Canad. Math. Bull. Vol. 61 (2), 2018 pp. 423–437
http://dx.doi.org/10.4153/CMB-2017-018-5

Stability of Traveling Wavefronts for a Two-Component Lattice Dynamical System Arising in Competition Models

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Abstract. In this paper, we study a two-component Lotka–Volterra competition system on a one-dimensional spatial lattice. By the comparison principle, together with the weighted energy, we prove that the traveling wavefronts with large speed are exponentially asymptotically stable, when the initial perturbation around the traveling wavefronts decays exponentially as $j + ct \to -\infty$, where $j \in \mathbb{Z}$, $t > 0$, but the initial perturbation can be arbitrarily large on other locations. This partially answers an open problem by J.-S. Guo and C.-H. Wu.

1 Introduction

In this paper, we study the following two-component Lotka–Volterra competition system on a one-dimensional spatial lattice

$$\begin{aligned} \frac{du_j}{dt} &= d_1(u_{j+1} + u_{j-1} - 2u_j) + r_1 u_j(1 - b_1 u_j - kv_j), \\ \frac{dv_j}{dt} &= d_2(v_{j+1} + v_{j-1} - 2v_j) + r_2 v_j(1 - b_2 v_j - hu_j), \end{aligned} \tag{1.1}$$

with the initial data $u_j(0) = u_{j0}$, and $v_j(0) = v_{j0}$, where $j \in \mathbb{Z}$, $t > 0$, d_i, r_i, b_i, $i = 1, 2$, h, and k are some positive constants. This model describes how two species u and v living in a discrete habitat compete with each other. Here u_j and v_j stand for the populations of two species at time t and position j, respectively, d_i is the migration coefficient, r_i is the net birth rate, $1/b_i$ is the carrying capacity of species i for $i = 1, 2$, and h, k are inter-specific competition coefficients.

With a certain normalization, we assume that the diffusion coefficients of species u, v are given by 1, d, the birth rates of species u, v are given by 1, r, and the carrying capacities are equal to 1. Then system (1.1) is reduced to the system

$$\begin{aligned} \frac{du_j}{dt} &= (u_{j+1} + u_{j-1} - 2u_j) + u_j(1 - u_j - kv_j), \\ \frac{dv_j}{dt} &= d(v_{j+1} + v_{j-1} - 2v_j) + rv_j(1 - v_j - hu_j), \end{aligned} \tag{1.2}$$

where all coefficients are positive.

Received by the editors March 6, 2017.
Published electronically May 29, 2017.
G.-B. Zhang is the corresponding author. Author G.-B. Z. was supported by NSF of China (11401478).
AMS subject classification: **34A33**, 34K20, 92D25.
Keywords: lattice dynamical system, competition model, traveling wavefront, stability.

It is easy to see that the system (1.2) has four constant equilibria: $(0,0)$, $(0,1)$, $(1,0)$, and coexistence equilibrium $(\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$ provided that $hk \neq 1$. From [5], we have the following asymptotic behaviors as $t \to +\infty$.

(i) $(u,v) \to (1,0)$ if $0 < k < 1 < h$.
(ii) $(u,v) \to (0,1)$ if $0 < h < 1 < k$.
(iii) $(u,v) \to$ one of $(0,1)$, $(1,0)$ (depending on the initial condition) if $k, h > 1$.
(iv) $(u,v) \to (\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$ (u and v coexist) if $0 < k, h < 1$.

We need to point out that case (ii) can be reduced to case (i) by exchanging the positions of u and v.

A biologically and mathematically interesting problem is the traveling wave solution connecting two equilibrium points. We note that a traveling wave solution of (1.2) is a special translation invariant solution of the form

$$u(x,t) = \varphi(\xi), \quad v(x,t) = \psi(\xi), \quad \xi = j + ct$$

that connect two different equilibria from $\{(0,0), (0,1), (1,0), (\frac{1-k}{1-hk}, \frac{1-h}{1-hk})\}$, where $c > 0$ is the wave speed. If φ and ψ are monotone, then (φ, ψ) is called a traveling wavefront. Substituting $(\varphi(j+ct), \psi(j+ct))$ into (1.2), we obtain the following wave profile system

$$\begin{aligned} c\varphi'(\xi) &= (\varphi(\xi+1) + \varphi(\xi-1) - 2\varphi(\xi)) + \varphi(\xi)(1 - \varphi(\xi) - k\psi(\xi)), \\ c\psi'(\xi) &= d(\psi(\xi+1) + \psi(\xi-1) - 2\psi(\xi)) + r\psi(\xi)(1 - \psi(\xi) - h\varphi(\xi)). \end{aligned}$$

Recently, Guo and Liang [3] and Guo and Wu [4–6] studied traveling wave solutions of system (1.2), which connect two boundary equilibria $(0,1)$ and $(1,0)$. It is easy to see that when $0 < k < 1 < h$, the equilibrium $(0,1)$ is unstable and the equilibrium $(1,0)$ is stable. Hence, system (1.2) is called a monostable system. When $h, k > 1$, both equilibria $(0,1)$ and $(1,0)$ are stable. Hence, system (1.2) is called a bistable system. For the bistable system (1.2), Guo and Wu [4] first showed that the propagation failure phenomenon occurs, and then proved the monotonicity of a traveling wave solution with nonzero speed and the uniqueness of nonzero wave speed. For the monostable system (1.2), Guo and Wu [5] proved the existence, monotonicity, and uniqueness of traveling wave solutions. Meanwhile, Guo and Liang [3] gave the characterization of the minimal speed for certain ranges of h, k, r, d. We note that Guo and Wu [6] showed the recent results on the wave propagation of (1.2), and gave some open problems. The second open problem is the stability of traveling wave solutions for both monostable and bistable cases. In this paper, we are devoted to proving the stability of traveling wave solutions for the monostable case.

It should be mentioned that the stability of traveling wave solutions of reaction-diffusion equations, nonlocal dispersal equations and lattice differential equations with monostable nonlinearity has been extensively studied in the literature. We refer the readers to [11, 18–24] for reaction-diffusion equations, [7, 8, 12, 32, 33] for nonlocal dispersal equations, and [2, 14–16, 29–31] for lattice differential equations. To the best of our knowledge, there were only a few papers studying the stability of traveling wave solutions of reaction-diffusion systems and nonlocal dispersal systems, see [9,13,26–28]. For the lattice differential system, there are still no results on the stability of traveling wave solutions. More recently, we [25] studied the following continuum version of (1.1) (where $d_i = b_i = 1$, $i = 1, 2$).

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \mathcal{D}[u](t,x) + r_1 u(t,x)[1-u(t,x)-kv(t,x)], \\ \frac{\partial v}{\partial t}(t,x) &= \mathcal{D}[v](t,x) + r_2 v(t,x)[1-v(t,x)-hu(t,x)], \end{aligned} \tag{1.3}$$

where $t > 0$, $x \in \mathbb{R}$, r_1, r_2, h, k are all positive constants, and

$$\mathcal{D}[\phi](t,x) = \phi(t,x+1) - 2\phi(t,x) + \phi(t,x-1).$$

By using the comparison principle and the weighted energy method, we proved the stability of traveling wavefronts of (1.3). Motivated by [25], in this paper, we still take the comparison principle and the weighted energy method to prove the stability of traveling wavefronts of (1.2). We should point out that the key step is to establish the l^2-energy estimates for the solutions of the perturbed system. Although (1.2) and (1.3) (choose $r_1 = 1$ and $r_2 = r$ in (1.3), and let the diffusion coefficient for v be d) take the same wave profile system, the technical details for obtaining *a priori* estimates are different. We should remark that by this method, the obtained stability results only hold for large wave speed. We leave the stability of traveling wavefronts with low speeds, especially the critical speed, for future study.

We now briefly describe the organization of this paper. In Section 2, we will give the notations, the existence of traveling wavefronts, some necessary assumptions, and the main theorem. Section 3 is mainly devoted to the proof of the stability theorem.

2 Preliminaries and Main Result

In this section, we first recall some known results, then define a weight function, and state our main result.

To study the stability of the traveling wavefront of (1.2), it is convenient to work on (u_j, v_j^*), where $v_j^* = 1 - v_j$. For the sake of convenience, we drop the star. Then (1.2) can be represented as

$$\begin{aligned} \frac{du_j}{dt} &= (u_{j+1} + u_{j-1} - 2u_j) + u_j(1 - u_j - k(1-v_j)), \\ \frac{dv_j}{dt} &= d(v_{j+1} + v_{j-1} - 2v_j) + r(1-v_j)(hu_j - v_j), \end{aligned} \tag{2.1}$$

with the initial data

$$u_j(0) = u_{j0}, \quad v_j(0) = 1 - v_{j0}. \tag{2.2}$$

Let $u_j(t) = \varphi(\xi)$, $v_j(t) = \psi(\xi)$, where $\xi = j + ct$. Then the wave profile system of (2.1) is

$$\begin{aligned} c\varphi'(\xi) &= \big(\varphi(\xi+1) + \varphi(\xi-1) - 2\varphi(\xi)\big) + \varphi(\xi)\big(1 - \varphi(\xi) - k(1-\psi(\xi))\big), \\ c\psi'(\xi) &= d\big(\psi(\xi+1) + \psi(\xi-1) - 2\psi(\xi)\big) + r(1-\psi(\xi))(h\varphi(\xi) - \psi(\xi)), \end{aligned} \tag{2.3}$$

with the boundary conditions

$$(\varphi(-\infty), \psi(-\infty)) = (0,0) \quad \text{and} \quad (\varphi(+\infty), \psi(+\infty)) = (1,1). \tag{2.4}$$

For the existence of a traveling wavefront of (2.1), we refer to Guo and Wu [4].

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Proposition 2.1 *Assume that* $0 < k < 1 < h$, $d > 0$, *and* $r > 0$. *Then there exists* $c^* > 0$ *such that for any* $c \ge c^*$, (2.1) *admits a traveling wavefront* $(\varphi(\xi), \psi(\xi))$ *connecting* $(0,0)$ *and* $(1,1)$, *and satisfying* $\varphi'(\cdot) > 0$ *and* $\psi'(\cdot) > 0$ *on* $\mathbb{R}$. *For any* $c < c^*$, *there is no such traveling wave.*

Before stating our main result, let us make the following notation. Throughout the paper, l_w^2 denotes a weighted l^2-space with a weighted function $0 < w(\xi) \in C(\mathbb{R})$, *i.e.*, $l_w^2 := \left\{ \zeta = \{\zeta_i\}_{i\in\mathbb{Z}}, \zeta_i \in \mathbb{R} \mid \sum_i w(i+ct)\zeta_i^2 < \infty \right\}$, and its norm is defined by

$$\|\zeta\|_{l_w^2} = \Big(\sum_i w(i+ct)\zeta_i^2 \Big)^{\frac{1}{2}}, \quad \text{for } \zeta \in l_w^2.$$

In particular, when $w \equiv 1$, we denote l_w^2 by l^2.

In order to obtain our stability result, we need the following assumption.

$$\text{(H)} \qquad 0 < k < \frac{2}{3}, \quad h > 2 + \frac{k}{2r}.$$

Define two functions on λ as follows:

$$\mathcal{M}_1(\lambda) = 4 - 3k - (e^\lambda + 1), \quad \mathcal{M}_2(\lambda) = 2d - 4r + 2rh - k - d(e^\lambda + 1).$$

By assumption (H), we get

$$\mathcal{M}_1(0) = 2 - 3k > 0, \quad \mathcal{M}_2(0) = -4r + 2rh - k > 0.$$

Then by the continuity of $\mathcal{M}_1(\lambda)$ and $\mathcal{M}_2(\lambda)$ with respect to λ, there exists $\lambda_0 > 0$ such that $\mathcal{M}_1(\lambda_0) > 0$ and $\mathcal{M}_2(\lambda_0) > 0$.

Furthermore, define

$$\mathcal{N}_1(\xi) = 4\varphi(\xi) - 3k + rh\psi(\xi) - rh - (e^{\lambda_0} + 1),$$
$$\mathcal{N}_2(\xi) = 2d - 4r + 2rh\varphi(\xi) - k + rh\psi(\xi) - rh - d(e^{\lambda_0} + 1),$$

where $(\varphi(\xi), \psi(\xi))$ is the traveling wavefront given in Proposition 2.1.

By (2.4), we have

$$\lim_{\xi\to+\infty} \mathcal{N}_1(\xi) = \mathcal{M}_1(\lambda_0) > 0 \quad \text{and} \quad \lim_{\xi\to+\infty} \mathcal{N}_2(\xi) = \mathcal{M}_2(\lambda_0) > 0.$$

Hence, there exists a number $\xi_0 > 0$ large enough such that

$$\mathcal{N}_1(\xi_0) = 4\varphi(\xi_0) - 3k + rh\psi(\xi_0) - rh - (e^{\lambda_0} + 1) > 0,$$
$$\mathcal{N}_2(\xi_0) = 2d - 4r + 2rh\varphi(\xi_0) - k + rh\psi(\xi_0) - rh - d(e^{\lambda_0} + 1) > 0.$$

Define the weighted function as follows:

$$\text{(2.5)} \qquad w(\xi) = \begin{cases} e^{-\lambda_0(\xi-\xi_0)} & \xi \le \xi_0, \\ 1 & \xi > \xi_0. \end{cases}$$

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Theorem 2.2 (Stability) *Assume that* (H) *holds. For any given traveling wavefront* $\big(\varphi(\xi(t,j)),\psi\big(\xi(t,j)\big)\big)$ *with the wave speed* $c > \max\{c^*,\widetilde{c}\}$, *where* $\widetilde{c} = \max\{c_1,c_2\}/\lambda_0$, *and*

$$c_1 = 3k + rh + (e^{\lambda_0} + e^{-\lambda_0} + 1), \tag{2.6}$$

$$c_2 = 4r + k + rh + d(e^{\lambda_0} + e^{-\lambda_0} + 1). \tag{2.7}$$

If the initial data satisfies $(0,0) \le (u_j(0), v_j(0)) \le (1,1)$, $j \in \mathbb{Z}$, *and the initial perturbations satisfy* $u_j(0) - \varphi(j) \in C(l_w^2)$ *and* $v_j(0) - \psi(j) \in C(l_w^2)$, *then the nonnegative solution of the Cauchy problems* (2.1) *and* (2.2) *uniquely exists and satisfies* $(0,0) \le \big(u_j(t), v_j(t)\big) \le (1,1)$, *for all* $j \in \mathbb{Z}$, $t > 0$, *and*

$$u_j(t) - \varphi(j+ct) \in C((0,+\infty); l_w^2), \quad v_j(t) - \psi(j+ct) \in C((0,+\infty); l_w^2),$$

where $w(\xi)$ *is defined by* (2.5). *Moreover,* $\big(u_j(t), v_j(t)\big)$ *converges to the traveling wavefront* $\big(\varphi(j+ct), \psi(j+ct)\big)$ *exponentially in time* t, i.e.,

$$\sup_{j\in\mathbb{Z}} |u_j(t) - \varphi(j+ct)| \le Ce^{-\mu t},$$
$$\sup_{j\in\mathbb{Z}} |v_j(t) - \psi(j+ct)| \le Ce^{-\mu t}$$

for all $t > 0$, *where* C *and* μ *are some positive constants.*

3 Stability of Traveling Wavefronts

3.1 Existence and Comparison of Solutions

In this subsection, we will establish the existence and comparison principle of solutions for the initial value problems (2.1) and (2.2).

Let $X = l_w^2 \times l_w^2$, where $w(x)$ is given by (2.5). Let

$$X^+ = \big\{ w = (u_j, v_j) \in X : u_j \ge 0, v_j \ge 0, j \in \mathbb{Z} \big\}.$$

It is easy to say that X^+ is a closed cone of X.

Let $T_1(t) = e^{-\mu_1 t}$ and $T_2 = e^{-\mu_2 t}$, where $\mu_1 = 2 + k$ and $\mu_2 = 2d + r + rh$. It is obvious that $T_i(t)$ is a linear C_0 semigroup on X, $i = 1, 2$. In particular, it is strongly positive. Let

$$f_1(w) = u_{j+1} + u_{j-1} + (\mu_1 - 1)u_j - u_j^2 - ku_j + ku_jv_j,$$
$$f_2(w) = v_{j+1} + v_{j-1} + (\mu_2 - 2d - r)v_j + rv_j^2 + rhu_j - rhu_jv_j.$$

Then the system (2.1) with the initial value (2.2) has an equivalent form as follows:

$$u_j(t) = T_1(t)u_j(0) + \int_0^t T_1(t-s) f_1(w(s))\, ds,$$
$$v_j(t) = T_2(t)v_j(0) + \int_0^t T_2(t-s) f_2(w(s))\, ds.$$

That is, $w(t) = T(t)w_0 + \int_0^t T(t-s)F(w(s))\, ds$, where

$$w(t) = \begin{pmatrix} u_j(t) \\ v_j(t) \end{pmatrix}, \quad T(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}, \quad w_0(x) = \begin{pmatrix} u_j(0) \\ v_j(0) \end{pmatrix},$$

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and

$$F(w)=\begin{pmatrix} f_1(w)\\ f_2(w)\end{pmatrix}.$$

We now state the definition of super-sub solutions of (2.1) with (2.2) as follows.

Definition 3.1 A continuous function $w=(u_j,v_j):[\tau,b)\to X$, $\tau<b$, is called a supersolution (subsolution) of (2.1) on $[\tau,b)$ if

$$u_j(t)\geq T_1(t-s)u_j(s)+\int_s^t T_1(t-r)f_1(w(r))\,dr,\quad \text{respectively } (\leq),$$
$$v_j(t)\geq T_2(t-s)v_j(s)+\int_s^t T_2(t-r)f_2(w(r))\,dr,\quad \text{respectively } (\leq),$$

for any $\tau\leq s<t<b$.

By the property of the semigroups $T_1(t)$ and $T_2(t)$, standard super-sub solutions technique, and the theory of abstract functional differential equations [17, Corollary 5], we obtain the boundedness and the comparison principle for the Cauchy problems (2.1) and (2.2), see also [1,10].

Lemma 3.2 (Boundedness) *Assume that* (H) *holds and that the initial data*

$$(u_j(0),v_j(0))$$

satisfies $(0,0)\leq\big(u_j(0),v_j(0)\big)\leq(1,1)$ *for* $j\in\mathbb{Z}$. *Then the solution* $\big(u_j(t),v_j(t)\big)$ *of the Cauchy problems* (2.1) *and* (2.2) *exists and satisfies* $(0,0)\leq(u_j(t),v_j(t))\leq(1,1)$, *for* $t\in(0,+\infty)$, $j\in\mathbb{Z}$.

Lemma 3.3 (Comparison principle) *Assume that* (H) *holds. Let* $\big(u_j^-(t),v_j^-(t)\big)$ *and* $\big(u_j^+(t),v_j^+(t)\big)$ *be the solution of* (2.1) *with the initial data* $(u_j^-(0),v_j^-(0))$ *and* $(u_j^+(0),v_j^+(0))$, *respectively. If*

$$(0,0)\leq\big(u_j^-(0),v_j^-(0)\big)\leq\big(u_j^+(0),v_j^+(0)\big)\leq(1,1)\qquad j\in\mathbb{Z},$$

then $(0,0)\leq\big(u_j^-(t),v_j^-(t)\big)\leq\big(u_j^+(t),v_j^+(t)\big)\leq(1,1)$ *for* $t\in(0,+\infty)$, $j\in\mathbb{Z}$.

3.2 Proof of Theorem 2.2

In this subsection, we are devoted to the proof of the stability result. Our proof relies on the weighted energy method combined with the comparison principle.

Let the initial data $\big(u_j(0),v_j(0)\big)$ be such that $(0,0)\leq(u_j(0),v_j(0))\leq(1,1)$ for $j\in\mathbb{Z}$, and for $j\in\mathbb{Z}$ let

$$u_j^-(0)=\min\{u_j(0),\varphi(j)\},\qquad u_j^+(0)=\max\{u_j(0),\varphi(j)\},$$
$$v_j^-(0)=\min\{v_j(0),\psi(j)\},\qquad v_j^+(0)=\max\{v_j(0),\psi(j)\}.$$

This implies that

$$\begin{aligned}&0\leq u_j^-(0)\leq u_j(0)\leq u_j^+(0)\leq 1, &&0\leq u_j^-(0)\leq\varphi(j)\leq u_j^+(0)\leq 1,\\ &0\leq v_j^-(0)\leq v_j(0)\leq v_j^+(0)\leq 1, &&0\leq v_j^-(0)\leq\psi(j)\leq v_j^+(0)\leq 1.\end{aligned}\tag{3.1}$$

Define $u_j^+(t), u_j^-(t), v_j^+(t), v_j^-(t)$ as the corresponding solutions of (2.1) with the initial data $u_j^+(0), u_j^-(0), v_j^+(0)$, and $v_j^-(0)$, respectively. Then by the comparison principle in Lemma 3.3, it follows that

$$
\begin{aligned}
&0 \le u_j^-(t) \le u_j(t) \le u_j^+(t) \le 1, \qquad 0 \le u_j^-(t) \le \varphi(j+ct) \le u_j^+(t) \le 1,\\
&0 \le v_j^-(t) \le v_j(t) \le v_j^+(t) \le 1, \qquad 0 \le v_j^-(t) \le \psi(j+ct) \le v_j^+(t) \le 1,
\end{aligned}
\tag{3.2}
$$

where $t \in (0,+\infty)$, $j \in \mathbb{Z}$.

Let

$$U_j(t) = u_j^+(t) - \varphi(j+ct), \quad U_{j0}(0) = u_j^+(0) - \varphi(j),$$

and

$$V_j(t) = v_j^+(t) - \psi(j+ct), \quad V_{j0}(0) = v_j^+(0) - \psi(j),$$

where $t \in (0,+\infty)$, $j \in \mathbb{Z}$. Then by (2.1) and (2.3), $\big(U_j(t), V_j(t)\big)$ satisfies

$$
\begin{aligned}
\frac{dU_j(t)}{dt} &= [U_{j+1}(t) + U_{j-1}(t) - 2U_j(t)]\\
&\quad + U_j(t)[1 - k - 2\varphi(\xi(t,j)) + kV_j(t) + k\psi(\xi(t,j))]\\
&\quad - U_j^2(t) + k\varphi(\xi(t,j))V_j(t),\\
\frac{dV_j(t)}{dt} &= d[V_{j+1}(t) + V_{j-1}(t) - 2V_j(t)]\\
&\quad + V_j(t)[2r\psi(\xi(t,j)) - r - rhU_j(t) - rh\varphi(\xi(t,j))]\\
&\quad + rV_j^2(t) + rh(1-\psi(\xi(t,j)))U_j(t),
\end{aligned}
\tag{3.3}
$$

with the initial data $U_j(0) = U_{j0}(0)$, $V_j(0) = V_{j0}(0)$, $j \in \mathbb{Z}$. It is easy to see from (3.1) and (3.2) that

$$(0,0) \le (U_j(t), V_j(t)) \le (1,1) \quad \text{and} \quad (0,0) \le (U_{j0}(0), V_{j0}(0)) \le (1,1).$$

Define

$$B_{\mu,w}^1(t,j) = A_w^1(t,j) - 2\mu, \quad B_{\mu,w}^2(t,j) = A_w^2(t,j) - 2\mu, \tag{3.4}$$

where

$$
\begin{aligned}
A_w^1(t,j) &= 2\Big(2 - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 1 + k + 2\varphi(\xi(t,j)) - kV_j(t) - k\psi(\xi(t,j))\Big)\\
&\quad - k\varphi(\xi(t,j)) - rh(1-\psi(\xi(t,j)))\\
&\quad - \Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big),\\
A_w^2(t,j) &= 2\Big(2d - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2r\psi(\xi(t,j)) + r + rhU_j(t) + rh\varphi(\xi(t,j))\Big)\\
&\quad - k\varphi(\xi(t,j)) - 2rV_j(t) - rh(1-\psi(\xi(t,j)))\\
&\quad - d\Big(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\Big).
\end{aligned}
$$

It is easy to see that $\xi(t,j+1) = \xi(t,j) + 1$ and $\xi(t,j-1) = \xi(t,j) - 1$.

Next we will establish some key inequalities.

Lemma 3.4 *Assume that* (H) *holds. For any* $c > \max\{c^*, \widetilde{c}\}$*, there exist some positive constants* C_i *such that* $A_w^i(t,j) \geq C_i$*,* $i = 1, 2$*, for all* $t > 0$ *and* $j \in \mathbb{Z}$*.*

Proof Since $c > \max\{c^*, \widetilde{c}\}$, we get $c\lambda_0 > c_1$ and $c\lambda_0 > c_2$, where c_1 and c_2 can be seen in (2.6) and (2.7). Clearly, the following hold.

$$c\lambda_0 - 3k - rh - (e^{\lambda_0} + e^{-\lambda_0} + 1) > 0 \quad \text{and} \quad c\lambda_0 - 4r - k - rh - d(e^{\lambda_0} + e^{-\lambda_0} + 1) > 0.$$

We first show that $A_w^1(t,j) \geq C_1$ for some positive constant C_1.

Case 1: $\xi(t,j) < \xi_0 - 1$. It is clear that $\xi(t,j) < \xi_0$, $\xi(t,j+1) < \xi_0$ and $\xi(t,j-1) < \xi_0$. Hence,

$$\begin{aligned} w(\xi(t,j)) &= e^{-\lambda_0(\xi(t,j)-\xi_0)} \\ w(\xi(t,j-1)) &= e^{-\lambda_0(\xi(t,j)-1-\xi_0)} \\ w(\xi(t,j+1)) &= e^{-\lambda_0(\xi(t,j)+1-\xi_0)}. \end{aligned}$$

Then one has

$$\begin{aligned} A_w^1(t,j) = 4 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} &- 2 + 2k + 4\varphi(\xi(t,j)) - 2kV_j(t) - 2k\psi(\xi(t,j)) \\ &- k\varphi(\xi(t,j)) - rh\big(1 - \psi(\xi(t,j))\big) \\ &- \left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\ &> c\lambda_0 - 3k - rh - (e^{\lambda_0} + e^{-\lambda_0}) = c\lambda_0 - 3k - rh - (e^{\lambda_0} + e^{-\lambda_0} + 1) + 1 \\ &> 0. \end{aligned}$$

Case 2: $\xi_0 - 1 \leq \xi(t,j) \leq \xi_0$. In this case, $\xi(t,j-1) < \xi_0$ and $\xi(t,j+1) \geq \xi_0$. Then $w(\xi(t,j)) = e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$, and $w(\xi(t,j+1)) = 1$. Hence, we get

$$\begin{aligned} A_w^1(t,j) = 4 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} &- 2 + 2k + 4\varphi(\xi(t,j)) - 2kV_j(t) - 2k\psi(\xi(t,j)) \\ &- k\varphi(\xi(t,j)) - rh\big(1 - \psi(\xi(t,j))\big) \\ &- \left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\ &> c\lambda_0 - 3k - rh - (e^{\lambda_0} + e^{\lambda_0(\xi(t,j)-\xi_0)}) \geq c\lambda_0 - 3k - rh - (e^{\lambda_0} + 1) \\ &= c\lambda_0 - 3k - rh - (e^{\lambda_0} + e^{-\lambda_0} + 1) + e^{-\lambda_0} \\ &> 0. \end{aligned}$$

Case 3: $\xi_0 < \xi(t,j) \le \xi_0 + 1$. In this case, $\xi(t,j-1) \le \xi_0$ and $\xi(t,j+1) > \xi_0$. Then $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j)) = w(\xi(t,j+1)) = 1$. Thus, we obtain

$$\begin{aligned}
A_w^1(t,j) &= 4 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2 + 2k + 4\varphi(\xi(t,j)) - 2kV_j(t) - 2k\psi(\xi(t,j)) \\
&\quad - k\varphi(\xi(t,j)) - rh\big(1-\psi(\xi(t,j))\big) \\
&\quad - \left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> 4\varphi(\xi_0) - 3k + rh\psi(\xi_0) - rh - (e^{-\lambda_0(\xi(t,j)-1-\xi_0)} + 1) \\
&> 4\varphi(\xi_0) - 3k + rh\psi(\xi_0) - rh - (e^{\lambda_0} + 1) \\
&= \mathcal{N}_1(\xi_0) > 0.
\end{aligned}$$

Case 4: $\xi(t,j) > \xi_0 + 1$. In this case, $\xi(t,j) > \xi_0$, $\xi(t,j+1) > \xi_0$, and $\xi(t,j-1) > \xi_0$. Then $w(\xi(t,j)) = w(\xi(t,j-1)) = w(\xi(t,j+1)) = 1$. Hence, we have

$$\begin{aligned}
A_w^1(t,j) &= 4 - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 2 + 2k + 4\varphi(\xi(t,j)) - 2kV_j(t) - 2k\psi(\xi(t,j)) \\
&\quad - k\varphi(\xi(t,j)) - rh\big(1-\psi(\xi(t,j))\big) \\
&\quad - \left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> 4\varphi(\xi_0) - 3k + rh\psi(\xi_0) - rh - 2 \\
&= \mathcal{N}_1(\xi_0) + (e^{\lambda_0} + 1) - 2 \\
&> e^{\lambda_0} - 1 > 0.
\end{aligned}$$

We can obtain $A_w^1(t,j) \ge C_1 > 0$ by choosing a suitable C_1 small enough.

Next we prove $A_w^2(t,j) \ge C_2$ for some positive constant C_2.

Case 1: $\xi(t,j) < \xi_0 - 1$. It is clear that $\xi(t,j) < \xi_0$, $\xi(t,j+1) < \xi_0$ and $\xi(t,j-1) < \xi_0$. Hence,

$$w(\xi(t,j)) = e^{-\lambda_0(\xi(t,j)-\xi_0)}, \quad w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)},$$
$$w(\xi(t,j+1)) = e^{-\lambda_0(\xi(t,j)+1-\xi_0)}.$$

Then one has

$$\begin{aligned}
A_w^2(t,j) &= 4d - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - 4r\psi(\xi(t,j)) + 2r + 2rhU_j(t) + 2rh\varphi(\xi(t,j)) \\
&\quad - k\varphi(\xi(t,j)) - 2rV_j(t) - rh\big(1-\psi(\xi(t,j))\big) \\
&\quad - d\left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> 2d + c\lambda_0 - 4r - k - rh - d(e^{\lambda_0} + e^{-\lambda_0}) \\
&= c\lambda_0 - 4r - k - rh - d(e^{\lambda_0} + e^{-\lambda_0} + 1) + d + 2d \\
&> 3d > 0.
\end{aligned}$$

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Case 2: $\xi_0 - 1 \le \xi(t,j) \le \xi_0$. In this case, $\xi(t,j-1) < \xi_0$ and $\xi(t,j+1) \ge \xi_0$. Then $w(\xi(t,j)) = e^{-\lambda_0(\xi(t,j)-\xi_0)}$, $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$, and $w(\xi(t,j+1)) = 1$. Hence, we get

$$\begin{aligned}
A_w^2(t,j) = 4d - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} &- 4r\psi(\xi(t,j)) + 2r + 2rhU_j(t) + 2rh\varphi(\xi(t,j)) \\
&- k\varphi(\xi(t,j)) - 2rV_j(t) - rh\big(1-\psi(\xi(t,j))\big) \\
&- d\left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> 2d + c\lambda_0 - 4r - k - rh - d(e^{\lambda_0} + e^{\lambda_0(\xi(t,j)-\xi_0)}) \\
&\ge c\lambda_0 - 4r - k - rh - d(e^{\lambda_0} + 1 + e^{-\lambda_0}) + de^{-\lambda_0} + 2d \\
&> de^{-\lambda_0} + 2d \\
&> 0.
\end{aligned}$$

Case 3: $\xi_0 < \xi(t,j) \le \xi_0 + 1$. In this case, $\xi(t,j-1) \le \xi_0$ and $\xi(t,j+1) > \xi_0$. Then $w(\xi(t,j-1)) = e^{-\lambda_0(\xi(t,j)-1-\xi_0)}$ and $w(\xi(t,j)) = w(\xi(t,j+1)) = 1$. Thus, we obtain

$$\begin{aligned}
A_w^2(t,j) = 4d - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} &- 4r\psi(\xi(t,j)) + 2r + 2rhU_j + 2rh\varphi(\xi(t,j)) \\
&- k\varphi(\xi(t,j)) - 2rV_j(t) - rh\big(1-\psi(\xi(t,j))\big) \\
&- d\left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> 2d - 4r + 2rh\varphi(\xi_0) - k - rh + rh\psi(\xi_0) - d(e^{-\lambda_0(\xi(t,j)-1-\xi_0)} + 1) \\
&> 2d - 4r + 2rh\varphi(\xi_0) - k - rh + rh\psi(\xi_0) - d(e^{\lambda_0} + 1) \\
&= \mathcal{N}_2(\xi_0) > 0.
\end{aligned}$$

Case 4: $\xi(t,j) > \xi_0 + 1$. In this case, $\xi(t,j) > \xi_0$, $\xi(t,j+1) > \xi_0$, and $\xi(t,j-1) > \xi_0$. Then $w\big(\xi(t,j)\big) = w\big(\xi(t,j-1)\big) = w\big(\xi(t,j+1)\big) = 1$. Hence, we have

$$\begin{aligned}
A_w^2(t,j) = 4d - c\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} &- 4r\psi(\xi(t,j)) + 2r + 2rhU_j + 2rh\varphi(\xi(t,j)) \\
&- k\varphi(\xi(t,j)) - 2rV_j(t) - rh\big(1-\psi(\xi(t,j))\big) \\
&- d\left(2 + \frac{w(\xi(t,j-1))}{w(\xi(t,j))} + \frac{w(\xi(t,j+1))}{w(\xi(t,j))}\right) \\
&> -4r + 2rh\varphi(\xi_0) - k - rh + rh\psi(\xi_0) \\
&= \mathcal{N}_2(\xi_0) + d(e^{\lambda_0} + 1) - 2d \\
&> d(e^{\lambda_0} - 1) > 0.
\end{aligned}$$

We can obtain $A_w^2(t,j) \ge C_2 > 0$ by choosing a suitable C_2 small enough. This completes the proof. ∎

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Lemma 3.5 *Assume that* (H) *holds. For any* $c > \max\{c^*, \widetilde{c}\}$*, there exist some positive constants* C_i *such that* $B^i_{\mu,w}(t,j) \ge C_i$*,* $i = 1,2$*, for all* $t > 0$*,* $j \in \mathbb{Z}$*, and* $0 < \mu < \frac{\min_{i=1,2}\{C_i\}}{2}$.

Proof The proof can be easily obtained by Lemma 3.4, so we omit it here. ∎

Next we will give the energy estimates.

Lemma 3.6 *Assume that* (H) *holds and* $0 < k < 1 < h$*. For any* $c > \max\{c^*, \widetilde{c}\}$*, it holds*

$$\|U_j(t)\|^2_{l^2_w} + \|V_j(t)\|^2_{l^2_w} + \int_0^t e^{-2\mu(t-s)}\big(\|U_j(s)\|^2_{l^2_w} + \|V_j(s)\|^2_{l^2_w}\big)\,ds \le Ce^{-2\mu t}\big(\|U_{j0}(0)\|^2_{l^2_w} + \|V_{j0}(0)\|^2_{l^2_w}\big) \tag{3.5}$$

for some positive constant C.

Proof Multiplying (3.3) by $e^{2\mu t}w(\xi(t,j))U_j(t)$ and $e^{2\mu t}w(\xi(t,j))V_j(t)$, respectively, where $\mu > 0$ is defined in Lemma 3.5, we obtain

$$\begin{aligned}
&\Big(\frac{1}{2}e^{2\mu t}w(\xi(t,j))U_j^2(t)\Big)_t - e^{2\mu t}w(\xi(t,j))U_j(t)(U_{j+1}(t) + U_{j-1}(t))\\
&\qquad + \Big(2 - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - \mu - 1 + k + 2\varphi(\xi(t,j)) - kV_j(t) - k\psi(\xi(t,j))\Big)\\
&\qquad\quad \times e^{2\mu t}w(\xi(t,j))U_j^2(t)\\
&= -e^{2\mu t}w(\xi(t,j))U_j^3(t) + k\varphi(\xi(t,j))e^{2\mu t}w(\xi(t,j))U_j(t)V_j(t)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\Big(\frac{1}{2}e^{2\mu t}w(\xi(t,j))V_j^2(t)\Big)_t - de^{2\mu t}w(\xi(t,j))V_j(t)(V_{j+1}(t) + V_{j-1}(t))\\
&\qquad + \Big(2d - \frac{c}{2}\frac{w'_\xi(\xi(t,j))}{w(\xi(t,j))} - \mu - 2r\psi(\xi(t,j)) + r + rhU_j(t) + rh\varphi(\xi(t,j))\Big)\\
&\qquad\quad \times e^{2\mu t}w(\xi(t,j))V_j^2(t)\\
&= re^{2\mu t}w(\xi(t,j))V_j^3(t) + rh(1 - \psi(\xi(t,j)))e^{2\mu t}w(\xi(t,j))U_j(t)V_j(t).
\end{aligned} \tag{3.7}$$

By the Cauchy–Schwarz inequality $2ab \le a^2 + b^2$, we obtain

$$2U_{j+1}(t)U_j(t) \le U_{j+1}^2(t) + U_j^2(t), \quad 2V_{j+1}(t)V_j(t) \le V_{j+1}^2(t) + V_j^2(t).$$

Summing about all $j \in \mathbb{R}$ for (3.6) and (3.7), then integrating over $[0,t]$, yields

$$
\begin{aligned}
(3.8)\quad & e^{2\mu t}\|U_j(t)\|_{l_w^2}^2 \\
&+ \int_0^t \sum_j \Big[2\Big(2 - \frac{c}{2}\frac{w'_\xi(\xi(s,j))}{w(\xi(s,j))} - \mu - 1 + k + 2\varphi(\xi(s,j)) \\
&\qquad - kV_j(s) - k\psi(\xi(s,j))\Big) - \frac{w(\xi(s,j+1))}{w(\xi(s,j))} - \frac{w(\xi(s,j-1))}{w(\xi(s,j))} - 2\Big] \\
&\qquad \times e^{2\mu s} w(\xi(s,j)) U_j^2(s)\, ds \\
&\le \|U_{j0}(0)\|_{l_w^2}^2 + k\int_0^t \sum_j \varphi(\xi(s,j)) e^{2\mu s} w(\xi(s,j))(U_j^2(s) + V_j^2(s))\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\quad & e^{2\mu t}\|V_j(t)\|_{l_w^2}^2 \\
&+ \int_0^t \sum_j \Big[2\Big(2d - \frac{c}{2}\frac{w'_\xi(\xi(s,j))}{w(\xi(s,j))} - \mu - 2r\psi(\xi(s,j)) + r + rhU_j(s) \\
&\qquad + rh\varphi(\xi(s,j))\Big) - d\frac{w(\xi(s,j+1))}{w(\xi(s,j))} - d\frac{w(\xi(s,j-1))}{w(\xi(s,j))} - 2d\Big] \\
&\qquad \times e^{2\mu s} w(\xi(s,j)) V_j^2(s)\, ds \\
&\le \|V_{j0}(0)\|_{l_w^2}^2 + 2r\int_0^t \sum_j e^{2\mu s} w(\xi(s,j)) V_j(s) V_j^2(s)\, ds \\
&\quad + \int_0^t \sum_j rh(1 - \psi(\xi(s,j))) e^{2\mu s} w(\xi(s,j))(U_j^2(s) + V_j^2(s))\, ds.
\end{aligned}
$$

Adding the two inequalities (3.8) and (3.9), we have

$$
\begin{aligned}
& e^{2\mu t}\big(\|U_j(t)\|_{l_w^2}^2 + \|V_j(t)\|_{l_w^2}^2\big) \\
&\quad + \int_0^t \sum_j e^{2\mu s}\big(B_{\mu,w}^1(s,j)U_j^2(s) + B_{\mu,w}^2(s,j)V_j^2(s)\big) w(\xi(s,j))\, ds \\
&\qquad\qquad \le \|U_{j0}(0)\|_{l_w^2}^2 + \|V_{j0}(0)\|_{l_w^2}^2,
\end{aligned}
$$

where $B_{\mu,w}^1(t,j)$ and $B_{\mu,w}^2(t,j)$ are defined in (3.4). According to Lemma 3.5, we can obtain (3.5), *i.e.*,

$$
\begin{aligned}
& \|U_j(t)\|_{l_w^2}^2 + \|V_j(t)\|_{l_w^2}^2 + \int_0^t e^{-2\mu(t-s)}\big(\|U_j(s)\|_{l_w^2}^2 + \|V_j(s)\|_{l_w^2}^2\big)\, ds \\
&\qquad \le Ce^{-2\mu t}\big(\|U_{j0}(0)\|_{l_w^2}^2 + \|V_{j0}(0)\|_{l_w^2}^2\big)
\end{aligned}
$$

for some positive constant C. This completes the proof. ∎

Proof of Theorem 2.2 By Sobolev's embedding inequality, $l^2 \hookrightarrow l^\infty$ and $\|\cdot\|_{l^2} \leq \|\cdot\|_{l^2_w}$ due to $w(\xi) \geq 1$ defined by (2.5), one has

$$\sup_{j\in\mathbb{Z}} |U_j(t)| \leq C\|U_j(t)\|^2_{l^2} \leq C\|U_j(t)\|^2_{l^2_w},$$
$$\sup_{j\in\mathbb{Z}} |V_j(t)| \leq C\|V_j(t)\|^2_{l^2} \leq C\|V_j(t)\|^2_{l^2_w}.$$

Then we obtain

$$\sup_{j\in\mathbb{Z}} |u_j^+(t) - \varphi(j+ct)| = \sup_{j\in\mathbb{Z}} |U_j(t)| \leq Ce^{-\mu t},$$
$$\sup_{j\in\mathbb{Z}} |v_j^+(t) - \psi(j+ct)| = \sup_{j\in\mathbb{Z}} |V_j(t)| \leq Ce^{-\mu t},$$

where $t > 0$. Similarly, we can also have

$$\sup_{j\in\mathbb{Z}} |u_j^-(t) - \varphi(j+ct)| = \sup_{j\in\mathbb{Z}} |U_j(t)| \leq Ce^{-\mu t},$$
$$\sup_{j\in\mathbb{Z}} |v_j^-(t) - \psi(j+ct)| = \sup_{j\in\mathbb{Z}} |V_j(t)| \leq Ce^{-\mu t}.$$

Thus, in view of the squeezing technique, we have

$$\sup_{j\in\mathbb{Z}} |u_j(t) - \varphi(j+ct)| \leq Ce^{-\mu t}, \quad t > 0,$$
$$\sup_{j\in\mathbb{Z}} |v_j(t) - \psi(j+ct)| \leq Ce^{-\mu t}, \quad t > 0.$$

This completes the proof of Theorem 2.2. ■

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