PARSEVAL RELATIONS FOR KONTOROVICH-LEBEDEV TRANSFORMS

by J. S. LOWNDES
(Received 15th July, 1961)

1. Kontorovich-Lebedev Transforms

The Kontorovich-Lebedev Transform of a function $f(r)$, $0<r<\infty$, may be written in a general form as $f_g(\mu)$ where,

$$f_g(\mu) = \int_0^\infty f(r)G_\mu(kr)r^{-1}dr. \qquad (1)$$

$G_\mu(kr)$ is a Bessel function of order μ and argument kr and $k = k_1+ik_2$ is a complex constant. If $f(r)$ satisfies certain conditions then the inversion formula corresponding to equation (1) may be written as

$$f(r) = [f_g,\ M_\mu(kr)] = \int_{\mu_1-i\infty}^{\mu_1+i\infty} \mu f_g(\mu)M_\mu(kr)d\mu \qquad (2)$$

where the path of integration is some abscissa μ_1 in the plane of the variable $\mu = \mu_1+i\mu_2$. $M_\mu(kr) = \phi(\mu)N_\mu(kr)$ where $\phi(\mu)$ is a constant or a function of μ alone and $N_\mu(kr)$ is a Bessel function of order μ and argument kr.

Particular transforms of $f(r)$ may be written as

$$\left.\begin{aligned} &\text{(a) } f_j(\mu) = \int_0^\infty f(r)J_\mu(kr)r^{-1}dr, &&\text{(b) } f_y(\mu) = \int_0^\infty f(r)Y_\mu(kr)r^{-1}dr \\ &\text{(c) } f_1(\mu) = \int_0^\infty f(r)H_\mu^{(1)}(kr)r^{-1}dr, &&\text{(d) } f_2(\mu) = \int_0^\infty f(r)H_\mu^{(2)}(kr)r^{-1}dr \end{aligned}\right\}, \qquad (3)$$

where $J_\mu(kr)$ and $Y_\mu(kr)$ are Bessel functions of the first and second kinds respectively and $H_\mu^{(1)}(kr)$, $H_\mu^{(2)}(kr)$ are Bessel functions of the third kind. Snow (**1**, § VIII (3)) has shown that provided $f(r)$ satisfies a number of well-defined conditions then there exist the following inversion formulæ corresponding to the transforms given in equations (3).

$$f(r) = \left[f_j, \frac{i}{2}Y_\mu\right] = \left[f_j, \frac{1}{2}H_\mu^{(1)}\right] = \left[f_j, \frac{-1}{2}H_\mu^{(2)}\right], \quad -\delta<\mu_1<\infty \qquad (4a,\ b,\ c)$$

$$= \left[f_y, \frac{i}{2}J_\mu\right] = \left[f_y, \frac{i}{2}H_\mu^{(1)}\right] = \left[f_y, \frac{i}{2}H_\mu^{(2)}\right], \quad -\delta<\mu_1<\delta \qquad (5a,\ b,\ c)$$

$$= \left[f_1, \frac{1}{4}H_\mu^{(1)}\right] = \left[f_1, \frac{1}{2}J_\mu\right] = \left[f_1, \frac{i}{2}Y_\mu\right]$$

$$= \left[f_1, \frac{1}{4i} e^{i\mu\pi} \sin(\mu\pi) H_\mu^{(1)}\right], \qquad -\delta < \mu_1 < \delta \qquad (6a,\ b,\ c,\ d)$$

$$= \left[f_2, \frac{-1}{4} H_\mu^{(2)}\right] = \left[f_2, \frac{-1}{2} J_\mu\right] = \left[f_2, \frac{i}{2} Y_\mu\right]$$

$$= \left[f_2, \frac{1}{4i} e^{-i\mu\pi} \sin(\mu\pi) H_\mu^{(2)}\right], \ -\delta < \mu_1 < \delta. \qquad (7a,\ b,\ c,\ d)$$

The value of δ determines the strip or half plane of μ in which the corresponding transform of $f(r)$ is an analytic function of μ. In equation (4) δ is arbitrary and in equations (5), (6) and (7) δ is positive. Some further useful formulæ given by Snow are

$$[f_j, J_\mu] = 0, \quad -\delta < \mu_1 < \infty \quad (8)$$

$$[f_1, H_\mu^{(2)}] = [f_2, H_\mu^{(1)}] = [f_y, Y_\mu] = 0, \quad -\delta < \mu_1 < \delta \quad (9a,\ b,\ c)$$

$$[f_1, g(\mu) H_\mu^{(2)}] = [f_2, g(\mu) H_\mu^{(1)}] = 0, \quad -\delta < \mu_1 < \delta \quad (10a,\ b)$$

$$[f_1, g(\mu) e^{i\mu\pi} H_\mu^{(1)}] = [f_2, g(\mu) e^{-i\mu\pi} H_\mu^{(2)}] = 0, \quad -\delta < \mu_1 < \delta \quad (11a,\ b)$$

where $g(\mu)$ is an even function of μ.

The imaginary axis is a possible path for the integrals (5, 6, 7, 9, 10, 11) and also for the integrals (4) and (8) when $\delta > 0$. In some cases the integrals may be reduced to those along the upper half of this axis. For example in equation (6*d*) $e^{i\mu\pi} \sin \mu\pi H_\mu^{(1)}(kr)$ is an even function of μ and taking the path $\mu = 0 + i\mu_2 = i\lambda$ we have

$$f(r) = -\tfrac{1}{2} \int_0^\infty \lambda e^{-\lambda\pi} \sinh(\lambda\pi) H_{i\lambda}^{(1)}(kr) d\lambda \int_0^\infty f(\xi) H_{i\lambda}^{(1)}(k\xi) \xi^{-1} d\xi. \quad \text{......(12)}$$

Similarly (7*d*) may be reduced to

$$f(r) = -\tfrac{1}{2} \int_0^\infty \lambda e^{\lambda\pi} \sinh(\lambda\pi) H_{i\lambda}^{(2)}(kr) d\lambda \int_0^\infty f(\xi) H_{i\lambda}^{(2)}(k\xi) \xi^{-1} d\xi. \quad \text{......(13)}$$

It is also possible to express the above results in terms of the modified Bessel functions $I_\lambda(z)$ and $K_\lambda(z)$. If, for example, we set $k = i\eta$ in equation (4*b*) we find

$$f(r) = \frac{1}{\pi i} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu K_\mu(\eta r) d\mu \int_0^\infty f(\xi) I_\mu(\eta\xi) \xi^{-1} d\xi. \quad \text{...............(14)}$$

Examples of other formulæ which may be obtained in a similar fashion are

$$f(r) = \frac{i}{\pi} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu I_{-\mu}(\eta r) \Phi(\mu) d\mu \quad \text{.............................(15)}$$

$$= \frac{1}{\pi i} \int_{\mu_1 - i\infty}^{\mu_1 - i\infty} \mu I_\mu(\eta r) \Phi(\mu) d\mu \text{................................(16)}$$

https://doi.org/10.1017/S0013091500014437 Published online by Cambridge University Press

$$= \frac{i}{\pi^2}\int_{\mu_1-i\infty}^{\mu_1-i\infty} \mu \sin(\mu\pi) K_\mu(\eta r)\Phi\ \mu) d\mu \quad \text{.................(17)}$$

$$= \frac{2}{\pi^2}\int_0^\infty \lambda \sinh(\lambda\pi) K_{i\lambda}(\eta r)\Phi(i\lambda) d\lambda \quad \text{....................(18)}$$

where

$$\Phi(\mu) = \int_0^\infty f(\xi) K_\mu(\eta\xi)\xi^{-1} d\xi.$$

Kontorovich and Lebedev (**2**) first reported formulæ (7*b*, *d*) and Lebedev (**3**), (**4**), (**5**) has given formulæ (6*b*, *d*), (14) and (18). Turner (**6**), Wells and Leitner (**7**) and Lowndes (**8**) have applied formula (16) to solve certain diffraction problems.

2. General Parseval Relation

In this section we obtain formally a general Parseval relation for the transform pair given by equations (1) and (2).

If $f_g(\mu)$ and $h_g(\mu)$ are Kontorovich-Lebedev transforms of the functions $f(r)$ and $h(r)$ and if L be the path of integration over which the inversion formulæ for $f(r)$ and $h(r)$ are defined then

$$\int_L \mu\psi(\mu) f_g(\mu) h_g(\mu) d\mu = \int_L \mu\psi(\mu) f_g(\mu) d\mu \int_0^\infty h(r) G_\mu(kr) r^{-1} dr \quad \text{......(19)}$$

by the definition of $h_g(\mu)$. Assuming that an interchange of the orders of integration is permissible we have

$$\int_L \mu\psi(\mu) f_g(\mu) h_g(\mu) d\mu = \int_0^\infty h(r) r^{-1} dr \int_L \mu\psi(\mu) f_g(\mu) G_\mu(kr) d\mu. \quad \text{......(20)}$$

If we now define $\psi(\mu)$ such that

$$\int_L \mu\psi(\mu) f_g(\mu) G_\mu(kr) d\mu = \int_L \mu f_g(\mu) M_\mu(kr) d\mu = f(r) \quad \text{..............(21)}$$

we find that the general Parseval relation is

$$\int_L \mu\psi(\mu) f_g(\mu) h_g(\mu) du = \int_0^\infty f(r) h(r) r^{-1} dr. \quad \text{.................(22)}$$

3. Particular Parseval Relations

The particular Parseval relations are characterised by the function $\psi(\mu)$ and the path of integration L. These are listed in table 1 for some of the transforms defined in § **1** and they are derived by means of the general method outlined in § **2**.

To obtain the Parseval relation for the transform (6*b*) for example, we proceed in the following way. Since

$$f(r) = [f_1, \tfrac{1}{2} J_\mu(kr)]$$

https://doi.org/10.1017/S0013091500014437 Published online by Cambridge University Press

we have, by equation (21), to find a function $\psi(\mu)$ such that

$$\int_L \mu f_1(\mu)\psi(\mu)H_\mu^{(1)}(kr)d\mu = \tfrac{1}{2}\int_L \mu f_1(\mu)J_\mu(kr)d\mu \quad \text{..............(23)}$$

where L is the path $(\mu_1 - i\infty,\ \mu_1 + i\infty)$.

TABLE 1

Equation Number	$\psi(\mu)$	L
6 a, b, c	$\frac{1}{4}$	$(\mu_1 - i\infty,\ \mu_1 + i\infty)$
6 d	$\frac{1}{4i} e^{i\mu\pi} \sin \mu\pi$	
7 a, b, c	$-\frac{1}{4}$	
7 d	$\frac{1}{4i} e^{-i\mu\pi} \sin \mu\pi$	
12	$-\frac{1}{2}e^{-\lambda\pi} \sinh \lambda\pi$	$(0, \infty)$
13	$-\frac{1}{2}e^{\lambda\pi} \sinh \lambda\pi$	
15, 16, 17	$\frac{i}{\pi^2} \sin \mu\pi$	$(\mu_1 - i\infty,\ \mu_1 + i\infty)$
18	$\frac{2}{\pi^2} \sinh \lambda\pi$	$(0, \infty)$

Now

$$2J_\mu(kr) = H_\mu^{(1)}(kr) + H_\mu^{(2)}(kr)$$

and therefore substituting for $H_\mu^{(1)}(kr)$ in the left-hand side of equation (23) we have

$$\int_L \mu f_1(\mu)\psi(\mu)H_\mu^{(1)}(kr)d\mu = 2\int_L \mu f_1(\mu)\psi(\mu)J_\mu(kr)d\mu - \int_L \mu f_1(\mu)\psi(\mu)H_\mu^{(2)}(kr)d\mu. \quad \text{.........(24)}$$

Equation (24) reduces to (23) if we choose $\psi = \frac{1}{4}$ since, in this case, by equation (9a) the second integral on the right-hand side of (24) is zero.

In the following section a number of integrals will be evaluated with the aid of two of the Parseval relations indicated in table 1.

https://doi.org/10.1017/S0013091500014437 Published online by Cambridge University Press

4. Evaluation of Integrals

Consider the transforms given in equations (13) and (18), i.e.

$$F_1(\mu) = \int_0^\infty f(r) H_{i\mu}^{(2)}(r) r^{-1} dr \quad \text{.........................(25)}$$

$$F_2(\mu) = \int_0^\infty f(r) K_{i\mu}(r) r^{-1} dr \quad \text{.........................(26)}$$

where for simplicity we have taken $k = 1$, $\eta = 1$.

We shall use the following transform pairs

$$f(r) = \frac{(\alpha r)^{\frac{1}{2}}}{\pi(\alpha+r)} e^{-i(\alpha+r)}, \qquad F_1(\mu) = \text{sech}\,(\mu\pi) H_{i\mu}^{(2)}(\alpha) \quad \text{.........(27)}$$

$$f(r) = \tfrac{1}{2} \exp\left\{\frac{-r}{2}\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{r^2}\right)\right\}, \qquad F_2(\mu) = K_{i\mu}(\alpha) K_{i\mu}(\beta) \quad \text{..............(28)}$$

$$f(r) = \frac{2}{\pi^2}\left(\frac{\pi r}{2}\right)^{\frac{1}{2}} e^{-\phi r}, \qquad F_2(\mu) = \text{sech}\,(\mu\pi) P_{-\frac{1}{2}+i\mu}(\phi) \quad \text{......(29)}$$

$$f(r) = \frac{\alpha}{4(2\pi r)^{\frac{1}{2}}} \exp\left(-r - \frac{\alpha^2}{8r}\right), \qquad F_2(\mu) = K_{2i\mu}(\alpha) \quad \text{....................(30)}$$

$$f(r) = \frac{(\alpha r)^{\frac{1}{2}}}{\pi(\alpha+r)} e^{-(\alpha+r)}, \qquad F_2(\mu) = \text{sech}\,(\mu\pi) K_{i\mu}(\alpha), \quad \text{.........(31)}$$

where $P_\lambda(z)$ is the Legendre function of the first kind. The result (27) is given in (**9**, p. 381) and the results (28) to (31) can be found in (**9**, pp. 175-177).

Consider the integral

$$I_1 = \tfrac{1}{2} \int_0^\infty \mu e^{\mu\pi}\, \text{sech}\,(\mu\pi) \tanh\,(\mu\pi) H_{i\mu}^{(2)}(\alpha) H_{i\mu}^{(2)}(\beta) d\mu. \quad \text{...........(32)}$$

Using the Parseval relation given by (13, table 1), i.e.

$$\tfrac{1}{2} \int_0^\infty \mu \sinh\,(\mu\pi) e^{\mu\pi} F_1(\mu) H_1(\mu) d\mu = -\int_0^\infty f(r) h(r) r^{-1} dr$$

and putting

$$F_1(\mu) = \text{sech}\,(\mu\pi) H_{i\mu}^{(2)}(\alpha), \quad H_1(\mu) = \text{sech}\,(\mu\pi) H_{i\mu}^{(2)}(\beta)$$

we see from the results (27) that

$$\begin{aligned} I_1 &= \tfrac{1}{2} \int_0^\infty \mu e^{\mu\pi}\, \text{sech}\,(\mu\pi) \tanh\,(\mu\pi) H_{i\mu}^{(2)}(\alpha) H_{i\mu}^{(2)}(\beta) d\mu \\ &= -\frac{(\alpha\beta)^{\frac{1}{2}}}{\pi^2} e^{-i(\alpha+\beta)} \int_0^\infty \frac{e^{-i2r}}{(\alpha+r)(\beta+r)} dr \\ &= -\frac{(\alpha\beta)^{\frac{1}{2}}}{\pi^2} \left\{ \frac{e^{i(\alpha-\beta)}}{\alpha-\beta} Ei(-i2\alpha) + \frac{e^{i(\beta-\alpha)}}{\beta-\alpha} Ei(-i2\beta) \right\} \end{aligned} \quad \text{...........(33)}$$

https://doi.org/10.1017/S0013091500014437 Published online by Cambridge University Press

where
$$Ei(-ix) = -\int_x^\infty \frac{e^{-i\lambda}}{\lambda}\,d\lambda.$$

A second integral is

$$I_2 = \frac{2}{\pi^2}\int_0^\infty \mu\,\text{sech}\,(\mu\pi)\tanh(\mu\pi)P_{-\frac{1}{2}+i\mu}(\phi)K_{i\mu}(\alpha)d\mu. \quad \text{.........(34)}$$

Setting

$$F_2(\mu) = \text{sech}\,(\mu\pi)P_{-\frac{1}{2}+i\mu}(\phi), \quad H_2(\mu) = \text{sech}\,(\mu\pi)K_{i\mu}(\alpha)$$

we find from equations (29) and (31) that

$$f(r) = \frac{2}{\pi^2}\left(\frac{\pi r}{2}\right)^{\frac{1}{2}} e^{-\phi r}, \quad h(r) = \frac{(r\alpha)^{\frac{1}{2}}}{\pi(r+\alpha)} e^{-(r+\alpha)}.$$

Hence, using the Parseval relation (18, table 1) we have

$$I_2 = \frac{2}{\pi^2}\int_0^\infty \mu\sinh(\mu\pi)F_2(\mu)H_2(\mu)d\mu$$
$$= \frac{2}{\pi^2}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{-\alpha}\int_0^\infty \frac{e^{-r(\phi+1)}}{r+\alpha}dr = -\frac{2}{\pi^2}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{\alpha\phi}Ei[-\alpha(\phi+1)] \quad \text{......(35)}$$

which is in agreement with a result given in (**10**).

Other integrals which may be evaluated using the Parseval relation for the transform (18) are as follows:

$$I_3 = \frac{2}{\pi^2}\int_0^\infty \mu\tanh(\mu\pi)P_{-\frac{1}{2}+i\mu}(\phi)K_{i\mu}(\alpha)K_{i\mu}(\beta)d\mu = \left(\frac{2\alpha\beta}{z\pi^3}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(z) \quad \text{...(36)}$$

where $z^2 = \alpha^2+\beta^2+2\alpha\beta\phi$.

$$I_4 = \frac{2}{\pi^2}\int_0^\infty \mu\sinh(\mu\pi)K_{2i\mu}(\alpha)K_{i\mu}(\beta)K_{i\mu}(\sigma)d\mu = \frac{\alpha}{4y}\exp\left[-\frac{(\sigma+\beta)}{2\sqrt{\sigma\beta}}y\right] \quad \text{...(37)}$$

where $y^2 = 4\sigma\beta+\alpha^2$. The values for I_3 and I_4 are in agreement with results given in (**9**).

$$I_5 = \frac{2}{\pi^2}\int_0^\infty \mu\sinh(\mu\pi)K_{i\mu}(\alpha)K_{i\mu}(\beta)K_{i\mu}(x)K_{i\mu}(y)d\mu = \tfrac{1}{2}K_0(R) \quad \text{......(38)}$$

where
$$R^2 = (\alpha\beta+xy)\left[\frac{1}{\alpha\beta}(\alpha^2+\beta^2)+\frac{1}{xy}(x^2+y^2)\right].$$

$$I_6 = \frac{2}{\pi^2}\int_0^\infty \mu\tanh(\mu\pi)P_{-\frac{1}{2}+i\mu}(\phi)K_{2i\mu}(\alpha)d\mu = \frac{\alpha}{2\pi^2}K_0\left[\alpha\left(\frac{\phi+1}{2}\right)^{\frac{1}{2}}\right]. \quad \text{...(39)}$$

The integral I_3 is evaluated using the transform pairs (28) and (29), I_4 using the pairs (28) and (30), I_5 using the pair (28) and I_6 using the pairs (29) and (30).

REFERENCES

(1) C. Snow, Hypergeometric and Legendre Functions etc., *N.B.S. Appl. Maths.* Series 19 (1952).

(2) M. J. Kontorovich and N. N. Lebedev, *J. Phys. U.S.S.R.*, **1** (1939), 229.

(3) N. N. Lebedev, (*Doklady*), *Akad. Sci. U.S.S.R.* (N.S.), **52** (1946), 655.

(4) N. N. Lebedev, (*Doklady*), *Akad. Nauk. S.S.S.R.* (N.S.), **58** (1947), 1007.

(5) N. N. Lebedev, (*Doklady*), *Akad. Nauk. S.S.S.R., Prinkl. Mat. Meh.* **13**, (1949), 465.

(6) R. D. Turner, *Quart. Appl. Math.*, **14** (1956), 63.

(7) C. P. Wells and A. Leitner, *Quart. Appl. Math.*, **15** (1958), 430.

(8) J. S. Lowndes, *Proc. Edin. Math. Soc.*, **11**, 3 (1959), 135.

(9) Erdelyi *et al.*, *Tables of Integral Transforms*, Vol. 2 (McGraw-Hill, 1954).

(10) J. S. Lowndes, *Siam Review* (April, 1961).

Royal College of Advanced Technology
Salford 5, Lancashire

https://doi.org/10.1017/S0013091500014437 Published online by Cambridge University Press