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On the Prime Ideals in a Commutative Ring

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Abstract. If n and m are positive integers, necessary and sufficient conditions are given for the existence of a finite commutative ring R with exactly n elements and exactly m prime ideals. Next, assuming the Axiom of Choice, it is proved that if R is a commutative ring and T is a commutative R-algebra which is generated by a set I, then each chain of prime ideals of T lying over the same prime ideal of R has at most $2^{|I|}$ elements. A polynomial ring example shows that the preceding result is best-possible.

1 Introduction

Throughout, all rings are commutative, with nonzero multiplicative identity element; and all algebras are unital. Our concern in this note is the relationship between $|R|$, the (cardinal) number of elements in a ring R, and $|\operatorname{Spec}(R)|$, the number of prime ideals of R. The answer for finite rings appears in Theorem 2.3 which, for any positive integers n and m, gives a necessary and sufficient condition that there exist a ring R such that $|R| = n$ and $|\operatorname{Spec}(R)| = m$. One consequence is that $|\operatorname{Spec}(R)| = o(|R|)$ as $|R| \to \infty$. The situation is far different for infinite rings. Indeed, as explained in Section 2, it follows from [3, Theorem 2] that (assuming the Continuum Hypothesis) there exists a denumerable ring T with a set S of pairwise incomparable prime ideals such that $|S| = \aleph_1$; thus, in this example, $|T| < |\operatorname{Spec}(T)|$. While the pairwise incomparable structure of S arises because the incomparability property (denoted INC, as in [10, p. 28]) is satisfied by any integral ring extension [10, Theorem 44], it should be noted that the structures studied in Theorem 2.3 also exhibit INC because all finite rings have (Krull) dimension 0. Accordingly, it seems natural to ask whether, if one restricts attention to "comparable" prime ideals, infinite rings support a conclusion with some of the flavour of Theorem 2.3. An affirmative answer is given in our main result, Theorem 2.6: if R is a ring and T is a commutative R-algebra which is generated by a set I, then each chain of prime ideals of T lying over the same prime ideal of R has cardinality at most $2^{|I|}$. (As usual, by a "chain" of prime ideals, we mean a set of prime ideals which is totally ordered by inclusion.) As noted via Proposition 2.7 and Remark 2.8(a), a polynomial ring example shows that Theorem 2.6 is best-possible.

The following comments about the literature help to place our results in perspective. The work in [3] on pairwise incomparability within infinite fibers was motivated, to some extent, by the role of INC in the work in [6] on finite integral extensions. Also, for singly-generated algebra extensions, INC is equivalent to a weak form of integrality called "primitivity": see [2, Theorem]. Thus, our emphasis on possibly infinite chains in Theorem 2.6 may be viewed as an effort to accommodate the general case without assuming even "weak" forms of integrality. Another motivation for studying possibly infinite chains is to ask if a weak form of INC is valid for arbitrary ring extensions. In this regard, we note that an

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infinite-chain analogue of the classical going-up property GU was given recently for domains in [9, Corollary]; and, for arbitrary (infinite) chains, analogues of the GU and GD (going-down) properties appear in [4, Theorem and Remark] for the more general ring-theoretic context. Given the classical importance of INC, GU and GD, we may thus view Theorem 2.6 as a generalization of INC-theoretic studies in the spirit of the GU- and GD-theoretic work in [9] and [4].

To ease the exposition, we place the technical details supporting the proof of Theorem 2.6 into Lemma 2.5. The proofs of the latter result and of Proposition 2.7 use facts about cardinal numbers and ordinal numbers which are recalled as needed. (Suitable references for such foundational material are the Appendix of [11] and the last third of [8].) In particular, we assume the Axiom of Choice in order to have the usual facts [8, pp. 96–98] about the arithmetic of infinite cardinal numbers.

2 Results

We begin by counting the prime ideals of a finite ring R. It will be useful to note that, since finite rings are artinian, it follows from standard structure theory [12, Theorem 3, p. 205; Remark 1, p. 208] that any such R is uniquely expressible as an internal direct product of finitely many finite local rings.

Proposition 2.1 *Let R be a finite ring, of cardinality $n := |R|$. Express R as the internal direct product $\prod\{R_j : 1 \leq j \leq m\}$ of finitely many local rings R_j. Then:*

(a) The number of prime ideals of R is $|\operatorname{Spec}(R)| = m \leq \log_2(n)$. Moreover, $m = \log_2(n)$ if and only if $R_j \cong \mathbb{Z}/2\mathbb{Z}$ for each j.
(b) Each nonempty chain of prime ideals in R has cardinality 1.

Proof Any artinian ring has Krull dimension 0 (*cf.* [12, Theorem 2, p. 203]). This fact gives (b) immediately and also ensures that each of the rings R_j has a unique prime ideal. Since

$$\operatorname{Spec}(R) = \operatorname{Spec}\Bigl(\prod\{R_j : 1 \leq j \leq m\}\Bigr) = \dot{\bigcup}\{\operatorname{Spec}(R_j) : 1 \leq j \leq m\},$$

we therefore have that

$$|\operatorname{Spec}(R)| = |\dot{\bigcup}\{\operatorname{Spec}(R_j) : 1 \leq j \leq m\}| = \sum |\operatorname{Spec}(R_j)| = \sum 1 = m.$$

Moreover, $n = |R| = \prod\{|R_j| : 1 \leq j \leq m\} \geq 2^m$; that is, $m \leq \log_2(n)$. Equality holds if and only if $|R_j| = 2$ for each j; that is, if and only if $R_j \cong \mathbb{Z}/2\mathbb{Z}$ for each j. Thus, (a) has been established, and the proof is complete. ■

Before giving our realization result for finite rings in Theorem 2.3, we include a known fact and sketch its proof (for lack of a suitable reference).

Lemma 2.2 *Any finite local ring has prime-power cardinality.*

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Proof Let R be a finite local ring, of cardinality $|R|$. Since R is module-finite, and hence integral, over its prime subring A, it follows via GU and INC [10, Theorem 44] (and the hypothesis that R is local) that A is local. Therefore, we may identify $A = \mathbb{Z}/p^k\mathbb{Z}$, for some prime number p and some positive integer k. Hence, by Lagrange's Theorem, $|R| = [R : A]|A| = [R : A]p^k$. It suffices to show that $|R/A| = [R : A]$ is an integral power of p.

Suppose not. Then, by Sylow's Theorem, the additive group R/A has a Sylow-q subgroup, say Q, for some prime number $q \neq p$. As Q is solvable, Q has an element of order q. If $\xi + A$ is such an element, then $\xi \in R \setminus A$ and $q\xi \in A$. Now, since q and p are distinct prime numbers, $\alpha q + \beta p^k = 1$ for some integers α, β. Hence, $\xi = \alpha q\xi + \beta p^k \xi$. However, $p^k\xi = 0$ since A being the prime subring of R implies that R has characteristic p^k. Thus, $\xi = \alpha q\xi \in A$, the desired contradiction, to complete the proof. ■

Theorem 2.3 *Let $n = \prod\{q_i^{e_i} : 1 \leq i \leq s\}$ be the prime-power decomposition of a positive integer $n \geq 2$, and let m be a positive integer. Then there exists a ring R such that $|R| = n$ and $|\operatorname{Spec}(R)| = m$ if and only if $s \leq m \leq \sum e_i$.*

Proof Suppose that a ring R satisfies $|R| = n$ and $|\operatorname{Spec}(R)| = m$. By Proposition 2.1(a), $R = \prod\{R_j : 1 \leq j \leq m\}$ for finitely many local rings R_j. For each j, Lemma 2.2 gives that $|R_j| = p_j^{f_j}$, for some prime number p_j and some positive integer f_j. Hence, $\prod\{q_i^{e_i} : 1 \leq i \leq s\} = n = |R| = \prod\{|R_j| : 1 \leq j \leq m\} = \prod\{p_j^{f_j} : 1 \leq j \leq m\}$. Relabel the rings R_j so that $p_1, \dots, p_d$ are all the distinct elements in the list $p_1, \dots, p_m$. For each $k = 1, \dots, d$, let

$$g_k = \sum\{f_j : 1 \leq j \leq m \text{ and } p_j = p_k\}.$$

Observe that $\sum\{g_k : 1 \leq k \leq d\} = \sum\{f_j : 1 \leq j \leq m\}$ and that the above equality of products may be rewritten as

$$\prod\{q_i^{e_i} : 1 \leq i \leq s\} = \prod\{p_k^{g_k} : 1 \leq k \leq d\}.$$

By unique factorization, $s = d$ and, after relabeling the rings $R_1, \dots, R_d$, we have that $q_i = p_i$ and $e_i = g_i$ for each $i = 1, \dots, s$. It follows that

$$s = d \leq m \leq \sum\{f_j : 1 \leq j \leq m\} = \sum\{g_k : 1 \leq k \leq d\} = \sum\{e_i : 1 \leq i \leq s\}.$$

This completes the proof of the "only if" assertion.

Conversely, suppose that $s \leq m \leq \sum e_i$. We shall produce a ring $R = \prod\{R_{ij} : 1 \leq i \leq s, 1 \leq j \leq k_i\}$ such that $|R| = n$ and $|\operatorname{Spec}(R)| = m$. For this, Proposition 2.1(a) shows that it suffices to let

$$R_{ij} := \mathbb{Z}/q_i^{e_{ij}}\mathbb{Z},$$

where the positive integers k_i, e_{ij} are chosen so as to satisfy

$$\sum\{e_{ij} : 1 \leq j \leq k_i\} = e_i \quad \text{for} \quad 1 \leq i \leq s \quad \text{and} \quad \sum\{k_i : 1 \leq i \leq s\} = m.$$

It remains only to prove that such a choice is possible. We do this by "induction" on m, subject to the condition that $s \leq m \leq \sum e_i$. If $m = s$, this is done easily: for each i, take

$k_i = 1$ and $e_{i1} = e_i$. For the "induction step", suppose that $s \le m-1 < m \le \sum e_i$ and that positive integers k_i^*, e_{ij}^* exist such that

$$\sum\{e_{ij}^* : 1 \le j \le k_i^*\} = e_i \quad \text{for} \quad 1 \le i \le s \quad \text{and} \quad \sum\{k_i^* : 1 \le i \le s\} = m-1.$$

Observe that $k_i^* \le e_i$ for each i. If $k_i^* = e_i$ for each i, then

$$m \le \sum e_i = \sum k_i^* = m-1,$$

a contradiction. Hence, $k_\lambda^* < e_\lambda$ for some λ, and so $e_{\lambda\mu}^* > 1$ for some μ. To produce the integers k_i, e_{ij}, make only the following changes to the data k_i^*, e_{ij}^*: replace k_λ^* with $k_\lambda := k_\lambda^* + 1$ and replace $e_{\lambda\mu}^*$ with two entries, namely $e_{\lambda\mu}^* - 1$ and 1. Evidently,

$$\sum\{e_{ij} : 1 \le j \le k_i\} = \sum\{e_{ij}^* : 1 \le j \le k_i^*\} = e_i \quad \text{for} \quad 1 \le i \le s \quad \text{and}$$
$$\sum\{k_i : 1 \le i \le s\} = \sum\{k_i^* : 1 \le i \le s\} + 1 = m,$$

as desired. This completes the "induction step" and the proof of the "if" assertion. ■

Remark 2.4

(a) Theorem 2.3 leads to another proof of the inequality $m \le \log_2(n)$ in Proposition 2.1. Indeed, given R as in Theorem 2.3, we have that

$$n = \prod q_i^{e_i} \ge \prod 2^{e_i} = 2^{\sum e_i} \ge 2^m.$$

(b) Another feature of the above inequality deserves to be stressed, namely that $|\operatorname{Spec}(R)| = o(|R|)$ as $|R| \to \infty$; *i.e.*, $\lim_{|R|\to\infty} \frac{|\operatorname{Spec}(R)|}{|R|} = 0$. By the above results, this assertion reduces to the fact that $\lim_{n\to\infty} \frac{\log_2(n)}{n} = 0$. However, as we recall below, infinite rings R need not satisfy $|\operatorname{Spec}(R)| < |R|$.

The limit result established above is somewhat reminiscent of the Prime Number Theorem: if $\pi(x)$ denotes the number of positive prime numbers which are less than or equal to a given real number x, then $\pi(x)/\big(x/\ln(x)\big) \to 1$ as $x \to \infty$. In particular, $\pi(x) = o(x)$. However, an exact analogy fails, since $|\operatorname{Spec}(R)|/\big(|R|/\ln(|R|)\big) = m/\big(n/\ln(n)\big) \to 0$ as $|R| \to \infty$. Indeed, it suffices to observe that $m \le \sum e_i \le \log_2(n)$, so that $m/\big(n/\ln(n)\big) \le \big(\ln(n)\big)^2/(n \ln 2)$, and the assertion follows from a standard limit theorem and L'Hôpital's rule. Moreover, the diversity of behaviour permitted by the bounds in Theorem 2.3 yields that as $|R| \to \infty$, $\liminf |\operatorname{Spec}(R)|/\log_2(|R|) = 0$ and $\limsup |\operatorname{Spec}(R)|/\log_2(|R|) = 1$. Indeed, for the assertion about lim inf, it suffices to consider, for each positive prime $p \ge 2$, the field with p elements; and for the lim sup assertion, it suffices to consider, for each integral power of 2, $n = 2^m \ge 2$, a ring which is the product of m copies of $\mathbb{Z}/2\mathbb{Z}$.

We turn now to the analogous questions for infinite rings. The situation here is more complex than for finite rings. For instance, any infinite cardinal number is realized as

$|K|$, for a suitable field K, although $|\operatorname{Spec}(K)| = 1$. More importantly, in contrast with the behaviour noted above for finite rings, an infinite ring T may satisfy $|T| < |\operatorname{Spec}(T)|$. Indeed, [3, Theorem 2] leads to an example of a denumerable ring T for which $|\operatorname{Spec}(T)| = 2^{|T|}$. In this example, T arises as a countably generated integral extension of a denumerable ring R; thus (assuming the Continuum Hypothesis), $2^{|T|} = \aleph_1$. Moreover, by the proof of [3, Theorem 2], it can be arranged that T has denumerably many fibers, each of cardinality $\aleph_1$, above R. Thus, since integral extensions satisfy INC, it can be arranged that $\operatorname{Spec}(T)$ has infinitely many subsets each of which consists of pairwise incomparable elements and each of which has cardinality greater than $|T|$.

We next give a technical result which will reduce the proof of our main result, Theorem 2.6, to standard techniques. Notice that Lemma 2.5 may be viewed as the transfinite counterpart of the result [10, p. 109] that if K is any field, then the polynomial ring $K[X_1, \ldots, X_n]$ has (Krull) dimension n.

Lemma 2.5 *Let K be a field, let I be an infinite set, let $\{X_i : i \in I\}$ be a set of algebraically independent indeterminates over K, and put $T := K[\{X_i : i \in I\}]$. Then any chain of prime ideals of T has cardinality at most $2^{|I|}$.*

Proof Let $\mathcal{S}$ denote the collection of finite subsets of $\{X_i : i \in I\}$. It will be useful to note, using basic facts about cardinal numbers, that $|\mathcal{S}| = |I|$ and $(\aleph_0)^{|I|} = 2^{|I|}$. For $s \in \mathcal{S}$, let $K[s] := K[\{X_i : i \in s\}]$. Note that $\mathcal{S}$ is partially ordered via inclusion and that $\{K[s] : s \in \mathcal{S}\}$ is a directed system.

Since T is then the direct limit of the polynomial rings $K[s]$ as s runs over $\mathcal{S}$, it follows (*cf.* [7, Proposition 6.1.2, p. 128]) that each prime ideal P of T is the direct limit (in fact, directed union, indexed by $s \in \mathcal{S}$) of $P \cap K[s]$. Now, consider any chain $\mathcal{C} = \{P_j : j \in J\}$ of prime ideals of T. For each $j \in J$ and $s \in \mathcal{S}$, put $P_{js} := P_j \cap K[s]$. By the above remark, if $j \in J$, then $P_j = \operatorname{dir lim} P_{js} = \bigcup P_{js}$. Moreover, for each $s \in \mathcal{S}$, $\mathcal{C}$ induces the chain $\mathcal{C}_s := \{P_{js} : j \in J\}$ of prime ideals of $K[s]$, and $|\mathcal{C}_s| \le \dim(K[s]) + 1 = |s| + 1 < \aleph_0$. By the above remark, $|\mathcal{C}|$ is the number of ways of choosing a "compatible family" of prime ideals from $\bigcup\{\mathcal{C}_s : s \in \mathcal{S}\}$. Hence,

$$|\mathcal{C}| \le \prod\{|\mathcal{C}_s| : s \in \mathcal{S}\} \le (\aleph_0)^{|\mathcal{S}|} = (\aleph_0)^{|I|} = 2^{|I|},$$

as asserted. ∎

Theorem 2.6 *Let R be a ring and let T be a ring which is generated as an R-algebra by a set I. Then each chain of prime ideals in T which lie over the same prime ideal of R has at most $2^{|I|}$ elements.*

Proof Since T is an R-algebra, there is a (unital) ring-homomorphism $f\colon R \to T$. By replacing R with $f(R)$, we may assume without loss of generality that R is a subring of T (as the generating set I remains unchanged). Now, consider any chain $\{Q_\alpha\}$ of prime ideals of T such that there exists a prime ideal P of R such that $Q_\alpha \cap R = P$ for each α. By [10, Theorem 9], $Q := \bigcap Q_\alpha$ is a prime ideal of T, and without loss of generality, we may suppose that Q is a member of the given chain. (Notice that $P = Q \cap R$.) By passing from $R \subset T$ to the canonical inclusion $R/P \subset T/Q$, we may suppose that R and T are integral domains and that the prime ideal $\{0\} = Q$ is the smallest member of the given chain in Spec(T). (In this reduction, I is replaced by the set of mod-Q cosets represented by the elements of the original set I; in particular, I is replaced by a set which does not have greater cardinality than that of I.) Next, we may replace $R \subset T$ with the canonical inclusion $R_{R\setminus\{0\}} \subset T_{R\setminus\{0\}}$. Observe that $K := R_{R\setminus\{0\}}$ is a field. (Also, I remains unchanged and the chain $\{Q_\alpha\}$ induces, and is replaced by, a chain $\{Q_\alpha T_{R\setminus\{0\}}\}$ of the same cardinality.) Now, if $T = K[\{u_i : i \in I\}]$, let $\{X_i : i \in I\}$ be a set of algebraically independent indeterminates over K, and let $g\colon K[\{X_i : i \in I\}] \to T$ be the K-algebra homomorphism sending X_i to u_i for each $i \in I$. Since g is surjective, it follows from a standard homomorphism theorem that $\{g^{-1}(Q_\alpha)\}$ is a chain of prime ideals of $K[\{X_i : i \in I\}]$ which has the same cardinality as $\{Q_\alpha\}$. We thus have our final reduction: $R \subset T$ may be replaced by $K \subset K[\{X_i : i \in I\}]$, where K is a field. If I is finite, say $\{1, 2, \ldots, n\}$, then

$$|\{Q_\alpha\}| \leq \dim(K[X_1, \ldots, X_n]) + 1 = n + 1 \leq 2^n = 2^{|I|}.$$

(The preceding assertion also holds trivially if I is empty, for then $T = K$ is a field and we may take $n = 0$.) Finally, if I is infinite, Lemma 2.5 yields that $|\{Q_\alpha\}| \leq 2^{|I|}$. ■

The next result, which is included for lack of a suitable reference, will be used in Remark 2.8(a). The latter result shows that the upper bounds in the statements of Lemma 2.5 and Theorem 2.6 are best-possible. Proposition 2.7 and the present formulation of Remark 2.8(a) for arbitrary infinite I are due to the referee. Our earlier version of Remark 2.8(a) had treated only the case of denumerable I, following an example of Sylvia and Roger Wiegand. We thank the referee and the Wiegands for kindly permitting us to include their contributions to Proposition 2.7 and Remark 2.8(a).

Proposition 2.7 *Let I be an infinite set. Then there is a chain of subsets of I having cardinality $2^{|I|}$.*

Proof Let $\alpha = |I|$. Then we can view α as an ordinal number with the property that each ordinal number less than α has strictly smaller cardinality. (In fact, if we use the formal approach of Kelley, any ordinal number γ may be identified with the set of all ordinal numbers less than γ [11, Theorem 119].) Thus, α is a well-ordered set whose members are the ordinal numbers less than α and such that for each $\beta < \alpha$ (*i.e.*, $\beta \in \alpha$), we have that the cardinality of the set β is strictly less than (the cardinality of) α. Now, let $\mathcal{A}$ denote the set α^α of all functions from α into α. Observe that $|\mathcal{A}| = |I|^{|I|} = 2^{|I|}$. Furthermore, $\mathcal{A}$ can be totally ordered via the lexicographic ordering: for distinct $f, g \in \mathcal{A}$, we declare $f < g$ if $f(\beta) < g(\beta)$, where $\beta \in \alpha$ is the smallest ordinal for which $f(\beta) \neq g(\beta)$.

Next, let $\mathcal{B}$ denote the set of "eventually constant" functions in $\mathcal{A}$; that is, $\mathcal{B} = \{f \in \mathcal{A} :$ there exists $\gamma \in \alpha$ such that $f(\delta_1) = f(\delta_2)$ whenever $\delta_1, \delta_2 > \gamma$ in $\alpha\}$. Since for any such γ, the cardinality of γ is less than that of α, it follows easily from the Generalized Continuum Hypothesis that $|\mathcal{B}| = |I|$, whence $|\mathcal{B}| < 2^{|I|} = |\mathcal{A}|$.

Now, let $\mathcal{A}_1$ denote the set of all functions in $\mathcal{A}$ which are not "eventually zero"; that is, $\mathcal{A}_1 := \{f \in \mathcal{A} :$ there does not exist $\gamma \in \alpha$ such that $f(\delta) = 0$ for all $\delta > \gamma$ in $\alpha\}$. Reasoning as above via the Generalized Continuum Hypothesis, we see that the number of functions in $\mathcal{A}$ which *are* "eventually zero" is $|I|$, and so, in view of the earlier calculation of

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$|\mathcal{A}|$, we conclude that $|\mathcal{A}_1| = 2^{|I|}$. We claim that $\mathcal{B}$ is "dense" in $\mathcal{A}_1$, in the following sense: for each $f < g$ in $\mathcal{A}_1$, there exists $h \in \mathcal{B}$ such that $f < h < g$. Indeed, if β is minimal such that $f(\beta) < g(\beta)$, then it suffices to define $h \in \alpha^\alpha$ by $h(\delta) := g(\delta)$ for all $\delta \leq \beta$ and $h(\xi) := 0$ if $\xi > \beta$. (Our later need for this property explains why we introduced $\mathcal{A}_1$, for if we knew only that $g \in \mathcal{A}$, then the above method would yield only that $f < h \leq g$.)

For each $f \in \mathcal{A}_1$, put $\mathcal{B}_f := \{k \in \mathcal{B} : k < f\}$. Clearly, $\mathcal{C} := \{\mathcal{B}_f : f \in \mathcal{A}_1\}$ is a chain of subsets of $\mathcal{B}$. In fact, by the "density" established above, if $f < g$ in $\mathcal{A}_1$, then $\mathcal{B}_f$ is a proper subset of $\mathcal{B}_g$; thus, the cardinality of $\mathcal{C}$ is $|\mathcal{A}_1| = 2^{|I|}$. Since $|\mathcal{B}| = |I|$, we can use a bijection $\mathcal{B} \to I$ to "translate" $\mathcal{C}$ into a chain of subsets of I having cardinality $2^{|I|}$, as desired. ■

Remark 2.8 (a) To show that Lemma 2.5 and Theorem 2.6 are best-possible, let R be any field K, let I be an infinite set, let $\{X_i : i \in I\}$ be a set of algebraically independent indeterminates over K, and let $T := K[\{X_i : i \in I\}]$. We claim that some chain of prime ideals of T has cardinality $2^{|I|}$.

To prove the claim, let $(J, \leq)$ be a well-ordered set of cardinality $2^{|I|}$. By Proposition 2.7, there exists a chain $\{s_j : j \in J\}$ of subsets of I, with s_j a proper subset of s_k whenever $j < k$ in J. For each $j \in J$, let A_j be the ideal of T which is generated by $\{X_i : i \in s_j\}$. An easy degree argument shows that A_j is a proper subset of A_k whenever $j < k$ in J. Thus, $\{A_j\}$ is a chain of prime ideals of T having cardinality $|J| = 2^{|I|}$, completing the proof of the claim.

(b) The proof of Lemma 2.5 used a result concerning compatible families of prime ideals in a directed system [7, Proposition 6.1.2, p. 128]. This result from [7] has often been cited or reestablished (*cf.* [1, Exercice 11, p. VIII.82], [5, Lemma 2.1]). In applying such results to Krull dimension, one should note that all infinite Krull dimensions are viewed as the same, namely ∞. An uncritical use of such results in which "∞" is replaced by various infinite cardinal numbers would lead to an absurdity in the proof of Lemma 2.5, in case I is denumerable, namely that $\dim(K[X_1, \dots, X_n, \dots]) \leq \sup\{\dim(K[X_1, \dots, X_n]) : n \in \mathbb{P}\} = \sup\{n : n \in \mathbb{P}\} = \aleph_0$, rather than ∞ (where $\mathbb{P}$ denotes the set of positive integers). If one fails to recognize that the classical theory of Krull dimension does not distinguish infinite values, one might well then be tempted to conclude that any chain of prime ideals of $K[X_1, \dots, X_n, \dots]$ has cardinality at most $(\aleph_0) + 1 = \aleph_0$. The example in (a) and Theorem 2.6 combine to show that such a conclusion is wrong: for denumerable I, the correct upper bound is $2^{\aleph_0}$.

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