradiction. This shows that case (a) does not arise.

In case (b) we have $V = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. From Lemma 2.5 this union is redundant; and from Proposition 2.3 just one term, say X_i, is omissible. Since $1 = \bigcap_{j \neq i} X_j$, by Lemma 2.2(b), it follows from Proposition 2.4 that $|V| = 9$. Now we apply Lemma 2.6. Firstly, by (c) of that result, the union (2.3) is redundant, and at most two terms on the right are omissible. If omitting S_5 say, leaves an irredundant union then, by Lemma 2.6(b), $V = X_1 \cup X_2 \cup X_3 \cup X_4$ and M_5 is omissible from (2.2), contradiction. If omitting S_4 and S_5 from (2.3) leaves $V = S_1 \cup S_2 \cup S_3$ then Lemma 2.6(a) yields $X_1 = X_2 = X_3 \subseteq D \cap V = 1$, another contradiction. ∎

Finally we note the following well known fact which is used repeatedly, and without explicit reference, throughout what follows: if M is a maximal subgroup, and U an Abelian minimal normal subgroup, of a group then either $U \subseteq M$ or $U \cap M = 1$.

3. Proof of Theorem 1.1

We have already determined the 2-groups which have maximal irredundant core-free intersection 5-covers. The next lemma addresses non-2-groups

Lemma 3.1. *Suppose that the intersection of a maximal irredundant cover of five subgroups of a group G is core-free. If G is not a 2-group then every minimal normal subgroup of G has order 4.*

Proof: By Lemma 2.2(a) G is a {2, 3}-group. Since G is soluble, by Burnside's Theorem, every minimal normal subgroup U of G is Abelian. Moreover, by Lemma 2.2(d), $|U| \leqslant 4$.

If $|U| = 2$ then, again by Lemma 2.2(d), U is contained in at most three of the subgroups A_i, say $U \not\subseteq A_4 \cup A_5$. Since U is central, and since $G = A_4 U = A_5 U$, every 3-element of G is in $A_4 \cap A_4$. However if $1 \neq u \in U$ and if y is a 3-element, then $uy \notin A_4 \cup A_5$. Hence $uy \in A_1 \cup A_2 \cup A_3$ and therefore $y \in A_1 \cup A_2 \cup A_3$. It follows that a Sylow 3-subgroup S of G is in $A_1 \cup A_2 \cup A_3$ and therefore, by Proposition 2.3, in one of A_i $(1 \leqslant i \leqslant 3)$, say in A_3. Therefore $S \subseteq A_3 \cap A_4 \cap A_5$ and so, by Lemma

2.2(c), $S \subseteq D$. Since, therefore, every 3-element of G is in D so is the subgroup T which they generate. Of course $T \trianglelefteq G$ so $T = 1$. But this contradicts the fact that G is not a 2-group. Therefore G has no normal subgroups of order 2.

If $|U| = 3$ then U is contained in at most two of the subgroups A_i, say $U \not\subseteq A_3 \cup A_4 \cup A_5$. It follows that $G = UA_i$ $(3 \leqslant i \leqslant 5)$. An argument similar to that of the last paragraph shows that every 2-element of $C := \mathbf{C}_G(U)$ is in D. Since the subgroup they generate is normal it is 1, and we see that C is a 3-group. Also, $\Phi(C) \subseteq \Phi(G) \subseteq D$, so $\Phi(C) = 1$. That is, C is elementary Abelian. By Lemma 2.5(b) $C \neq G$. That is, no minimal normal subgroup of G is central. However $|G : C| = 2$, and so G satisfies the hypotheses of Lemma 2.7, contradiction. ∎

Proof of Theorem 1.1: Let G be a group with a maximal irredundant cover $\bigcup_{i=1}^{5} A_i$ with core-free intersection D. By Lemma 2.5 we may suppose that G is not a 2-group. Suppose that U is a minimal normal subgroup of G. It follows from Lemma 3.1 that $|U| = 4$. Also, by Lemma 2.2(d), U is in at most one of the subgroups A_i, say $U \not\subseteq A_2 \cup A_3 \cup A_4 \cup A_5$. A familiar argument gives that $C := \mathbf{C}_G(U)$ is an elementary 2-group. Moreover G/C embeds into $\mathrm{Aut}\,(U) \cong \mathrm{Sym}_3$, and $O_3(G/C) \neq 1$. As G/C-module, C has no non-trivial fixed points for the action of $O_3(G/C)$, using Lemma 3.1. It follows that C is the first or second nilpotent residual of G. Therefore C is complemented in G, using the result in [**2**, (5.18) p.383], say $G = CH$ where $H \cong C_3$ or $H \cong \mathrm{Sym}_3$. As H-module C is completely reducible, and every minimal normal subgroup of G is of order 4.

If $C = U$ then $G \cong \mathrm{Alt}_4$ or $G \cong \mathrm{Sym}_4$. The first case is (b) of the theorem. The second does not arise because Sym_4 has no maximal irredundant cover of five subgroups. For, D is core-free, does not contain the monolith of Sym_4 so, by Lemma 2.2(d), four of the five subgroups of the cover are copies of Sym_3 whilst the fifth, therefore, contains all the elements of Sym_4 of order 4. However this is a contradiction because these elements generate Sym_4.

If $C \neq U$ then $\mathbf{C}_{A_i}(U) \neq 1$ $(2 \leqslant i \leqslant 5)$, and $C = U \times \mathbf{C}_{A_i}(U)$. Since D is core-free, it follows from Lemma 3.1 and Lemma 2.2(d) that $1 = \mathbf{C}_{A_i}(U) \cap \mathbf{C}_{A_j}(U)$ $(2 \leqslant i < j \leqslant 5)$. Then, for $i \neq j$,

$$\left|\mathbf{C}_{A_j}(U)\right| |U| = |C| \geqslant \left|\mathbf{C}_{A_i}(U)\mathbf{C}_{A_j}(U)\right| = \left|\mathbf{C}_{A_i}(U)\right| \left|\mathbf{C}_{A_j}(U)\right|$$

so that $|U| \geqslant \left|\mathbf{C}_{A_i}(U)\right| \geqslant |U|$. It follows that each $\mathbf{C}_{A_i}(U)$ is minimal normal in G. That is, C is the direct product of two minimal normal subgroups of G. If H were isomorphic to Sym_3 then C, as H-module, would contain just three proper non-zero submodules instead of the (at least) five it does contain. Hence $|H| = 3$.

Now we examine the nature of this cover for G. Choose $a \in G$ of order 3. Then $\langle a \rangle$ is a Sylow 3-subgroup of G, and every 3-element of G is conjugate either to a or to a^2. Define $S_i := \{w \in C\colon a^w \in A_i\}$ $(1 \leqslant i \leqslant 5)$ and $N_i := A_i \cap C$ $(1 \leqslant i \leqslant 5)$. S_i is a coset on N_i in C: it is not empty since otherwise A_i would contain no 3-element, would therefore be equal to C, and some N_j would be in two of the A_k whence, by Lemma 2.2(d), in D, which is core-free. We have

$$C = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \tag{3.1}$$

since every a^w is in some A_i. We may regard C as a space of dimension 2 over the field F of 4 elements, where $\langle a \rangle$ is the multiplicative group of F, and apply Lemma 2.6(b). If the union (3.1) is irredundant then $S_i = N_i c$ $(1 \leqslant i \leqslant 5)$ for some $c \in C$. Hence $a^c \in A_i$ $(1 \leqslant i \leqslant 5)$, so $|D| = 3$ and G has the structure required by (c) of the theorem. If, however, (3.1) is redundant then, by Lemma 2.6, at most one term, say S_5, is omissible and $N_i = N_1$ $(1 \leqslant i \leqslant 4)$. This gives $N_1 = \bigcap_{i=1}^{4} N_i \subseteq \bigcap_{i=1}^{4} A_i = D$, a contradiction to the core-freeness of D. □

4. Proof of Theorem 1.2

If the result is false, let G be a group with an irredundant cover $\mathcal{C}$ of five subgroups, with core-free intersection D, for which $|G : D| > 16$. In the light of Theorem 1.1, $\mathcal{C}$ is not maximal. Suppose $\mathcal{C}$ chosen from among such 5-covers of G with as many maximal subgroups as possible. Let $\mathcal{C}^*$ be a cover of G got from $\mathcal{C}$ by replacing one of its non-maximal subgroups by a maximal subgroup containing it. Write D^* for the intersection of $\mathcal{C}^*$: $D^* \supseteq D$. $\mathcal{C}^*$ is redundant; for, if not, $D^* = D$ by Lemma 2.2(b), and so is core-free, while $\mathcal{C}^*$ has more maximal subgroups than does $\mathcal{C}$. It follows that we may write $\mathcal{C} = \bigcup_{i=1}^{5} A_i$ where A_1 is not maximal, and if A_1^* is a maximal subgroup containing it, then $\mathcal{C}^* = \{A_1^*, A_2, A_3, A_4, A_5\}$ is redundant as a cover for G.

If G is an irredundant union of four of the subgroups in $\mathcal{C}^*$, then we may suppose that

$$G = A_1^* \cup A_2 \cup A_3 \cup A_4 \tag{4.1}$$

since A_1^* is certainly essential. If $D_1 := A_1^* \cap A_2 \cap A_3 \cap A_4$ then it follows from Proposition 2.4 that $|G : D_1| \leqslant 9$ with equality only if $A_1^* \cap A_i = D_1$ $(2 \leqslant i \leqslant 4)$. If we have equality therefore, it follows that

$$A_1^* = A_1 \cup D_1 \cup (A_5 \cap A_1^*), \tag{4.2}$$

an irredundant union. However from (4.1) we deduce that $|A_1^* : D_1| = 3$, and from (4.2) and Proposition 2.3 that $|A_1^* : D_1| = 2$, a contradiction. Hence $|G : D_1| \leqslant 8$. Then, since $D_1 = A_2 \cap A_3 \cap A_4$, we have $|D_1 : D| \leqslant 2$ by Lemma 2.2(c), so $|G : D| \leqslant 16$, a contradiction.

Lastly, if G is an irredundant union of three of the subgroups in $\mathcal{C}^*$, we may suppose that

$$G = A_1^* \cup A_2 \cup A_3 \tag{4.3}$$

since A_1^* is surely included. Let us write $N := A_2 \cap A_3$ $(= A_1^* \cap A_2 = A_1^* \cap A_3)$. Now

$$A_1^* = A_1 \cup N \cup (A_1^* \cap A_4) \cup (A_1^* \cap A_5). \tag{4.4}$$

If the union (4.4) is irredundant then $|A_1^* : D| = |A_1^* : A_1 \cap N \cap A_4 \cap A_5| \leqslant 9$. However, by (4.3), $|A_1^* : A_2 \cap A_3| = 2$, so $|A_1^* : D| \neq 9$. Hence $|G : D| = |G : A_1^*|\,|A_1^* : D| \leqslant 16$, a contradiction. On the other hand if the union (4.4) is redundant then three of the subgroups on the right side are essential, and the possible intersections I satisfy $|I : D| \leqslant 2$, using Lemma 2.2(c). Hence $|G : D| = |G : A_1^*|\,|A_1^* : I|\,|I : D| \leqslant 2.4.2 = 16$. This contradiction completes the proof of Theorem 1.2. □

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