# ON NON-CROSS VARIETIES OF $A$-GROUPS

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## 1. Introduction

The purpose of this paper is to provide a proof for a result announced in [3]. The result arose from a search for just-non-Cross varieties (recall that a Cross variety is one which can be generated by a finite group, and a just-non-Cross variety is a non-Cross variety every proper subvariety of which is Cross). For the motivation for this search, we refer the reader to [12]: for related results, see [1], [12], [13].

To state the result precisely, we need some further notation. An $A$-group is a soluble locally finite group whose Sylow subgroups are all abelian. $\mathfrak{A}_n$ denotes the variety of abelian groups of exponent dividing $n$.

THEOREM 1. *Any non-Cross variety of A-groups contains the product variety* $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ *for some set of distinct primes* $p$, $q$, $r$.

Since for distinct primes $p$, $q$, $r$ Graham Higman [7] has shown that $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ is non-Cross, we have as an immediate consequence

COROLLARY 2. *A variety of A-groups is just-non-Cross if and only if it has the form* $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ *for distinct primes* $p$, $q$, $r$.

The proof of Theorem 1 is divided into two steps:

(A) *any infinite set of (non-isomorphic) critical groups in* $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ *generates* $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ *($p$, $q$, $r$ distinct primes),*

(B) *any non-Cross variety of A-groups contains an infinite set of (non-isomorphic) critical groups in* $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$, *for some set of three distinct primes* $p$, $q$, $r$.

Our proof of (A) follows the lines of a proof of a more general result outlined by Graham Higman at the conference at which these results were announced (see [8]), and we shall not reproduce the proof here. Our concern here will be to give a proof of (B). These results formed part of my Ph. D. Thesis, submitted to the Australian National University [2]. This work was done under the supervision of Dr L. G. Kovács, and I welcome this opportunity to express my thanks for his advice and encouragement at that time.

## 2. Preliminaries

Our notation and terminology in general follows that of Hanna Neumann's book [14]. All groups will be finite unless otherwise stated. For a group $G$, $\zeta G$ will denote the centre of $G$, $\sigma G$ of the socle of $G$, $\sigma^* G$ the centraliser in $G$ of $\sigma G$, $\Phi(G)$ the Frattini subgroup of $G$, $F(G)$ the Fitting subgroup of $G$, $G'$ the derived group of $G$, and if $p$ is a prime, $G_p$ is to mean an arbitrary Sylow $p$-subgroup of $G$. We define $\delta_i G$ inductively by $\delta_0 G = G$, $\delta_{i+1} G = (\delta_i G)'$, $i \geqq 0$, and say $G$ is soluble of derived length $d$ if $\delta_{d-1} G \neq 1$ and $\delta_d G = 1$. We will adopt a right normed notation for compounds of these subgroups: thus, for example, $\sigma\zeta\delta_i G = \sigma(\zeta(\delta_i G))$.

If $\mathfrak{V}$ is a locally finite variety, the exponent of $\mathfrak{V}$, denoted by $e(\mathfrak{V})$, is the order of its cyclic relatively free group. $\mathfrak{A}^k$ will denote the variety of all soluble groups of derived length at most $k$.

We will need the following results.

LEMMA 2.1. ((Kovács and Newman [10] 2.1). *Let $H/K$ be a chief factor of the soluble group G, and C the centraliser of $H/K$ in G. If $|H/K| \leqq m$, $|G/C| \leqq m!$ and conversely if $|G/C| \leqq m$, $H/K$ is elementary abelian on at most m generators.*

LEMMA 2.2. *If $\mathfrak{V}$ is a variety of A-groups, then $\mathfrak{V}$ is a Cross variety if and only if there is a bound on the orders of chief factors of groups in $\mathfrak{V}$.*

This is an immediate consequence of the main result of Kovács and Newman [10].

The next two lemmas are results either contained in or easily derived from Taunt [15].

LEMMA 2.3. *For an A-group G, of derived length d,*

(1) $\sigma^* G = F(G)$,

(2) $\sigma G = \sigma\sigma^* G = \sigma\zeta G \times \cdots \times \sigma\zeta\delta_{d-2} G \times \sigma\delta_{d-1} G$,

(3) *if $L_i$ is any system normaliser of $\delta_i G$ relative to $G$, $0 \leqq i \leqq d-2$, then $G = \delta_{i+1} G L_i$, $\delta_{i+1} G \cap L_i = 1$, and*

$$\zeta\delta_i G \times \cdots \zeta G \leqq L_i.$$

LEMMA 2.4. *For a critical A-group G, of derived length d, $\sigma^* G$ is an indecomposable normal homocyclic subgroup of G, and for some prime p,*

$$F(G) = G_p = \sigma^* G = \delta_{d-1} G.$$

LEMMA 2.5. *An A-group is critical if and only if it is monolithic.*

This is a special case of Theorem 1.66 of Kovács and Newman [11].

From Lemmas 2.2 and 2.4 and Hilfssatz 2.2 of Huppert [9], we obtain

LEMMA 2.6. *Any metabelian variety of A-groups is a Cross variety.*

## 3. The proof of (B)

We first show that to prove (B) it is enough to prove (C) below. The main task of this section will then be to establish (C).

(C) *Let $\mathfrak{V}$ be a non-Cross variety of $A$-groups satisfying the following two conditions.*

(i) $\mathfrak{V} \leqq \mathfrak{A}_p\mathfrak{U}$, *where $\mathfrak{U}$ is a variety of $A$-groups of derived length at most $k$ ($\geqq 2$), and $p$ is a prime which does not divide the exponent of $\mathfrak{U}$, and*

(ii) $\mathfrak{V} \cap \mathfrak{A}^k$ *is a Cross variety.*

*Then there exist distinct primes $q, r$ dividing $e(\mathfrak{U})$ such that for any positive integer $n$, $\mathfrak{V}$ contains a critical group $G$ with $G \in \mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ and $|G_r| \geqq r^n$.*

(3.1) (B) *is true if* (C) *is true.*

Proof. Let $\mathfrak{W}$ be a non-Cross variety of $A$-groups, and suppose that

$$e(\mathfrak{W}) = e = p_1^{\alpha_1} \cdots p_t^{\alpha_t} (p_1, \cdots, p_t \text{ distinct primes}).$$

It follows from Theorem 8.3 of Taunt [15] and Lemma 2.6 that $t \geqq 3$.

Suppose $k+1$ is the smallest integer such that $\mathfrak{W} \cap \mathfrak{A}^{k+1} = \mathfrak{W}^*$ is non-Cross: then $k \geqq 2$, by Lemma 2.6. Let $\mathscr{D}_i$ denote the set of critical groups in $\mathfrak{W}^*$ whose monolith is a $p_i$-group. Each critical group falls into some $\mathscr{D}_i$ and hence at least one, $\mathscr{D}_1$ say, will contain an infinite set of non-isomorphic critical groups.

For each $G \in \mathscr{D}_i$, the chief factors of $G/\sigma^*G$ have bounded order (by the choice of $\mathfrak{W}^*$, and Lemmas 2.2 and 2.4), and so we get that the orders of the chief factors of $G$ in $\sigma^*G$ are not bounded. But from Lemma 6.4 of Taunt [15] and Lemma 2.4, $\sigma^*G/\Phi\sigma^*G$ is a chief factor of $G$, and all chief factors of $G$ in $\sigma^*G$ have the same order. Let $\mathscr{D} = \{G/\Phi\sigma^*G : G \in \mathscr{D}_1)$: the variety generated by $\mathscr{D}$ has groups with chief factors of arbitrarily large order, and so is non-Cross (by Lemma 2.2). Let $\mathfrak{V}$ be the variety generated by $\mathscr{D}$, $\mathfrak{U}$ the variety generated by $\{G/\sigma^*G : G \in \mathscr{D}_1\}$. Then $\mathfrak{V} \leqq \mathfrak{A}_{p_1}\mathfrak{U}$, $\mathfrak{U}$ is a variety of $A$-groups of derived length $k$, $p_1$ is a prime not dividing $e(\mathfrak{U})$ (by Lemma 2.4). Also $\mathfrak{V} \cap \mathfrak{A}^k \leqq \mathfrak{W} \cap \mathfrak{A}^k$ and so is Cross by assumption. This establishes 3.1.

The proof of (C) is carried out in several steps, numbered consecutively.

Let $\mathfrak{V}$ be a variety satisfying the conditions of (C). For notational convenience we introduce the following conventions: if $G \in \mathfrak{V}$, then $\eta G$ will denote a Hall $p'$-subgroup of $G$, $\lambda_i G$ will denote a system normaliser of $\delta_i \eta G$ relative to $\eta G$.

We have $\mathfrak{V} \leqq \mathfrak{A}_p\mathfrak{U}$, where $\mathfrak{U}$ is a variety of $A$-groups of derived length at most $k$,

$$e = e(\mathfrak{U}) = p_1^{\alpha_1} \cdots p_t^{\alpha_t}, \text{ where } p_1, \cdots, p_t \text{ are distinct primes and } t \geqq 2,$$

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and $\mathfrak{V} \cap \mathfrak{A}^k$ is a Cross variety. It follows that $\mathfrak{V}$ must contain an infinite set of non-isomorphic critical groups of derived length $k+1$: choose $\mathscr{D}$ to be some such infinite set: from now on we will be working with this fixed $\mathscr{D}$.

(3.2) *The following sets are not bounded*:

(i) $\mathscr{S}_1 = \{|\sigma G| : G \in \mathscr{D}\}$

(ii) $\mathscr{S}_2 = \{|\sigma\eta G| : G \in \mathscr{D}\}$

(iii) $\mathscr{S}_3 = \{|\eta G/\sigma^*\eta G| : G \in \mathscr{D}\}$

(iv) $\mathscr{S}_4 = \{|\eta G/\delta_{k-1}\eta G| : G \in \mathscr{D}\}$.

PROOF. (i) is an immediate consequence of the choice of $\mathscr{D}$ and Lemma 2.2.

(ii) Suppose $s$ is an upper bound for $\mathscr{S}_2$. Then $\sigma\eta G$ can be generated by $s$ elements, and hence so also can $\sigma^*\eta G$ (from Lemma 2.3). Also, $\eta G/\sigma^*\eta G$ is faithfully represented by automorphisms of $\sigma\eta G$, and hence

$$|\eta G| = |\sigma^*\eta G| \cdot |\eta G/\sigma^*\eta G| \leqq e^s s!.$$

But from Lemma 2.4, $\sigma G$ is a self centralising chief factor of $G$, and $\eta G = G/\sigma G$ and from Lemma 2.1 we get

$$|\sigma G| \leqq e^{e^s s!},$$

and $\mathscr{S}_1$ is bounded, a contradiction.

(iii) Suppose $s$ is an upper bound for $\mathscr{S}_3$. Since $\eta G$ is faithfully and irreducibly represented on $\sigma G$, $\sigma\eta G$ is the normal closure of a single element (Gaschütz [6]). Hence $|\sigma\eta G| \leqq e^s$, and $\mathscr{S}_2$ is bounded, a contradiction.

(iv) Since $\delta_{k-1}\eta G \leqq \sigma^*\eta G$ by Lemma 2.3, this is an immediate consequence of (iii).

(3.3) *The sets* $\mathscr{X}_j = \{|\sigma\zeta\delta_j\eta G| : G \in \mathscr{D}\}, 0 \leqq j \leqq k-2$, *are bounded.*

PROOF. By Lemma 2.3, we have $\sigma\zeta\delta_i\eta G \leqq \lambda_i G$. Also, since $i < k-1$, we have $\lambda_i G$ of derived length at most $k-1$. If $N$ is a normal subgroup of $\lambda_i G$ contained in $\sigma\zeta\delta_i\eta G$, then $N$ is a normal subgroup of $\eta G$ (from Lemma 2.3).

We consider $\sigma G$ as a $GF(p)(\lambda_i G)$-module. By Maschke's Theorem it is completely reducible: let $M$ be an irreducible submodule, and $K$ the kernel of the representation of $\lambda_i G$ on $M$. The possibility that $K \cap \sigma\zeta\delta_i\eta G \neq 1$ is ruled out by Clifford's theorem ([5] Theorem 49.2).

For each $G$ in $D$, construct a group $G^* = M\lambda_i G/K$, where as above $M$ is an irreducible component of $\sigma G$ considered as $GF(p)(\lambda_i G)$-module, and $K$ the kernel of the corresponding representation. Clearly, $G^*$ is monolithic, with monolith $MK/K$, and $G^* \in \mathfrak{A}^k$. Let $\mathfrak{X} = \text{var}\,\{G^* : G \in \mathscr{D}\}$: then $\mathfrak{X} \leqq \mathfrak{V} \cap \mathfrak{A}^k$ and so by assumption $\mathfrak{X}$ is a Cross variety. From Lemma 2.2 there is a bound, $s$ say, on the order of chief factors of groups in $\mathfrak{X}$. Since $MK/K$ is a self centralising chief factor of $G^*$, and since

$$\sigma\zeta\delta_i\eta G \cong (\alpha\zeta\delta_i\eta G)K/K \leqq \lambda_i G/K = \eta G^*,$$

we have from Lemma 2.1 $|\sigma\zeta\delta_i\eta G| \leqq s!$, and so $\mathscr{X}_i$ is bounded.

(3.4) *The set* $\mathscr{X}_{k-1} = \{\sigma\delta_{k-1}\eta G : G \in \mathscr{D}\}$ *is not bounded.*

Proof. Follows immediately from (3.2), (3.3) and Lemma 2.3.

(3.5) *The set* $\mathscr{L} = \{|\sigma\lambda_{k-2}G| : G \in \mathscr{D}\}$ *is not bounded.*

Proof. Suppose $s$ is an upper bound for $\mathscr{L}$. Then $\sigma\lambda_{k-2}G$ can be generated by $s$ elements, and hence so can $\sigma^*\lambda_{k-2}G$ giving $|\sigma^*\lambda_{k-2}G| \leqq e^s$. Also $\lambda_{k-2}G/\sigma^*_{k-2}G$ is faithfully represented by automorphisms of $\sigma\lambda_{k-2}G$ and so

$$|\lambda_{k-2}G/\sigma^*\lambda_{k-2}G| \leqq s!,$$

whence $|\lambda_{k-2}G| \leqq e^s s!$. But $\lambda_{k-2}G \cong \eta G/\delta_{k-1}\eta G$ (from Lemma 2.3), and so $S_4$ is bounded, contradicting (3.2) (iv).

We have

$$N = \sigma\zeta\delta_{k-2}\eta G \times \cdots \times \sigma\zeta\eta G \leqq \lambda_{k-2}G$$

by Lemma 2.3, and also $N$ is a direct product of minimal normal subgroups of $\eta G$, which are also minimal normal in $\lambda_{k-2}G$ since they are centralised by $\delta_{k-1}\eta G$. Hence $N \leqq \sigma\lambda_{k-2}G$. Let $\alpha G$ denote a $\lambda_{k-2}G$-normal complement of $N$ in $\sigma\lambda_{k-2}G$. From (3.3) and (3.5) we deduce

(3.6). *The set* $\mathscr{A} = \{|\alpha G| : G \in \mathscr{D}\}$ *is not bounded: further, for some prime* $p_i$ *dividing* $e$, *the set* $\mathscr{A}_i = \{|(\alpha G)_{p_i}| : G \in \mathscr{D}\}$ *is not bounded.*

(3.7) *The representation of* $\alpha G$ *as a group of automorphisms of* $\sigma\delta_{k-1}\eta G$ *is faithful.*

Proof. Suppose that $\sigma^*\eta G \cap \alpha G \neq 1$, and let $M$ be a minimal normal subgroup of $\lambda_{k-2}G$ contained in this intersection. Then $M$ is a minimal normal subgroup of $\eta G(= (\sigma^*\eta G)(\lambda_{k-2}G))$, and so

$$M \leqq \sigma\eta G \cap \lambda_{k-2}G = \sigma\zeta\delta_{k-2}\eta G \times \cdots \times \sigma\zeta\eta G.$$

But then, from the definition of $\alpha G$,

$$M \leqq (\sigma\eta G \cap \lambda_{k-2}G) \cap \alpha G = 1,$$

a contradiction. Since $\alpha G$ centralises $\sigma\eta G \cap \lambda_{k-2}G$, it must intersect the centraliser of $\sigma\delta_{k-1}\eta G$ in $\eta G$ trivially, and the result follows.

Now, choose a fixed prime $p_l$ for which $\mathscr{A}_l$ is not bounded. For $1 \leqq j \leqq t$, and for $G \in \mathscr{D}$, define $K_j$ to be the centraliser of $(\sigma\delta_{k-1}\eta G)_{p_j}$ in $(\alpha G)_{p_l}$: note that $K_l = (\alpha G)_{p_l}$ since $G$ is an $A$-group. Since $(\alpha G)_{p_l}$ is normal in $\lambda_{k-2}G$, and $(\sigma\delta_{k-1}\eta G)_{p_j}$ is normalised by $\lambda_{k-2}G$, we have $K_j$ normal in $\lambda_{k-2}G$, $1 \leqq j \leqq t$. Now let $\kappa_j G$ be a $\lambda_{k-2}G$-normal complement of $K_j$ in $(\alpha G)_{p_l}$, and put

$$\mathscr{K}_j = \{|\kappa_j G| : G \in \mathscr{D}\}, \qquad 1 \leqq j \leqq t.$$

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(3.8) *For some $j \neq l$, the set $\mathscr{K}_j$ is not bounded.*

PROOF. Suppose that $s$ is an upper bound for $\mathscr{K}_j$, $1 \leqq j \leqq t$. From (3.7) we have $\bigcap_{j=1}^{t} \kappa_j G = 1$, and hence $|(\alpha G)_{p_l}| \leqq s^t$, and $\mathscr{A}_l$ is bounded, contradicting the choice of $\mathscr{A}_l$. Thus at least one $\mathscr{K}_j$ is not bounded: since $\mathscr{K}_j = \{1\}$, $j \neq l$ for such a $j$.

Note that if $\mathscr{K}_j$ is not bounded, neither is the set $\{|(\sigma\delta_{k-1}\eta G)_{p_j}| : G \in \mathscr{D}\}$, using a by now familiar argument.

We choose a $j$ such that $\mathscr{K}_j$ is not bounded, and from now on will be working with this fixed $\mathscr{K}_j$. Our aim will be, given an arbitrary positive integer $n$, to construct a critical group $H \in \mathfrak{A}_p \mathfrak{A}_{p_j} \mathfrak{A}_{p_l}$ as a factor of some $G \in \mathscr{D}$ such that $|H_{p_l}| \geqq p^n$: if we can do this, (C) is proved.

Thus suppose we are given a positive integer $n$. Since $\mathfrak{B} \cap \mathfrak{A}^k$ is a cross variety, there is a bound, $m$ say, on the order of chief factors of groups in $\mathfrak{B} \cap \mathfrak{A}^k$ by Lemma 2.2. Choose $G \in \mathscr{D}$ such that $|\kappa_j(G)| \geqq (m!)^n$. From now on we will be working with this fixed $G$. Put $S = (\sigma\delta_{k-1}\eta G)_{p_j}$, and let $S_1 \times \cdots \times S_a$ be some decomposition of $S$ into minimal normal subgroups of $\eta G$: then $|S_i| \leqq m$, $1 \leqq i \leqq a$. Let $N_i$ denote the centraliser of $S_i$ in $\kappa_j G$: then $|\kappa_j G : N_i| \leqq m!$. Since $\kappa_j G$ is faithfully represented on $S$, we have $\bigcap_{i=1}^{a} N_i = 1$.

(3.9) *There exist centralisers $N_1, \cdots, N_n$, and minimal normal subgroups of $\lambda_{k-2}G$, $M_1, \cdots, M_n$, contained in $\kappa_j G$, such that for $1 \leqq i \leqq n$,*

$$M_i \leqq \bigcap_{\substack{u=1\\u\neq i}}^{n} N_u, \quad N_i \cap M_i = 1.$$

PROOF. Relabelling $S_1, \cdots, S_a$ if necessary, we can choose $N_1, \cdots, N_b$ satisfying

(i) the $N_i$ are minimal, $1 \leqq i \leqq b$: that is $N_w \leqq N_i$ implies $N_w = N_i$,

(ii) $N_i \neq N_w$ if $i \neq w$, $1 \leqq i, w \leqq b$,

(iii) $\bigcap_{i=1}^{b} N_i = 1$, $\bigcap_{\substack{i=1\\i\neq w}}^{b} N_i \neq 1$, $1 \leqq w \leqq b$.

Since $|\kappa_j G : N_i| \leqq m!$, $|\kappa_j G| \leqq (m!)^b$. But $|\kappa_j G| \geqq (m!)^n$, and so $b \geqq n$, and we take the first $n$ of these $N_i$'s.

Now choose $M_i$ to be any minimal normal subgroup of $\lambda_{k-2}G$ such that

$$M_i \leqq \bigcap_{\substack{w=1\\w\neq i}}^{n} N_w, \quad 1 \leqq i \leqq n.$$

Then $N_1, \cdots, N_n, M_1, \cdots, M_n$ satisfy the requirements of (3.9).

Now consider the subgroup $T$ of $\eta G$, where $T = \langle S_1, \cdots, S_n, M_1, \cdots, M_n \rangle$. If we put $T_i = \langle S_i, M_i \rangle$, we have $T = T_1 \times \cdots \times T_n$. To prove this observe that the $S_i$ are minimal normal $p_j$-subgroups of $\eta G$, the $M_i$ are minimal normal $p_l$-subgroups of $\lambda_{k-2}G$, and $[S_i, M_w] = 1$ for $i \neq w$, $1 \leqq i, w \leqq n$: hence it suffices to show that the products $S_1 \cdots S_n$, $M_1 \cdots M_n$ are direct. The first comes from the choice of $S_1, \cdots, S_n$, the second from the fact that $M_i \leqq N_w$ for $i \neq w$, and $M_i \cap N_i = 1$, giving $M_i \cap \prod_{w \neq 1} M_w = 1$.

Note that $S_1, \cdots, S_n$ are elementary abelian $p_j$-groups, $M_1, \cdots, M_n$ are elementary abelian $p_l$-groups, and hence $T \in \mathfrak{A}_{p_j}\mathfrak{A}_{p_l}$.

(3.10) $\zeta T_i = 1$, $\sigma T_i = S_i$, $1 \leqq i \leqq n$.

PROOF. We may consider $S_i$ as a non-trivial irreducible $GF(p_j)\lambda_{k-2}G$-module: since $M_i$ is a normal subgroup of $\lambda_{k-2}G$, Clifford's Theorem gives us that $S_i$, as $GF(p_j)M_i$-module, is completely reducible, and the representation of $M_i$ on each irreducible submodule is non-trivial. Hence $S_i \cap \zeta T_i = 1$.

If $(\zeta T_i)_{p_l} \neq 1$, then $(\zeta T_i)_{p_l} \cap M_i \neq 1$, contradicting the choice of $M_i$. We conclude $\zeta T_i = 1$.

That $\sigma T_i = S_i$ now follows from the fact that $T_i$ is metabelian and Lemma 2.3.

By Maschke's Theorem $\sigma G$ is completely reducible as a $GF(p)T$-module: let $R$ be an irreducible component, and let $C$ be the kernel of the representation of $T$ on $R$.

(3.11) $|(T/C)_{p_l}| \geqq p_l^n$.

PROOF. From Clifford's Theorem, we deduce $S_i \nleqq C$, for each $S_i$ is a minimal normal subgroup of $\eta G$, $1 \leqq i \leqq n$.

Let $\pi_i$ denote the canonic projection of $T$ onto $T_i$, $1 \leqq i \leqq n$, and put $C_i = C\pi_i$: note that $C_i$ is normal in $T_i$. Suppose that for some $i$, $|(C_i)_{p_l}| = |M_i|$. Then $(C_i)_{p_l}$ is also a Sylow $p_l$-subgroup of $T_i$, and hence $M_i$ is contained in $C_i$. If $Q$ is any minimal normal subgroup of $T_i$, we have $Q \leqq S_i$ (from (3.10)). Now $[M_i, Q] = Q$ (for otherwise $Q$ would be centralised by $M_i$ and $S_i$, and so by $T_i$, contradicting (3.10)), and so

$$\begin{aligned}[C, Q] &= [C\pi_i, Q] \\ &\geqq [M_i, Q] \\ &= Q.\end{aligned}$$

Thus $Q \leqq C$, and hence $S_i \leqq C$, a contradiction. Thus

$$\begin{aligned}|(T/C)_{p_l}| &\geqq \prod_{i=1}^{n} |(T_i/C_i)_{p_l}| \\ &\geqq p_l^n.\end{aligned}$$

Now, consider the split extension of $R$ by $T$: clearly $C$ is normal in $RT$. Put $H = RT/C$. Then $H$ is a critical group in $\mathfrak{A}_p\mathfrak{A}_{p_j}\mathfrak{A}_{p_l}$, and $|H_{p_l}| = |(T/C)_{p_l}| \geqq p_l^n$. Hence $(C)$ is proved.

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