SOME RESULTS CONCERNING THE STRUCTURE OF GRAPHS

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1. Introduction and terminology. The object of this paper is to present results concerning the structure of 3-connected graphs and of 5-chromatic and 6-chromatic graphs and also a theorem on contraction and a theorem of Turán type. The Axiom of Choice is assumed.

A graph Γ: a set $V(\Gamma)$ whose elements are called the vertices of the graph; with each pair of distinct vertices a and b there is associated a set $e(a, b, \Gamma)$ $(=e(b, a, \Gamma)$, $e(a, b, \Gamma) \cap V(\Gamma) = \emptyset)$ whose elements are called the edges joining a and b, $e(a, b, \Gamma) \cap e(a', b', \Gamma) = \emptyset$ if $\{a, b\} \neq \{a', b'\}$; the union of all the sets $e(a, b, \Gamma)$ is denoted by $E(\Gamma)$, and $\Gamma = V(\Gamma) \cup E(\Gamma)$. (a, b) denotes an element of $e(a, b, \Gamma)$. If $|e(a, b, \Gamma)| \leq 1$ for all $a, b \in V(\Gamma)$ then the graph contains no multiple edges. If Γ and Γ' are graphs and $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$ then Γ' is called a subgraph of Γ, $\Gamma' \subseteq \Gamma$; if in addition $V(\Gamma') \neq V(\Gamma)$ or $E(\Gamma') \neq E(\Gamma)$ then Γ' will be called a proper subgraph of Γ, $\Gamma' \subset \Gamma$. A planar graph is a graph which corresponds to a line complex imbedded in the plane without intersection of lines. If $W \subseteq V(\Gamma)$ then $\Gamma - W$ will denote the graph obtained from Γ by deleting all vertices of W and all edges incident with one or two vertices of W. The valency of a vertex is the number of edges incident with the vertex.

A graph Γ will be called λ-connected, where λ is an integer ≥ 1, if any two vertices a and b of Γ are connected by a set of λ (or more) paths of Γ, any two of which have no vertex other than a and b and no edge in common. (A λ-connected graph is also μ-connected for $1 \leq \mu < \lambda$.) For graphs without multiple edges this property is by Menger's theorem equivalent to the following: Γ is connected, $|V(\Gamma)| \geq \lambda+1$, and

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$\forall W[W \subset V(\Gamma)$ and $|W| < \lambda .\Rightarrow. \Gamma - W$ is connected] [1].

If $A, B \subset \Gamma$, $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$ then a path which has one end-vertex in A and the other in B and has no other vertex in common with $A \cup B$ will be called an (A) (B)-path.

$\Gamma(W)$, where $W \subseteq V(\Gamma)$, will denote $\Gamma - (\Gamma - W)$, that is to say the subgraph of Γ generated or spanned by the vertices of W.

$((x_1, x_2, \ldots, x_i))$, where $i \geq 2$, will denote a circuit whose vertices in cyclic order are $x_1, x_2, \ldots, x_i$. The length of the circuit is i.

If Y is a path and p, q are vertices of Y then $Y[p, q]$ $(= Y[q, p])$ will denote that part of Y which has p and q as its two end-vertices; $Y[p, p] = p$.

A complete k-graph or $\langle k \rangle$ will denote a graph with k (≥ 1) vertices in which each pair of distinct vertices are joined by exactly one edge, a $\langle 1 \rangle$ is a single vertex. A $\langle k- \rangle$ will denote a $\langle k \rangle$ with exactly one edge missing.

A wheel will denote a graph which consists of a circuit together with a vertex not belonging to the circuit and joined to each vertex of the circuit by at least one edge.

K or $K(x_1, x_2, x_3; y_1, y_2, \ldots, y_i)$, where $i \geq 3$, will denote a graph with the $i + 3$ vertices $x_1, x_2, x_3, y_1, \ldots, y_i$ in which x_1, x_2, x_3 are each joined to $y_1, \ldots, y_i$ by exactly one edge and there are no more edges. $K(x_1, x_2, x_3; y_1, \ldots, y_i)$ together with one or more edges joining x_1 and x_2 will be denoted by K_1 or $K_1(x_1, x_2, x_3; y_1, \ldots, y_i)$, $K_1(x_1, x_2, x_3; y_1, \ldots, y_i)$ together with one or more edges joining x_2 and x_3 will be denoted by K_2 or $K_2(x_1, x_2, x_3; y_1, \ldots, y_i)$, and $K_2(x_1, x_2, x_3; y_1, \ldots, y_i)$ together with one or more edges

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joining x_3 and x_1 by K_3 or $K_3(x_1, x_2, x_3; y_1, \ldots, y_i)$. A K with six vertices is called a *Kuratowski graph* and will be denoted by K^6; a K_i with six vertices will be denoted by K_i^6 for $i = 1, 2, 3$.

A *prism-graph* P or $P(x_1, x_2, x_3, y_1, y_2, y_3)$ will denote a graph consisting of the two disjoint circuits $((x_1, x_2, x_3))$ and $((y_1, y_2, y_3))$ together with the edges (x_1, y_1), (x_2, y_2), (x_3, y_3). $((x_1, x_2, x_3))$ and $((y_1, y_2, y_3))$ will be called the *ends* of P.

If Γ is a graph then ΓU will denote a graph obtained from Γ through the process of subdividing edges by inserting new vertices having valency 2; the vertices having valency ≥ 3 in ΓU will be called *branch-vertices.* For convenience it will be assumed that $\Gamma U \neq \Gamma$. <kU> will denote a graph obtained from a <k> by this process, $KU(x_1, x_2, x_3; y_1, \ldots, y_i)$ a graph so obtained from $K(x_1, x_2, x_3; y_1, \ldots, y_i)$. A path of ΓU connecting two branch-vertices will be called a *rib.*

If $\Gamma_1, \ldots, \Gamma_n$ are mutually disjoint connected graphs each of which contains at least three vertices, then any graph constructed from them by the following procedure will be called a cockade composed of $\Gamma_1, \ldots, \Gamma_n$: an edge (a_1, b_1) of Γ_1 and an edge (a_2, b_2) of Γ_2 are selected, a_1 is identified with a_2, b_1 with b_2, and (a_1, b_1) with (a_2, b_2); if $n > 2$ an edge (a, b) of the resulting graph and an edge (a_3, b_3) of Γ_3 are selected, a is identified with a_3, b with b_3, and (a, b) with (a_3, b_3); and so on with $\Gamma_4, \ldots, \Gamma_n$.

2. *The 3-connected graphs which do not contain two disjoint circuits.* Recently K. Corrádi and A. Hajnal have proved that if a finite graph without multiple edges has at least 3k vertices and each vertex has valency $\geq 2k$, where k is an integer ≥ 1, then the graph contains k (or more) mutually

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disjoint circuits [2]. P. Erdős and the writer have proved that (a) if the number of vertices is at least 6, all vertices have valency ≥ 3, and at least four vertices have valency ≥ 4, then the graph contains two disjoint circuits, (b) for $k \geq 3$, if the number of vertices having valency $\geq 2k$ exceeds the number having valency $\leq 2k - 2$ by at least $k^2 + 2k - 4$, then the graph contains k (or more) mutually disjoint circuits [3]. These results suggest the question, — which (2k-1)-connected graphs do not contain k mutually disjoint circuits? This question will here be answered for the case $k = 2$, graphs with multiple edges being allowed.

THEOREM 1. The only 3-connected graphs with at least four vertices which do not contain two disjoint circuits are the <4>'s, the <4>'s with additional edges which are either all incident with the same vertex, or with two of three vertices (so that the fourth vertex has valency 3), the <5>'s, the <5->'s, the <5->'s with additional edges joining vertices having valency 4 in the <5->, the wheels, the K's, the K_1's, the K_2's and the K_3's.

Proof. The theorem can easily be verified for graphs having fewer than six vertices. The proof for graphs with at least six vertices follows.

The following result is a special case of an extension of Menger's Theorem proved by the writer [4]:

If Γ is a λ-connected graph, $a \in V(\Gamma)$, $A \subset \Gamma$, $a \notin A$ and A contains at least λ vertices, then Γ contains λ (a)(A)-paths any two of which have only a in common. ...(1)

The next step is to prove that

If a 3-connected graph contains a K^6U then it contains two disjoint circuits. ...(2)

Proof. Let Γ denote the graph and $K_0 = K^6U(x_1, x_2, x_3; y_1, y_2, y_3)$ a K^6U contained in Γ. For $1 \leq i, j \leq 3$ let L_{ij} denote the rib of K_0 which connects x_i and y_j, the notation

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being chosen so that L_{11} contains more than two vertices, and let a denote a vertex in $L_{11}-x_1-y_1$. By (1) Γ contains at least three $(a)(K_0-a)$-paths any two of which have only a in common, so it contains one to which neither of the two neighbours of a in K_0 belong, L say. Let b denote the end-vertex of L other than a. There are four alternatives: (i) $b \in L_{11}$, (ii) $b \notin L_{11}$, b is a branch vertex of K_0, (iii) $b \notin L_{11}$, b is an intermediate vertex of a rib of K_0 incident with x_1 or with y_1, (iv) b is an intermediate vertex of a rib of K_0 incident neither with x_1 nor with y_1. These alternatives will be considered in turn.

If (i) is the case then Γ contains the two disjoint circuits $L \cup L_{11}[a,b]$ and $L_{22} \cup L_{32} \cup L_{33} \cup L_{23}$. If (ii) is the case then it may be assumed that $b = x_2$, in which case Γ contains the two disjoint circuits $L \cup L_{21} \cup L_{11}[a,y_1]$ and $L_{12} \cup L_{32} \cup L_{33} \cup L_{13}$. If (iii) is the case then it may be assumed that $b \in L_{12}$, in which case Γ contains the two disjoint circuits $L \cup L_{12}[x_1,b] \cup L_{11}[x_1,a]$ and $L_{22} \cup L_{32} \cup L_{33} \cup L_{23}$. If (iv) is the case then it may be assumed that $b \in L_{22}$, in which case Γ contains the two independent circuits $L \cup L_{22}[x_2,b] \cup L_{21} \cup L_{11}[y_1,a]$ and $L_{12} \cup L_{32} \cup L_{33} \cup L_{13}$.

So in each case Γ contains two disjoint circuits. This proves (2).

If a 3-connected graph contains a <5U> then it contains two disjoint circuits. ...(3)

Proof. Let Γ denote the graph, let Ω denote a <5U> contained in Γ whose branch vertices are w_1, w_2, w_3, w_4, w_5 and for $1 \leq i \neq j \leq 5$ let W_{ij} $(=W_{ji})$ denote the rib of Ω

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connecting w_i and w_j, the notation being chosen so that W_{12} contains more than two vertices, and let a denote an intermediate vertex of W_{12}. By (1) Γ contains at least three $(a)(\Omega - a)$-paths any two of which have only a in common, so it contains one to which neither of the two neighbours of a in Ω belong, W say. Let b denote the end-vertex of W different from a. There are four alternatives: (i) $b \in W_{12}$, (ii) $b \notin W_{12}$, b is a branch-vertex of Ω, (iii) $b \notin W_{12}$, b is an intermediate vertex of a rib of Ω incident with w_1 or with w_2, (iv) b is an intermediate vertex of a rib of Ω incident neither with w_1 nor with w_2. These four alternatives will be considered in turn.

If (i) is the case then Γ contains the two disjoint circuits $W \cup W_{12}[a, b]$ and $W_{34} \cup W_{45} \cup W_{53}$. If (ii) is the case then it may be assumed that $b = w_3$, in which case Γ contains the two disjoint circuits $W \cup W_{23} \cup W_{12}[a, w_2]$ and $W_{14} \cup W_{45} \cup W_{51}$. If (iii) is the case then it may be assumed that $b \in W_{13}$, in which case Γ contains the two disjoint circuits $W \cup W_{13}[b, w_1] \cup W_{12}[w_1, a]$ and $W_{34} \cup W_{45} \cup W_{53}$. If (iv) is the case then it may be assumed that $b \in W_{34}$, in which case Γ contains the two disjoint circuits $W \cup W_{34}[b, w_3] \cup W_{23} \cup W_{12}[a, w_2]$ and $W_{14} \cup W_{45} \cup W_{51}$.

So in each case Γ contains two disjoint circuits, and (3) is proved.

If a 3-connected graph contains at least six vertices and a <5> then it contains two disjoint circuits. ...(4)

Proof. Let Γ denote the graph, let Φ denote a <5> with vertices $f_1, f_2, \ldots, f_5$ contained in Γ, and let f denote a vertex of Γ which does not belong to Φ. By (1) Γ contains three $(f)(\Phi)$-paths any two of which have only f in common, let F_1, F_2, F_3 denote three such paths, the notation being

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chosen so that f_1, f_2, f_3, respectively, are their end-vertices. Then Γ contains the two disjoint circuits $F_1 \cup F_2 \cup (f_1, f_2)$ and $((f_3, f_4, f_5))$. This proves (4).

The only planar 3-connected graphs with more than five vertices which do not contain two disjoint circuits are the wheels. ...(5)

Proof. Let Γ denote a planar 3-connected graph with more than five vertices which does not contain two disjoint circuits.

Γ contains a <4U>. For let a denote a vertex of Γ, Γ-a contains at least five vertices, by (4) Γ does not contain a <5>, therefore Γ-a contains two vertices not joined by an edge, b and c say. Γ-a is 2-connected because Γ is 3-connected, so Γ-a contains two (b)(c)-paths which have only b and c in common. Two such paths together constitute a circuit C with at least four vertices. By (1) Γ contains three (a)(C)-paths any two of which have only a in common. These and C together constitute a <4U>. (This is true whether Γ is planar or not.)

Suppose that Θ is a <4U> contained in Γ. If $g \in V(\Gamma)$ and $g \notin \Theta$ then by (1) Γ contains three $(g)(\Theta)$-paths any two of which have only g in common. if G_1, G_2 and G_3 are any three such $(g)(\Theta)$-paths, then the end-vertices of G_1, G_2, G_3 belonging to Θ are branch-vertices of Θ. For let f_1, f_2, f_3, f_4 denote the branch-vertices of Θ, F_{ij} the rib of Θ which connects f_i and f_j $(1 \leq i \neq j \leq 4,\ F_{ij} = F_{ji})$, g_1, g_2, g_3 the end-vertices of G_1, G_2, G_3 belonging to Θ, respectively, and assume that g and f_4 are separated by the circuit $F_{12} \cup F_{23} \cup F_{31}$. If at most one of g_1, g_2, g_3 is a branch-vertex of Θ then it may be supposed that $g_1 \in F_{12} - f_1 - f_2$ and $g_2 \in F_{23} - f_2 - f_3$, in which case Γ contains the two disjoint circuits $G_1 \cup G_2 \cup F_{12}[f_2, g_1] \cup F_{23}[f_2, g_2]$ and

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$F_{13} \cup F_{34} \cup F_{41}$ contrary to hypothesis. If two of g_1, g_2, g_3 are branch-vertices of Θ and one is not then it may be supposed that $f_1 = g_1$ and $f_2 = g_2$ and either $g_3 \in F_{12} - f_1 - f_2$ or $g_3 \in F_{23} - f_2 - f_3$. Then if $g_3 \in F_{12} - f_1 - f_2$ Γ contains the two disjoint circuits $G_1 \cup F_{12}[f_1, g_3] \cup G_3$ and $F_{23} \cup F_{34} \cup F_{42}$, and if $g_3 \in F_{23} - f_2 - f_3$ then Γ contains the two disjoint circuits $G_2 \cup F_{23}[f_2, g_3] \cup G_3$ and $F_{13} \cup F_{34} \cup F_{41}$, which is contrary to the hypothesis that Γ does not contain two disjoint circuits. So the three end-vertices of G_1, G_2, G_3 are branch-vertices of Θ.

Since Θ is a <4U> the notation may be chosen so that F_{12} contains at least one vertex besides f_1 and f_2, f_5 say. By (1) Γ contains three $(f_5)(\Theta - f_5)$-paths any two of which have only f_5 in common, so it contains one to which neither of the two neighbours of f_5 in Θ belong. Let F denote such a path and f the end-vertex of F different from f_5. $f = f_3$ or $f = f_4$, for otherwise, since $f \notin F_{34} - f_3 - f_4$ because Γ is planar, it may be assumed without loss of generality that $f \in F_{12}$ or $f \in F_{13} - f_1 - f_3$; if $f \in F_{12}$ then it may be assumed without loss of generality that $f \neq f_1$, in which case Γ contains the two disjoint circuits $F \cup F_{12}[f_5, f]$ and $F_{13} \cup F_{34} \cup F_{41}$ contrary to hypothesis; if $f \in F_{13} - f_1 - f_3$ then Γ contains the two disjoint circuits $F \cup F_{13}[f_1, f] \cup F_{12}[f_1, f_5]$ and $F_{23} \cup F_{34} \cup F_{42}$ contrary to hypothesis. Therefore $f = f_3$ or $f = f_4$. Suppose that $f = f_4$. F contains only one edge. For suppose on the contrary that f' is an intermediate vertex of F. By (1) Γ contains three $(f')(\Theta \cup F - f')$ paths any two of which have only f' in common, so it contains one to which neither of the two neighbours of f' in F belong, E say. Let e denote the end-vertex of E different from f'. The following four alternatives have to be distinguished:

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(i) $e \in F$, $e \neq f_4$, (ii) $e = f_4$, (iii) $e \in F_{12}[f_1, f_5] - f_5$, (iv) $e \in F_{14} - f_1 - f_4$. If (i) holds then Γ contains the two disjoint circuits $E \cup F[e, f']$ and $F_{13} \cup F_{34} \cup F_{41}$. If (ii) holds then Γ contains the two disjoint circuits $E \cup F[e, f']$ and $F_{12} \cup F_{23} \cup F_{31}$. If (iii) holds then Γ contains the two disjoint circuits $E \cup F_{12}[e, f_5] \cup F[f', f_5]$ and $F_{23} \cup F_{34} \cup F_{42}$. If (iv) holds then Γ contains the two disjoint circuits $E \cup F_{14}[e, f_1] \cup F_{12}[f_1, f_5] \cup F[f', f_5]$ and $F_{23} \cup F_{34} \cup F_{42}$. But by hypothesis Γ does not contain two disjoint circuits, so F contains only one edge. By symmetry F_{14}, F_{24}, F_{34} contain only one edge.

Every vertex of Γ belongs to Θ. For suppose on the contrary that $c \in V(\Gamma)$ and $c \notin \Theta$. There are two alternatives to consider: (i) c and f_4 are separated by the circuit $F_{12} \cup F_{23} \cup F_{31}$, (ii) (i) is not the case and c and f_2 are separated by the circuit $F_{14} \cup F_{43} \cup F_{31}$. If (i) holds then Γ contains three $(c)(\Theta)$-paths G_1, G_2, G_3, any two of which have only c in common, and the notation can be chosen so that f_1, f_2, f_3, respectively, are their end-vertices. Γ then contains the two disjoint circuits $G_1 \cup G_3 \cup F_{13}$ and $F_{24} \cup F \cup F_{12}[f_2, f_5]$. If (ii) holds then Γ contains three $(c)(\Theta)$-paths G'_1, G'_3, G'_4 any two of which have only c in common, and the notation can be chosen so that f_1, f_3, f_4, respectively, are their end-vertices. Γ then contains the two disjoint circuits $G'_1 \cup G'_3 \cup F_{13}$ and $F_{24} \cup F \cup F_{12}[f_2, f_5]$. But by hypothesis Γ does not contain two disjoint circuits, so every vertex of Γ belongs to Θ.

Θ contains one or more vertices other than f_1, f_2, f_3, f_4, f_5 because $|V(\Gamma)| \geq 6$. They all belong to $F_{12} \cup F_{23} \cup F_{31}$ because F_{14}, F_{24}, F_{34} and F contain only one edge. Each

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of them is joined to f_4. For let f_6 denote such a vertex, it may be assumed that $f_6 \in F_{13}$, then f_6 is joined to f_4 or to f_2 by what was said above, but if $(f_2, f_6) \in \Gamma$ then Γ contains the two disjoint circuits $F_{13}[f_3, f_6] \cup F_{23} \cup (f_2, f_6)$ and $F_{12}[f_1, f_5] \cup F \cup F_{14}$ contrary to hypothesis, so $(f_2, f_6) \notin \Gamma$ and $(f_4, f_6) \in \Gamma$. Γ contains no edge which joins two vertices of $F_{12} \cup F_{23} \cup F_{31}$ but does not belong to $F_{12} \cup F_{23} \cup F_{31}$. For $F_{12} \cup F_{23} \cup F_{31}$ contains at least five vertices and each of them is joined to f_4, and Γ does not contain two disjoint circuits. (5) is now proved.

The graph which consists of a $K^6(x_1, x_2, x_3; y_1, y_2, y_3)$ together with an edge which does not belong to the K^6 and joins e.g. x_1 and y_1 contains the two disjoint circuits $((x_1, y_1))$ and $((x_2, y_2, x_3, y_3))$ ...(6)

$K^6(x_1, x_2, x_3; y_1, y_2, y_3) \cup (x_1, x_2) \cup (y_1, y_2)$ contains the two disjoint circuits $((x_1, x_2, y_3))$ and $((y_1, y_2, x_3))$ ...(7)

$K(x_1, x_2, x_3; y_1, y_2, y_3, y_4) \cup (y_1, y_4)$ contains the two disjoint circuits $((x_1, y_1, y_4))$ and $((x_2, y_2, x_3, y_3))$ ...(8)

If each of the vertices x_1, x_2, x_3, x_4 is joined to each of the vertices y_1, y_2, y_3, y_4 then the graph contains the two disjoint circuits $((x_1, y_1, x_2, y_2))$ and $((x_3, y_3, x_4, y_4))$ ...(9)

To complete the proof of Theorem 1 let Γ denote a 3-connected graph which has at least six vertices and does not contain two disjoint circuits. If Γ is planar then Γ is a wheel by (5). Suppose that Γ is not planar. Then by Kuratowski's theorem and (2), (3) and (4) Γ contains a $K^6 = K^6(x_1, x_2, x_3; y_1, y_2, y_3)$, where by (7) the notation can be chosen so that no two of y_1, y_2, y_3 are joined by an edge. If $|V(\Gamma)| = 6$ then it follows from (6) that Γ is a K^6 or a K_1^6 or a K_2^6 or a K_3^6.

Suppose that $|V(\Gamma)| \geq 7$ and let z denote a vertex of Γ different from $x_1, x_2, x_3, y_1, y_2, y_3$. By (1) Γ contains three $(z)(K^6)$-paths Z_1, Z_2, Z_3, any two of which have only z in common. If e.g. Z_1 has x_1 and Z_2 has y_1 as end-vertex then $(K^6 - (x_1, y_1)) \cup Z_1 \cup Z_2$ is a K^6U, which is not the case by (2); therefore, since no two of y_1, y_2, y_3 are joined by an edge, it may be assumed that Z_1, Z_2, Z_3 have x_1, x_2, x_3 respectively as end-vertices. Z_1, Z_2, Z_3 contain only one edge each, for otherwise Γ would contain a $K^6U(x_1, x_2, x_3, y_1, y_2, z)$ contrary to (2). By (8) $e(y_1, z, \Gamma) = e(y_2, z, \Gamma) = e(y_3, z, \Gamma) = \emptyset$. So if $|V(\Gamma)| = 7$ then $\Gamma = K(x_1, x_2, x_3; y_1, y_2, y_3, z)$ or $\Gamma = K_1(x_1, x_2, x_3; y_1, y_2, y_3, z)$ or $\Gamma = K_2(x_1, x_2, x_3; y_1, y_2, y_3, z)$ or $\Gamma = K_3(x_1, x_2, x_3; y_1, y_2, y_3, z)$. If $|V(\Gamma)| \geq 8$ then let u denote any vertex of Γ different from $x_1, x_2, x_3, y_1, y_2, y_3, z$. By what has just been said (with u in place of z) and by (6), (7), (8) and (9) u is joined to each of x_1, x_2, x_3 by exactly one edge and u is not joined to y_1, y_2, y_3, z. It follows by (6), (8) and (9) that Γ is a K or a K_1 or a K_2 or a K_3. The proof of Theorem 1 is now complete.

3. A property of λ-connected graphs. The following two theorems clarify the structure of λ-connected graphs with more than λ vertices.

THEOREM 2. If Γ is a λ-connected graph with more than λ vertices and with some multiple edges then the graph without multiple edges obtained from Γ by deleting all but one of each set of multiple edges is λ-connected.

Theorem 2 is clearly equivalent to

THEOREM 2'. If Γ is a λ-connected graph with more than λ vertices then any two vertices a and b of λ are connected by λ $(a)(b)$-paths contained in Γ and such that any

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two of them have no vertex other than a and b and no edge in common and all or all but one contain more than one edge.

Note concerning Theorem 2'. If $|e(a, b, \Gamma)| \leq 1$ then the existence of such a set of paths in Γ follows from the definition of λ-connectedness.

Proof of Theorem 2'. Each vertex of Γ is joined to at least λ different vertices of Γ. For let x denote a vertex of Γ. If x is joined to all the other vertices then the assertion is true because $|V(\Gamma)| \geq \lambda + 1$; if x is not joined to the vertex y then Γ contains λ $(x)(y)$-paths any two of which have only x and y in common and each of which contains three or more vertices, so again the assertion is true. It follows that a is joined to at least $\lambda - 1$ vertices other than b. Let $a_1, \ldots, a_{\lambda-1}$ denote vertices different from b to which a is joined. By (1) Γ contains λ $(b)(\{a, a_1, \ldots, a_{\lambda-1}\})$-paths any two of which have only b in common. These together with a and the edges $(a, a_1), (a, a_2), \ldots, (a, a_{\lambda-1})$ constitute λ $(a)(b)$-paths with the required properties.

4. P-s and PU-s in graphs.

THEOREM 3. If a 3-connected graph has at least six vertices and is neither a K nor a K_1 nor a K_2 nor a K_3 nor a wheel nor obtainable from a K, K_1, K_2, K_3 or a wheel by duplicating edges already present, then corresponding to any two vertices of the graph there is a P or a PU contained in the graph such that either both the vertices belong to the same end of the P or PU, or one belongs to one end and one to the other.

Proof. Let a and b denote two arbitrary vertices of the graph, and if the graph has no multiple edges then let Γ denote the graph while if the graph has multiple edges then let Γ denote the graph without multiple edges obtained from it by deleting all but one of each set of multiple edges; by Theorem 2 Γ is a 3-connected graph without multiple edges. Γ is not isomorphic to any of the graphs mentioned in Theorem 1, therefore by Theorem 1 Γ contains two disjoint circuits,

C_1 and C_2 say, each of which contains three or more vertices because Γ contains no multiple edges.

Γ contains two disjoint circuits whose union contains a and b. Proof: to see that Γ contains two disjoint circuits whose union contains at least one of a, b suppose that $a, b \notin C_1, C_2$. By (1) Γ contains three $(a)(C_1 \cup C_2)$-paths Y_1, Y_2, Y_3 any two of which have only a in common; let y_1, y_2, y_3, respectively, denote their end-vertices other than a. The notation can be chosen so that $y_1, y_2 \in C_1$. Let the union of Y_1 and Y_2 and one of the two arcs of C_1 connecting y_1 and y_2 be denoted by C'_1. C'_1 is a circuit containing a, and $C'_1 \cap C_2 = \emptyset$. To see that Γ contains two disjoint circuits whose union contains a and b suppose that $b \notin C'_1 \cup C_2$. By (1) Γ contains three $(b)(C'_1 \cup C_2)$-paths Z_1, Z_2, Z_3 any two of which have only b in common; let z_1, z_2, z_3, respectively, denote their end-vertices other than b. If at least two of z_1, z_2, z_3 belong to C_2 then let the notation be chosen so that $z_1, z_2 \in C_2$. Let the union of Z_1 and Z_2 and one of the two arcs of C_2 connecting z_1 and z_2 be denoted by C'_2. C'_2 is a circuit containing b, and $C'_1 \cap C'_2 = \emptyset$, so C'_1 and C'_2 are two disjoint circuits in Γ whose union contains a and b. The remaining alternative is that at least two of z_1, z_2, z_3 belong to C'_1. In that case let the notation be chosen so that $z_1, z_2 \in C'_1$, and let the union of Z_1 and Z_2 and an arc of C'_1 connecting z_1 and z_2 and containing a be denoted by C''_1. C''_1 is a circuit containing a and b, and $C''_1 \cap C_2 = \emptyset$. The assertion is thereby proved.

The following result is a special case of an extension of Menger's Theorem [4]:

If Γ is a λ-connected graph and $A \subset \Gamma$, $B \subset \Gamma$,

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$|V(A)| \geq \lambda$, $|V(B)| \geq \lambda$ and $A \cap B = \emptyset$ then Γ contains λ or more mutually disjoint (A)(B)-paths. ...(10)

To complete the proof of Theorem 3, let C and C' be two disjoint circuits contained in Γ such that $a, b \in C \cup C'$. $|V(C)| \geq 3$ and $|V(C')| \geq 3$ because Γ contains no multiple edges. Therefore by (10) with $\lambda = 3$, Γ contains three mutually disjoint (C)(C')-paths. These C and C' together constitute a P or PU with C and C' as its two ends. Theorem 3 is thereby proved.

THEOREM 4. A finite graph with at least three vertices which contains neither a P nor a PU is either a <3>, <4>, <5>, K_3 or wheel, or a <3>, <4>, <5>, K_3 or wheel with some or all edges duplicated an arbitrary number of times, or a cockade composed of such graphs, or else it can be obtained from a graph coming under one of these categories by deleting edges.

Proof by induction over the number of vertices. The theorem is obviously true for graphs with fewer than six vertices. Let Γ denote a finite graph with at least six vertices which contains neither a P nor a PU, and suppose that the theorem is true for graphs which have fewer vertices than Γ. Let Γ^+ denote a graph with the following properties: $V(\Gamma^+) = V(\Gamma)$, $\Gamma \subseteq \Gamma^+$, Γ^+ contains neither a P nor a PU, if a and b are any two vertices of Γ^+ not joined by an edge in Γ^+ then $\Gamma^+ \cup (a, b)$ contains a P or a PU; $\Gamma^+ = \Gamma$ possibly, and if $\Gamma^+ \neq \Gamma$ then Γ can be obtained from Γ^+ by deleting edges.

Γ^+ is 2-connected. Proof (by reductio ad absurdum): Suppose that Γ^+ is not 2-connected. Γ^+ is obviously connected. Therefore, since $|V(\Gamma)| \geq 6$, by Theorem 2 and Menger's Theorem Γ^+ contains a cut-vertex, c say. It follows that $\Gamma^+ = \Gamma' \cup \Gamma''$, where $|V(\Gamma')| \geq 2$ and $|V(\Gamma'')| \geq 2$ and $\Gamma' \cap \Gamma'' = c$. Let a' and a'', respectively, denote vertices of Γ' and Γ'' joined to c. $(a', a'') \notin \Gamma^+$, therefore

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$\Gamma^+ \cup (a', a'')$ contains a P or a PU, P_o say, to which (a', a'') belongs. The branch vertices of P_o either all belong to Γ' or all belong to Γ'' because a P is 3-connected. Suppose that the branch-vertices of P_o all belong to Γ'. Then one of the ribs of P_o includes (a', a'') and an $(a'')(c)$-path Y belonging to Γ''. It follows that $(a', c) \notin P_o$, because if two ribs join the same pair of branch-vertices then at least one of them passes through a third branch-vertex but all branch-vertices of P_o belong to Γ'. Since $(a', c) \notin P_o$ and all branch-vertices of P_o belong to Γ', the graph obtained from P_o through replacing $(a', a'') \cup Y$ by (a', c) is a P or a PU contained in Γ', and therefore in Γ^+. This contradicts the definition of Γ^+, therefore Γ^+ is 2-connected.

If Γ^+ is 3-connected then by Theorem 3 Γ^+ is either a <4>, <5>, K_3 or wheel, or a <4>, <5>, K_3 or wheel with some or all edges duplicated, so Theorem 4 is true for Γ. Suppose in what follows that Γ^+ is not 3-connected. Then by Theorem 2 and Menger's Theorem Γ^+ contains two vertices a and b such that $\Gamma^+ - a - b$ is disconnected. Since Γ^+ is 2-connected it follows that $\Gamma^+ = \Gamma_1 \cup \Gamma_2$, where $|V(\Gamma_1)| \geq 3$, $|V(\Gamma_2)| \geq 3$, and $V(\Gamma_1 \cap \Gamma_2) = \{a, b\}$. a and b are joined by at least one edge in Γ^+. Proof (by reductio ad absurdum): If $(a, b) \notin \Gamma^+$ then $\Gamma^+ \cup (a, b)$ contains a P or a PU, P_1 say, to which (a, b) belongs. The branch-vertices of P_1 either all belong to Γ_1 or all belong to Γ_2 because a P is 3-connected. Suppose that the branch vertices of P_1 all belong to Γ_1. It follows that $P_1 \cap (\Gamma_2 - a - b) = \emptyset$ because $(a, b) \in P_1$. Let d denote a vertex of $\Gamma_2 - a - b (\neq \emptyset)$. By (1) Γ^+ contains an (a)(d)-path and a (b)(d)-path which have only d in common, let Z denote the union

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of these two paths. Since $P_1 \cap (\Gamma_2 - a - b) = \emptyset$ the graph obtained from P_1 through replacing (a, b) by Z is a PU contained in Γ^+. This contradicts the definition of Γ^+, so a and b are joined by at least one edge in Γ^+. It may therefore be assumed that $(a, b) \in \Gamma_1 \cap \Gamma_2$. From this and the induction hypothesis it follows that Theorem 4 is true for Γ. The theorem is therefore proved.

Remark concerning Theorem 4. Not every cockade composed of the graphs described in Theorem 4 has the property that if two independent vertices are joined by an edge then the resulting graph contains a P or a PU!

THEOREM 5. (a) If a planar graph with at least six vertices is 3-connected and is neither a wheel nor obtainable from a wheel by duplicating edges, then corresponding to any two vertices there is a P or PU contained in the graph such that the union of its two ends includes the two vertices.
(b) If a planar graph with at least six vertices has no multiple edges and triangulates the whole plane, then corresponding to any two circuits, in particular corresponding to any two disjoint <3>-s, there is a P or a PU contained in the graph which has the two circuits as its ends.

Proof. (a) follows from Theorem 3 because a K is not planar. (b) follows from (10) with $\lambda = 3$ provided the graph is 3-connected. Now a graph which triangulates the whole plane and contains at least six vertices is obviously 2-connected, and if it is not 3-connected then it contains a cut-set $\{a, b\}$, and since the graph triangulates the whole plane it follows that $|e(a, b)| \geq 2$, which is contrary to hypothesis; therefore Theorem 5 is proved.

5. A theorem of Turán type concerning K^6-s, K^6U-s, P-s and PU-s. The following theorem includes as a particular case the graphs obtained from planar graphs without multiple edges which triangulate the whole plane by adding an edge joining two non-neighbouring vertices.

THEOREM 6. If a graph without multiple edges has

$n \geq 6$ vertices and at least $3n-5$ edges then it contains a K^6 or a K^6U and a P or a PU, unless it is a cockade composed of <5>-s (such a cockade has exactly $3n-5$ edges).

Proof. A theorem of K. Wagner [5] states that any finite graph without multiple edges with at least three vertices which contains neither a K^6 nor a K^6U is either a <3,>, <4>, <5> or a graph with at least six vertices which triangulates the whole plane, or a cockade composed of such graphs, or else it can be obtained from a graph belonging to one of these categories by deleting edges. A <k> has less than $3k-5$ edges if $3 \leq k \leq 4$ and exactly $3k-5$ edges if $k = 5$, while a graph with $m(\geq 3)$ vertices which has no multiple edges and triangulates the whole plane has exactly $3m-6$ edges. A cockade with n vertices composed of such graphs contains at most $3n-6$ edges, unless the cockade is composed entirely of <5>-s, in which case the total number of edges is $3n-5$ — this can be proved very easily by induction over the number of graphs of which the cockade is composed. Therefore a graph which satisfies the conditions of Theorem 6 contains a K^6 or a K^6U unless it is a cockade composed of <5>-s.

By Theorem 4 any finite graph without multiple edges which contains at least three vertices and neither a P nor a PU is either a <3>, <4>, <5>, K_3 without multiple edges or wheel without multiple edges, or a cockade composed of such graphs, or else it can be obtained from a graph belonging to one of these categories by deleting edges. A wheel without multiple edges having $m(\geq 4)$ vertices contains exactly $2m-2$ edges, a K_3 without multiple edges having $m(\geq 6)$ vertices contains exactly $3m-6$ edges. A cockade with n (≥ 6) vertices composed of <3>-s, <4>-s, <5>-s, K_3-s without multiple edges and wheels without multiple edges contains at most $3n-6$ edges unless it is composed of <5>-s only. So by Theorem 4 a graph which satisfies the conditions of Theorem 6 contains a P or PU unless it is a cockade composed of <5>-s.

6. A theorem concerning homomorphism.

Definitions. The graph Γ can be contracted into the graph Δ if there exists a mapping ϕ of $V(\Gamma)$ onto $V(\Delta)$ such that 1. $(\forall x)[x \in V(\Delta) \Rightarrow \Gamma(\phi^{-1}(x))$ is connected],

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2. $(\forall x, x')\,[x, x' \in V(\Delta) \Rightarrow \Gamma$ contains $|e(x, x', \Delta)|\; (\phi^{-1}(x))$ $(\phi^{-1}(x'))$-edges]. The graph Γ is <u>homomorphic</u> to the graph Δ, for short Γ hom. Δ, if Γ can be contracted into a graph of which Δ is a subgraph. These definitions differ from the analogous definitions for graphs without multiple edges [6][7] in that multiple edges of Δ are here significant; if Δ contains no multiple edges then the present definition is equivalent to the definitions in [6] and [7].

The following is a generalisation of a result of K. Wagner [8].

THEOREM 7. If Δ is a subgraph of a graph into which the graph Γ is contracted by the mapping ϕ, and if Δ contains no vertex of valency > 3, then $\Gamma \supseteq \Delta'$ or $\Gamma \supseteq \Delta' U$, where there is an isomorphism I between Δ and Δ' such that for each vertex x of Δ $I(x) \in \phi^{-1}(x)$.

<u>Proof</u>. Let $\Gamma' = \bigcup_{x \in V(\Delta)} \Gamma(\phi^{-1}(x))$ and let Γ'' be a subgraph of Γ obtained by adding $|e(x, x', \Delta)|\; (\phi^{-1}(x))(\phi^{-1}(x'))$-edges of Γ to Γ' for all pairs $x, x' \in V(\Delta)$. Any vertex of $\phi^{-1}(x)$ which is joined to at least one vertex not in $\phi^{-1}(x)$ by one or more edges of Γ'' will be called a <u>clasp-vertex</u> of $\phi^{-1}(x)$. $\phi^{-1}(x)$ has at most three clasp-vertices because $v(x, \Delta) \leq 3$.

Let Γ''' be a subgraph of Γ'' obtained as follows: For each vertex x of Δ

(i) If $\phi^{-1}(x)$ contains only one clasp-vertex, $X(x)$ say, then every vertex of $\phi^{-1}(x)$ other than $X(x)$ is deleted from Γ''.

(ii) If $\phi^{-1}(x)$ contains two clasp-vertices then let these be $Y_1(x)$ and $Y_2(x)$, the notation being chosen so that $Y_2(x)$ is in Γ'' joined to one vertex only outside $\phi^{-1}(x)$; a $(Y_1(x))$ $(Y_2(x))$-path is selected in $\Gamma(\phi^{-1}(x))$, and all vertices of

$\phi^{-1}(x)$ which do not belong to this path are deleted from Γ''.

(iii) If $\phi^{-1}(x)$ contains three clasp-vertices then let them be $Z_1(x)$, $Z_2(x)$ and $Z_3(x)$. Either $\Gamma(\phi^{-1}(x))$ contains a path which joins two of them and passes through the third, or $\Gamma(\phi^{-1}(x))$ contains no such path. In the first case let the notation be chosen so that $Z_3(x)$ is an intermediate vertex of a path in $\Gamma(\phi^{-1}(x))$ joining $Z_1(x)$ and $Z_2(x)$; all the vertices of $\phi^{-1}(x)$ which do not belong to the path are deleted from Γ''. In the second case let $R(x)$ denote a $(Z_1(x))(Z_2(x))$-path and $S(x)$ an $(R(x))(Z_3(x))$-path contained in $\Gamma(\phi^{-1}(x))$ and let $Z(x)$ denote the vertex common to $R(x)$ and $S(x)$; all vertices of $\phi^{-1}(x)$ which belong neither to $R(x)$ nor to $S(x)$ are deleted from Γ''.

It is easy to see that Γ''' is isomorphic to Δ or to a ΔU, the vertex $X(x)$, $Y_1(x)$, $Z_3(x)$ or $Z(x)$ in Γ''', as the case may be, corresponding to the vertex x of Δ. This proves Theorem 7.

Note that Theorem 7 is true whether Δ is finite or infinite. The condition that Δ contains no vertex of valency ≥ 4 is essential, this is illustrated by the following very simple example: $V(\Delta) = \{x, y\}$, $|e(x, y, \Delta)| = 4$; $V(\Gamma) = \{x', y_1, y_2\}$, $|e(x', y_1, \Gamma)| = |e(x', y_2, \Gamma)| = 2$, $|e(y_1, y_2, \Gamma)| = 1$. Γ hom. Δ with $\phi(x') = x$, $\phi(y_1) = \phi(y_2) = y$, but Γ obviously does not contain a subgraph isomorphic to Δ or to a ΔU; other simple examples can easily be found, including ones in which Γ and Δ have no multiple edges.

7. Concerning the structure of 5-chromatic and 6-chromatic graphs.

Definitions. A graph is said to be k-colourable, k being a positive integer, if the vertices of the graph can be divided into k mutually disjoint (colour) classes in such a way that no two vertices in the same class are joined by an edge;

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such a partitioning of the vertices is called a k-colouring. A graph is said to have chromatic number k or to be k-chromatic if it is k-colourable and not (k-1)-colourable. A k-chromatic graph Γ is called vertex-critical if for each vertex a of Γ Γ-a is (k-1)-chromatic. A k-chromatic graph is called contraction-critical if it is connected and not homomorphic to any graph having fewer vertices and chromatic number $\geq k$. It is easy to see that if a graph is contraction-critical then it is vertex-critical. (Contraction-critical graphs are sometimes called irreducible graphs, particularly in the theory of 5-chromatic planar graphs.)

A theorem of de Bruijn and Erdős [9] states that if k is a positive integer and every finite subgraph of an infinite graph is k-colourable, then the whole graph is k-colourable. It follows that all vertex-critical k-chromatic graphs have a finite number of vertices and every k-chromatic graph contains a vertex-critical k-chromatic subgraph. It is easy to see that any vertex-critical k-chromatic graph is connected and contains no cut-vertex, and each of its vertices is joined to at least k-1 others.

The writer has proved elsewhere [10], [11], [12] that

If a vertex-critical k-chromatic graph contains an $<\ell>$, where $\ell < k$, then the graph is homomorphic to an $<\ell+1>$ ...(11)

Every contraction-critical k-chromatic graph with $k \geq 5$, other than a $<k>$, is 5-connected. ...(12)

Every 4-chromatic graph contains a $<4>$ or a $<4U>$. ...(13)

The following theorem is concerned with the case in which two vertices form a cut-set in a vertex-critical graph.

THEOREM 8. If Γ is a vertex-critical k-chromatic graph, where $k \geq 3$, and the two vertices p and q of Γ are such that Γ-p-q is disconnected, then $(p, q) \notin \Gamma$ and $\Gamma = \Gamma' \cup \Gamma''$ where $\Gamma' \cap \Gamma'' = \{p, q\}$, and the notation can be chosen so that

A. In every (k-1)-colouring of Γ' p and q have the same colour and in every (k-1)-colouring of Γ'' p and q have different colours.

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B. $\Gamma' \cup (p, q)$ is k-chromatic and vertex-critical.

C. The graph obtained from Γ'' by identifying p with q is k-chromatic and vertex-critical.

D. Γ-p-q consists of two connected components, both of which are joined to p and to q in Γ.

<u>Proof.</u> Let $\Gamma\text{-}p\text{-}q = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and let $\Gamma' = \Gamma - \Gamma_2$ and $\Gamma'' = \Gamma - \Gamma_1$. Then $\Gamma = \Gamma' \cup \Gamma''$ and $V(\Gamma' \cap \Gamma'') = \{p, q\}$. Γ' and Γ'' are both (k-1)-colourable because Γ is vertex-critical. Γ' and Γ'' can not be (k-1)-coloured with the same k-1 colours in such a way that the two colourings match over p and q, for Γ is not (k-1)-colourable. Therefore $(p, q) \notin \Gamma$ and the notation can be chosen so that in every (k-1)-colouring of Γ' p and q have the same colour and in every (k-1)-colouring of Γ'' the colour of p is different from the colour of q. This proves that $(p, q) \notin \Gamma$, $\Gamma' \cap \Gamma'' = \{p, q\}$, and A is true.

Hence $\Gamma' \cup (p, q)$ is k-chromatic. Let t denote an arbitrary vertex of $\Gamma' \cup (p, q)$, it will be shown that $\Gamma' \cup (p, q)\text{-}t$ is (k-1)-colourable. If $t = p$ or $t = q$ then $\Gamma \cup (p, q)\text{-}t \subset \Gamma - t$, and $\Gamma - t$ is (k-1)-colourable because Γ is vertex-critical. Suppose that $t \neq p$ and $t \neq q$. Γ-t is (k-1)-colourable, consequently since in every (k-1)-colouring of Γ'' the colour of p is different from the colour of q, $\Gamma' - t$ can be (k-1)-coloured in such a way that the colour of p is different from the colour of q, so $\Gamma' \cup (p, q)\text{-}t$ is (k-1)-colourable. This proves B.

Let Γ''' denote the graph obtained from Γ'' by identifying p with q, i.e. the graph obtained from Γ''-p-q by adjoining a vertex r not belonging to Γ and edges according to the rule $|e(r, x, \Gamma''')| = |e(p, x, \Gamma'')| + |e(q, x, \Gamma'')|$ for each vertex x of Γ''-p-q. Γ''-p-q is (k-1)-colourable. Γ''' is not (k-1)-colourable, because if it were then Γ'' could be (k-1)-coloured in such a way that p and q have the same colour by colouring $\Gamma''\text{-}p\text{-}q (= \Gamma''' - r)$ as it is coloured in a (k-1)-colouring of Γ''' and then giving p and q the colour of r. Hence Γ''' is k-chromatic. Let u denote an arbitrary vertex of Γ''', it will be shown that Γ'''-u is (k-1)-colourable. If u = r then

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$\Gamma''' - u = \Gamma'' - p - q$, and $\Gamma'' - p - q$ is (k-1)-colourable because Γ is vertex-critical. Suppose that $u \neq r$. $\Gamma - u$ is (k-1)-colourable, therefore since in every (k-1)-colouring of Γ' p and q have the same colour, $\Gamma'' - u$ can be coloured with k-1 colours in such a way that p and q have the same colour; a (k-1)-colouring of $\Gamma''' - u$ is obtained by giving r the colour which p and q have in such a (k-1)-colouring of $\Gamma'' - u$. This proves C.

Each connected component of $\Gamma - p - q$ is joined to p and to q because Γ is connected and contains no cut-vertex. If Γ_1 had more than one connected component then two vertices joined by an edge (namely p and q) would constitute a cut-set of the vertex-critical k-chromatic graph $\Gamma' \cup (p, q)$, but this contradicts what has already been proved; so Γ_1 is connected. If Γ_2 had more than one connected component then r would be a cut-vertex of Γ''', but Γ''' is vertex-critical and therefore contains no cut-vertex; so Γ_2 is connected. This proves D.

Note. A k-chromatic graph is called edge-critical if every proper subgraph is (k-1)-colourable. Theorem 8 remains true if 'vertex-critical' is everywhere replaced by 'edge-critical', the proof is practically the analogue of the above proof of Theorem 8.

THEOREM 9. Any vertex-critical 5-chromatic graph either contains a P or a PU, or else each edge of the graph belongs to some <5> or <5U> contained in the graph.

Proof (by induction over the number of vertices n): the theorem is clearly true for $n = 5$. Suppose that it is true for $5 \leq n \leq m-1$, where $m \geq 6$, and let Γ denote a vertex-critical 5-chromatic graph with m vertices. If Γ is 3-connected then it contains a P or a PU since by Theorem 3 all 3-connected graphs which contain neither a P nor a PU nor a <5> are 4-colourable ($\Gamma \not\supset$ <5> because Γ is 5-chromatic and vertex critical and $m \geq 6$). Suppose that Γ is not 3-connected. Then by Menger's Theorem Γ contains two vertices p and q such that $\Gamma - p - q$ is disconnected. In the notation of Theorem 8 $\Gamma' \cup (p, q)$ is 5-chromatic and vertex-critical; therefore by the induction hypothesis $\Gamma' \cup (p, q)$ either contains a P or a PU, or each edge of $\Gamma' \cup (p, q)$ belongs to some <5> or <5U> contained in $\Gamma' \cup (p, q)$. If

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$\Gamma' \cup (p,q)$ contains a P or a PU then so does Γ because (p,q) can be replaced by a $(p)(q)$-path contained in Γ''. (It follows at once from Theorem 8 D that Γ' and Γ'' are connected, therefore they contain $(p)(q)$-paths.) Suppose that $\Gamma' \cup (p,q)$ contains neither a P nor a PU, and let (a,b) denote any edge of Γ (a,b,p,q need not all be distinct). If $(a,b) \in \Gamma'$ then by the induction hypothesis $\Gamma' \cup (p,q)$ contains a $<5>$ or a $<5U>$ to which (a,b) belongs. It follows that Γ contains a $<5U>$ to which (a,b) belongs because (p,q) can be replaced by a $(p)(q)$-path contained in Γ''. Suppose that $(a,b) \notin \Gamma'$; then $(a,b) \in \Gamma''$. By the induction hypothesis $\Gamma' \cup (p,q)$ contains a $<5>$ or a $<5U>$ to which (p,q) belongs. If a,b,p,q are all distinct then, since Γ contains no cut-vertex, by (10) the notation can be chosen so that Γ'' contains an $(a)(p)$-path A and a $(b)(q)$-path B such that $A \cap B = \emptyset$. By replacing (p,q) with $A \cup B \cup (a,b)$ it is seen that Γ contains a $<5U>$ to which (a,b) belongs. There remains the alternative that $p = a$ and $b \in \Gamma''-p-q$. By Theorem 8 D $\Gamma''-p$ is connected, therefore $\Gamma''-p$ contains a $(b)(q)$-path, C say. By replacing (p,q) with $C \cup (a,b)$ it is seen that Γ contains a $<5U>$ to which (a,b) belongs.

Hence Theorem 9 is true for Γ, and therefore the theorem is true generally.

THEOREM 10. If Γ is any contraction-critical-5-chromatic graph other than a $<5>$ or a $<5>$ with some edges duplicated, and if a,b,c,d are any four vertices of Γ, then $\Gamma-a-b$ contains a P or PU whose two ends together include c and d.

Proof. It may be assumed that Γ contains no multiple edges, since replacing each set of multiple edges by a single edge does not change the chromatic number of a graph.

$|V(\Gamma)| \geq 8$. For if not, then $|V(\Gamma)| = 6$ or $|V(\Gamma)| = 7$. If $|V(\Gamma)| = 6$ then by (12) $\Gamma = <6>$, which is contrary to hypothesis. Suppose that $|V(\Gamma)| = 7$. By (12), and because the sum of the valencies of all the vertices is equal to $2|E(\Gamma)|$, it follows that some vertex of Γ, x say, is joined to all the others. $\Gamma - x$ is 4-chromatic, therefore by (13) it contains a $<4>$ or a $<4U>$. Consequently Γ contains a $<5>$ or a $<5U>$ and so Γ is not contraction-critical. This contradiction proves that $|V(\Gamma)| \geq 8$.

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It follows from (12) that Γ-a-b is 3-connected. Hence, by Theorem 3, Γ-a-b contains a P or PU whose two ends together include c and d unless Γ-a-b is a K, K_1, K_2, K_3 or a wheel. It will be shown below that Γ-a-b is not a K, K_1, K_2, K_3 or wheel. Γ-a-b is not a K because a K is 2-chromatic and Γ is 5-chromatic.

Suppose that $\Gamma\text{-a-b} = K_1(x_1, x_2, x_3; y_1, y_2, \ldots, y_i)$. Then $(a, b) \in \Gamma$ because a K_1 is 3-chromatic and Γ is 5-chromatic. a and b are both joined to x_1 and to x_2 in Γ because if e.g. $(a, x_2) \notin \Gamma$ then a 4-colouring C of Γ could be obtained thus: $C(x_2) = 1$, $C(x_1) = C(x_3) = 2$, $C(y_1) = C(y_2) = \ldots = C(y_i) = 3$, $C(a) = 1$, $C(b) = 4$, whereas Γ is 5-chromatic. Therefore $\Gamma(a, b, x_1, x_2) = \langle 4 \rangle$, hence by (11) Γ hom. $\langle 5 \rangle$, and this is contrary to hypothesis. So Γ-a-b is not a K_1.

Suppose that $\Gamma\text{-a-b} = K_2(x_1, x_2, x_3; y_1, y_2, \ldots, y_i)$. Then $(a, b) \in \Gamma$ because a K_2 is 3-chromatic and Γ is 5-chromatic. $(x_2, a) \in \Gamma$ and $(x_2, b) \in \Gamma$ because if e.g. $(x_2, a) \notin \Gamma$ then a 4-colouring C of Γ defined as above would exist. (y_1, a), $(y_1, b) \in \Gamma$ by (12). Therefore $\Gamma(a, b, x_2, y_1) = \langle 4 \rangle$, hence by (11) Γ hom. $\langle 5 \rangle$, contrary to hypothesis. So Γ-a-b is not a K_2.

Γ-a-b is not a K_3 because a K_3 contains $\langle 4 \rangle$-s and Γ does not.

Suppose that Γ-a-b is a wheel consisting of the circuit $((u_1, u_2, \ldots, u_i))$ together with the vertex v, which does not belong to the circuit and is joined to every vertex of the circuit by one edge. $u_1, \ldots, u_i$ are joined to a and to b in Γ by (12). It follows that $(a, v) \notin \Gamma$ and $(b, v) \notin \Gamma$, since $\Gamma \not\supseteq \langle 4 \rangle$. Hence $(a, b) \in \Gamma$ for otherwise $u_1, u_2, \ldots, u_i$ could be

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coloured with the colours 1, 2, 3 and a, b, v with the colour 4, whereas Γ is 5-chromatic. Consequently $\Gamma(a, b, u_1, u_2) = <4>$, which leads to a contradiction. So $\Gamma - a - b$ is not a wheel.

Hence Theorem 10 is true.

THEOREM 11. A 5-chromatic graph is either homomorphic to a $<5>$, or else if any two of its vertices are deleted then the remaining graph contains a P or a PU.

Proof. It is sufficient to establish the theorem for vertex-critical graphs. Let Λ be a vertex-critical 5-chromatic graph and let m and n denote two vertices of Λ. Λ is finite and therefore homomorphic to a contraction-critical 5-chromatic graph, Γ say ($\Lambda = \Gamma$ possibly); let ϕ denote the mapping and let a denote $\phi(m)$ and b denote $\phi(n)$ ($a = b$ possibly). If $\Gamma \supseteq <5>$ then the theorem is true. Suppose that $\Gamma \not\supseteq <5>$. Then by Theorem 10 $\Gamma - a - b$ contains a P or a PU. Therefore $\Lambda - m - n$ is homomorphic to a P or a PU. Consequently by Theorem 7 Λ contains a P or a PU. Theorem 11 is thereby proved.

THEOREM 12. Corresponding to any vertex of a vertex-critical 6-chromatic graph there exists in the graph a P or a PU containing the vertex.

Proof (by induction over the number of vertices n). The theorem is clearly true for $n = 6$. Suppose that it is true for $6 \leq n \leq m-1$, where $m \geq 7$, and let Γ denote a vertex-critical 6-chromatic graph with m vertices. If Γ is 3-connected then the assertion of Theorem 12 for Γ follows from Theorem 3. Suppose that Γ is not 3-connected. Then by Menger's Theorem Γ contains two vertices p and q such that $\Gamma - p - q$ is disconnected. The notation of Theorem 8 will be adopted. Let f denote a vertex of Γ. To prove Theorem 12 it will be shown that Γ contains a P or a PU to which f belongs.

Suppose first that $f \in \Gamma'$. By Theorem 8 B $\Gamma' \cup (p, q)$ is 6-chromatic and vertex-critical, therefore by the induction hypothesis $\Gamma' \cup (p, q)$ contains a P or a PU to which f belongs. It follows that Γ contains a P or a PU to which f belongs, since (p, q) can be replaced by a $(p)(q)$-path

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contained in Γ'', if necessary. (It follows from Theorem 8 D that Γ' and Γ'' are connected.)

Suppose secondly that $f \notin \Gamma'$, so that $f \in \Gamma''-p-q$. Γ' is connected and therefore contains a (p)(q)-path, R say. Let Γ''' denote the graph obtained from Γ'' by identifying p with q, and let r denote the vertex of Γ''' not belonging to Γ'' (see the proof of Theorem 8), $r \neq f$. $\Gamma'' \cup R$ is contracted into a graph of which Γ''' is a subgraph by the mapping ϕ defined by $\phi(x) = x$ if $x \notin R$ and $\phi(x) = r$ if $x \in R$. By Theorem 8 C Γ''' is 6-chromatic and vertex-critical. Hence by the induction hypothesis Γ''' contains a P or a PU to which f belongs. Therefore by Theorem 7 $\Gamma'' \cup R$ contains a P or a PU to which f belongs.

Hence Γ contains a P or a PU to which f belongs. Theorem 12 is thereby proved.

The results established in this section may be applied to graphs with higher chromatic number with the help of the following general rule:

Let Γ denote a vertex-critical k-chromatic graph, where $k \geq 3$, and let g denote any vertex of Γ. Let Γ be coloured with the colours $1, 2, \ldots, k$ in any permissible way subject to the condition that colour 1 is given to g only, and let C_i denote the set of those vertices of Γ which have colour i for $i = 1, \ldots, k$. Then for $1 \leq \ell \leq k-1$ $\Gamma - C_{\ell+1} - C_{\ell+2} - \ldots - C_k$ contains a vertex-critical ℓ-chromatic graph to which g belongs. For $\Gamma - C_{\ell+1} - \ldots - C_k$ is ℓ-chromatic and $\Gamma - C_{\ell+1} - \ldots - C_k - g$ is $(\ell-1)$-chromatic. Consequently for example Theorem 12 can also be formulated thus: Corresponding to any vertex of a vertex-critical graph with chromatic number ≥ 6 there exists in the graph a P or a PU containing the vertex.

The following rule is the analogue of the above for edge-critical graphs: Let Γ denote an edge-critical k-chromatic graph, where $k \geq 3$, and let (a, b) denote any edge of Γ. Let $\Gamma-(a, b)$ be coloured with the colours $1, \ldots, k-1$ in any permissible way subject to the condition that a and b are given the colour 1, and let D_i denote the set of those vertices

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of Γ which have colour i for $i = 1, \ldots, k-1$. ($\Gamma-(a,b)$ is $(k-1)$-colourable because Γ is edge-critical, and in any $(k-1)$-colouring of $\Gamma-(a,b)$ the colour of a is the same as the colour of b because Γ is k-chromatic.) Then for $2 \leq \ell \leq k-1$ $\Gamma - D_\ell - \ldots - D_{k-1}$ contains an edge-critical ℓ-chromatic graph to which (a, b) belongs. For $\Gamma - D_\ell - \ldots - D_{k-1}$ is ℓ-chromatic and $\Gamma - (a,b) - D_\ell - \ldots - D_{k-1}$ is $(\ell-1)$-chromatic.

REFERENCES

1. K. Menger, Kurventheorie, Leipzig 1932, 221-228.
   G. Hajos, Zum Mengerschen Graphensatz, Acta Szeged 7(1934), 44.

2. K. Corradi and A. Hajnal, On the maximum number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar.

3. G. A. Dirac and P. Erdös, On the maximum number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar.

4. G. A. Dirac, Extensions du théoreme de Menger, C. R. Acad. Sci., Paris, 250, fasc. 26(1960), 4252.

5. K. Wagner, Über eine Erweiterung eines Satzes von Kuratowski, Deutsche Math. 2 (1937), 280.

6. K. Wagner, Bemerkungen zu Hadwiger's Vermutung, Math. Ann. 141 (1960), 433.

7. G. A. Dirac, A contraction theorem for abstract graphs, Math. Ann. 144 (1961), 93.

8. K. Wagner, loc. cit. 6, 436, Lemma 1. 4.

9. N. G. de Bruijn and P. Erdös, A colour problem for infinite graphs and a problem in the theory of relations, Proc. Koninkl. Akad. Wetenschappen A (1951), 371.

https://doi.org/10.4153/CMB-1963-019-5 Published online by Cambridge University Press

10. G. A. Dirac, Theorems related to the four colour conjecture, Journal London Math. Soc. 29(1954), 144, Theorem 1.

11. G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs. Journal London Math. Soc. 27(1952), 87. Bernhardine Zeidl, Über 4- und 5-chrome Graphen, Monatsh. Math. 62 (1958), 212.

12. G. A. Dirac, Trennende Knotenpunktmengen und Reduzibilität abstrakter Graphen, Crelle 204 (1960), 128.

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