Proceedings of the Edinburgh Mathematical Society (1979), 22, 263–269 ©

REAL PARTS OF QUASI-NILPOTENT OPERATORS

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(Received 15th May 1978)

1. Introduction

The purpose of this paper is to answer the question: which self-adjoint operators on a separable Hilbert space are the real parts of quasi-nilpotent operators? In the finite-dimensional case the answer is: self-adjoint operators with trace zero. In the infinite dimensional case, we show that a self-adjoint operator is the real part of a quasi-nilpotent operator if and only if the convex hull of its essential spectrum contains zero. We begin by considering the finite dimensional case.

Theorem 1. *A self-adjoint square matrix is the real part of a nilpotent matrix if and only if it has trace zero.*

Proof. Let A be the real part of a nilpotent matrix N. Since N is unitarily equivalent to an upper triangular matrix with main diagonal 0, we have $\operatorname{tr} N = 0$, $\operatorname{tr} N^* = 0$, and $\operatorname{tr} A = \frac{1}{2}(\operatorname{tr} N + \operatorname{tr} N^*) = 0$.

Conversely, if $\operatorname{tr} A = 0$, then A is unitarily equivalent to a matrix B with main diagonal zero. (Proof: by induction on the size of the matrix, using the observation that 0 belongs to the numerical range of A; for details see (1)). Then obviously $2B = N + N^*$ with N strictly upper triangular and hence nilpotent. But $B = \operatorname{Re} N$ and it follows that A is the real part of a nilpotent matrix.

2. The main result

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. We will denote the essential spectrum of an operator T by $\sigma_e(T)$, and the convex hull of a set S by conv S. If T is an operator on $\mathcal{H}$, then $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ denotes the matrix of T with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ into *infinite-dimensional* subspaces.

Theorem 2. *If A is the real part of a quasi-nilpotent operator Q, then* 0 *belongs to the convex hull of the essential spectrum of A.*

Proof. First we show that A cannot be positive and invertible. If A is positive and invertible, B self-adjoint, and $T = A + iB$, then $A^{-1/2}TA^{-1/2} = I + iA^{-1/2}BA^{-1/2}$ is invertible, and so T cannot be quasi-nilpotent. By considering $-A$ in place of A we also conclude that A cannot be negative and invertible, and therefore zero belongs to conv $\sigma(A)$. The same argument applies in any C^*-algebra, in particular in the Calkin algebra $\mathcal{A}$, and since the image of Q in $\mathcal{A}$ is quasi-nilpotent, we have $0 \in \operatorname{conv} \sigma_e(A)$.

To prove the converse of Theorem 2 we need several Lemmas.

Lemma 1. *If A is self-adjoint and $0 \in \operatorname{conv} \sigma_e(A)$, then $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $0 \in \operatorname{conv} \sigma_e(A_i)$ for $i = 1, 2$.*

Proof. If $\sigma_e(A)$ contains a positive number and a negative number, then $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ with $B \geq 0$ and $C \leq 0$. Take $\mathcal{M}$ to be the direct sum of a "half" of the first coordinate space which reduces B and a half of the second coordinate space which reduces C. The matrix of A relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ has the required property.

To complete the proof it is enough to consider the case where A is essentially positive. If the null space of A is infinite-dimensional, we can take $\mathcal{M}$ to be half of the null space. Otherwise there must exist a sequence $\{a_n\}$ of positive numbers decreasing to zero such that $E(a_{n+1}, a_n)\mathcal{H}$ is non-zero for every n, where E is the spectral resolution of A. If δ is the union of the intervals (a_{n+1}, a_n) for n even and $\mathcal{M} = E(\delta)\mathcal{H}$, then the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ has the required property.

Lemma 2. *If T is an operator with $0 \in \operatorname{conv} \sigma_e(T)$, then $T = \begin{pmatrix} * & * \\ * & K \end{pmatrix}$ with K a compact operator.*

Proof. A result of Fillmore, Stampfli, and Williams (**2**, Theorem 5.1) asserts this conclusion if 0 lies in the essential numerical range of T. But the essential numerical range contains the convex hull of the essential spectrum.

Lemma 3. *If A and B are real parts of quasi-nilpotent operators, then so is $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$.*

Proof. Let A (respectively B) be the real part of the quasi-nilpotent operator Q_1 (respectively Q_2), and let $Q = \begin{pmatrix} Q_1 & 2C \\ 0 & Q_2 \end{pmatrix}$. The real part of Q is obviously $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$, and furthermore $Q - \lambda$ is invertible for every nonzero λ, so Q is quasi-nilpotent.

Theorem 3. *If A is self-adjoint and $0 \in \operatorname{conv} \sigma_e(A)$, then A is the real part of a quasi-nilpotent operator.*

Proof. By Lemmas 1 and 2,

$$A = \begin{pmatrix} * & * & 0 & 0 \\ * & K_1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & K_2 \end{pmatrix}$$

with K_1 and K_2 compact. By interchanging the second and fourth direct summands the matrix becomes

$$\begin{pmatrix} * & 0 & * & * \\ 0 & K_2 & * & * \\ * & * & * & 0 \\ * & * & 0 & K_1 \end{pmatrix}.$$

https://doi.org/10.1017/S0013091500016436 Published online by Cambridge University Press

In view of Lemma 3, it is now enough to prove that if K is compact then $\begin{pmatrix} * & 0 \\ 0 & K \end{pmatrix}$ is the real part of a quasi-nilpotent. (The reduction of the general problem to this class of operators is similar to the one used by Sam Holby (3) in treating the Brown-Pearcy commutator theorem.) So let $A = \begin{pmatrix} B & 0 \\ 0 & K \end{pmatrix}$; since we may write K as a direct sum $K_1 \oplus K_2$ where K_2 is of trace-class, and then absorb K_1 into B, we will assume that K is of trace-class. Write K as a direct sum of operators K_n for $n \neq 0$, and let $K_0 = B$. We may assume that these operators act on a fixed Hilbert space $\mathcal{K}$, so that

$$A = \cdots \oplus K_{-1} \oplus K_0 \oplus K_1 \oplus \cdots$$

on $l^2(\mathcal{K})$. Let $M = \Sigma \|K_n\|$, and observe that $M < \infty$ since

$$\sum_{n \neq 0} \|K_n\| \leq \sum \operatorname{tr}(K_n^* K_n)^{1/2} = \operatorname{tr}(K^*K)^{1/2}.$$

Let $L^2([0, 1], \mathcal{K})$ be the Hilbert space of $\mathcal{K}$-valued weakly measurable functions on $[0, 1]$ with

$$\|f\|^2 = \int_0^1 \|f(t)\|^2 \, dt < \infty.$$

We will construct a quasi-nilpotent integral operator on $L^2([0, 1], \mathcal{K})$ whose real part is unitarily equivalent to A. To this end, let

$$\phi(s, t) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n(s-t)} K_n$$

for $0 \leq s$, $t \leq 1$. The series converges uniformly and defines a continuous function from $[0, 1] \times [0, 1]$ into $\mathcal{B}(\mathcal{K})$. Define an operator V on $L^2([0, 1], \mathcal{K})$ by

$$(Vf)(s) = 2 \int_0^s \phi(s, t) f(t) \, dt$$

(see (4), Sections 3.5 and 3.7 for basic facts about vector-valued (Bochner) integrals). Such Volterra-type integral operators are well-known to be quasi-nilpotent, at least in the scalar case. For completeness we include a proof of this fact in Lemma 4 below. It remains only to show that Re V is unitarily equivalent to A. First observe that

$$(V^*f)(s) = 2 \int_s^1 \phi(t, s)^* f(t) \, dt$$

and that $\phi(t, s)^* = \phi(s, t)$, and therefore

$$((\operatorname{Re} V)f)(s) = \int_0^1 \phi(s, t) f(t) \, dt.$$

If $f \in L^2([0, 1], \mathcal{K})$ has Fourier expansion

$$f(t) = \sum e^{2\pi i n t} x_n, \; x_n \in \mathcal{K}, \; \sum \|x_n\|^2 = \|f\|^2$$

then easily

$$((\operatorname{Re} V)f)(s) = \sum e^{2\pi i n s} K_n x_n,$$

and it follows that A is unitarily equivalent to Re V. This is because the subspaces $\mathscr{K}_n$ consisting of all $g(t) = e^{2\pi int}x$, $x \in \mathscr{K}$, are pairwise orthogonal, span $L^2([0, 1], \mathscr{K})$, and are invariant for Re V with Re $V|\mathscr{K}_n$ unitarily equivalent to K_n.

Corollary 1. *The set of real parts of quasi-nilpotent operators is closed in the uniform operator topology.*

Proof. If $A_n \to A$ in norm and F is a closed set with $\sigma_e(A_n) \cap F \neq \emptyset$ for every n, then $\sigma_e(A) \cap F \neq \emptyset$ by the upper semi-continuity of the spectrum. Applying this with $F = [0, +\infty)$ and $F = (-\infty, 0]$ gives the result.

Corollary 2. *If A is a compact self-adjoint operator then A is the real part of a compact quasi-nilpotent operator.*

Proof. We need only show that the operator V constructed in the proof of Theorem 3 (with K_0 compact) is a compact operator. Let V_n be the operator defined by

$$(V_n f)(s) = K_n \int_0^s e^{2\pi in(s-t)} f(t)\, dt.$$

Then V_n is compact for all n, $V = \Sigma_{n=-\infty}^{\infty} V_n$, and $\Sigma \|V_n\| \leq \Sigma \|K_n\| < \infty$, so V is compact.

Lemma 4. *Let $\mathscr{K}$ be a Hilbert space, and let $(s, t) \to K(s, t) \in \mathscr{B}(\mathscr{K})$ be weakly measurable on $0 \leq t \leq s \leq 1$ and satisfy $M = \int_0^1 \int_0^s \|K(s, t)\|^2\, dt\, ds < \infty$. Then the operator V defined on $L^2([0, 1], \mathscr{K})$ by*

$$(Vf)(s) = \int_0^s K(s, t) f(t)\, dt$$

is quasi-nilpotent.

Proof. Let $f \in L^2([0, 1], \mathscr{K})$. Then

$$\begin{aligned} \|(Vf)(s)\|^2 &= \Big\| \int_0^s K(s, t) f(t)\, dt \Big\|^2 \\ &\leq \int_0^s \|K(s, t)\|^2\, dt \int_0^s \|f(t)\|^2\, dt \\ &\leq \alpha(s) \|f\|^2, \end{aligned}$$

where $\alpha(s) = \int_0^s \|K(s, t)\|^2\, dt$. By induction

$$\|(V^n f)(s)\|^2 \leq \alpha(s) \int_0^s \alpha(s_1) \int_0^{s_1} \alpha(s_2) \ldots \int_0^{s_{n-2}} \alpha(s_{n-1})\, ds_{n-1} \ldots ds_1 \|f\|^2$$

for all $n \geq 1$, and therefore

$$\|V^n f\|^2 \leq \int_0^1 \alpha(s_1) \int_0^{s_1} \alpha(s_2) \ldots \int_0^{s_{n-1}} \alpha(s_n)\, ds_n \ldots ds_1 \|f\|^2$$
$$= \frac{1}{n!}\left(\int_0^1 \alpha(s)\, ds\right)^n \|f\|^2$$
$$= \frac{1}{n!} M^n \|f\|^2,$$

so that $\|V^n\|^{1/n} \to 0$.

3. Operators with imaginary spectrum and nilpotent operators

Which operators are the real parts of nilpotent operators? Which operators are the real parts of operators with imaginary spectrum? The following result answers the second question.

Theorem 4. *For any self-adjoint operator A the following conditions are equivalent*:

(a) *A is the real part of an operator with (purely) imaginary spectrum*;

(b) *A is the real part of a quasi-nilpotent operator*;

(c) $0 \in \operatorname{conv} \sigma_e(A)$.

Proof. The implication (c) $\Rightarrow$ (b) is Theorem 3, and (b) $\Rightarrow$ (a) is trivial. The proof of (a) $\Rightarrow$ (c) is similar to the proof of Theorem 2. If $T = A + iB$ and if π is the canonical projection of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra, let $t = \pi(T)$, $a = \pi(A)$, $b = \pi(B)$. If a is positive and invertible, and λ is a real number, then $a^{-1/2}(t - i\lambda)a^{-1/2} = 1 + ia^{-1/2}(b - \lambda)a^{-1/2}$ is invertible, and so $t - i\lambda$ is invertible for every λ. If t has purely imaginary spectrum, then a cannot be positive and invertible. The same applies to $-a$, so $0 \in \operatorname{conv} \sigma(a) = \operatorname{conv} \sigma_e(A)$.

Let $\mathcal{R}e\mathcal{N}$ denote the class of operators which are real parts of nilpotent operators. The next result shows that the norm-closure of $\mathcal{R}e\mathcal{N}$ is the set of real parts of quasi-nilpotent operators, and also gives another characterisation of the latter.

Theorem 5. *For any self-adjoint operator A the following conditions are equivalent*:

(a) *A belongs to the norm-closure of $\mathcal{R}e\mathcal{N}$,*

(b) *A is the real part of a quasi-nilpotent operator*;

(c) *for each $\epsilon > 0$, there exists a decomposition $I = E_1 + \cdots + E_m$ into pairwise orthogonal projections such that $\|E_k A E_k\| < \epsilon$ for all k.*

Proof. The implication (a) $\Rightarrow$ (b) follows from Corollary 1. To prove (b) $\Rightarrow$ (c) it suffices to show that every quasi-nilpotent operator satisfies condition (c). Let Q be a quasi-nilpotent operator and let $\epsilon > 0$. By a theorem of Rota (6), there is an invertible operator S such that $\|S^{-1}QS\| < \epsilon/2$. If $S = PU$ is the polar decomposition of S, where P is positive and U is unitary, then $\|P^{-1}QP\| = \|S^{-1}QS\| < \epsilon/2$. By the spectral theorem, there is a positive operator P_0 with finite spectrum $\{\lambda_1, \ldots, \lambda_m\}$ such that

$\|P_0^{-1}QP_0 - P^{-1}QP\| < \epsilon/2$. If E_k is the spectral projection for P_0 associated with λ_k, then $P_0E_k = \lambda_kE_k$ and $E_kP_0^{-1} = \lambda_k^{-1}E_k$, and so

$$\|E_kQE_k\| = \|E_kP_0^{-1}QP_0E_k\| \leq \|P_0^{-1}QP_0\| < \epsilon.$$

Finally we show (c) ⇒ (a). Let A be a self-adjoint operator, let $\epsilon > 0$, and let E_k be as in condition (c). Let $T = 2\sum_{j<k} E_jAE_k$. Then $T^m = 0$ and

$$\operatorname{Re} T = \sum_{j\neq k} E_jAE_k = A - \sum E_kAE_k.$$

Therefore

$$\|\operatorname{Re} T - A\| = \left\|\sum E_kAE_k\right\| = \max \|E_kAE_k\| < \epsilon$$

and (a) follows.

4. Final remarks

1. An operator A is called a self-commutator if $A = T^*T - TT^*$ for some operator T. Radjavi (5) showed that a self-adjoint operator A is a self-commutator if and only if $0 \in \operatorname{conv} \sigma_e(A)$. Consequently the class of self-commutators is the same as the class of real parts of quasi-nilpotents. It should be noted that the last assertion is also valid on finite-dimensional spaces (both coincide with the class of operators of trace zero).
2. The set of real parts of quasi-nilpotents is closed in the uniform operator topology. However, the set of real parts of nilpotent operators $\mathcal{R}e\mathcal{N}$ is not (except on finite-dimensional spaces). For example a projection of rank one is the real part of a quasi-nilpotent by Theorem 3, and hence belongs to $\overline{\mathcal{R}e\mathcal{N}}$ by Theorem 5. But if N is a nilpotent operator, then there is an orthonormal basis $\{e_n\}$ such that $(Ne_n, e_n) = 0$ for all n, so that if $\operatorname{Re} N$ is of trace class, it must have trace zero. In particular $\operatorname{Re} N$ can never be a projection of rank one.
3. We have confined our attention to separable Hilbert spaces. However, the techniques used here can be modified to allow us to describe the real parts of quasi-nilpotents on a non-separable Hilbert space $\mathcal{H}$. Let $\mathcal{J}$ be the largest closed ideal in $\mathcal{B}(\mathcal{H})$, i.e., the closure of the ideal of all operators A such that the dimension of the closure of the range of A is less than the dimension of $\mathcal{H}$. Let π_0 denote the canonical projection of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{J}$. The analogue of Theorems 2, 3 and 4 is the following theorem.

Theorem. *Let A be a self-adjoint operator on an infinite-dimensional Hilbert space. The following conditions are equivalent*:

(i) *A is in the closure of the set of real parts of nilpotent operators*;

(ii) *A is the real part of a quasi-nilpotent operator*;

(iii) *A is the real part of an operator with imaginary spectrum*;

(iv) $0 \in \operatorname{conv} \sigma(\pi_0(A))$;

(v) *A can be written as the direct sum $\Sigma \oplus A_\alpha$, with each A_α acting on a separable space and* $0 \in \operatorname{conv} \sigma_e(A_\alpha)$.

Finally, since Corollary 1 extends to the non-separable case, we note that Theorem 5 is valid (with the same proof) on non-separable Hilbert spaces.

REFERENCES

(1) P. A. FILLMORE, On similarity and the diagonal of a matrix, *Amer. Math. Monthly* **76** (1969), 167–169.

(2) P. A. FILLMORE, J. G. STAMPFLI, and J. P. WILLIAMS, On the essential numerical range, the essential spectrum, and a problem of Halmos, *Acta Sci. Math.* (*Szeged*) **33** (1972), 179–192.

(3) SAM HOLBY, A note on a note on commutators, (unpublished manuscript).

(4) E. HILLE and R. S. PHILLIPS, *Functional analysis and semi-groups*, rev. ed., (Amer. Math. Soc. Colloq. Publ. vol. 31, Amer. Math. Soc., Providence, R.I., 1957).

(5) H. RADJAVI, Structure of A^*A-AA^*, *J. Math. Mech.* **16** (1966), 19–26.

(6) G. -C. ROTA, On models for linear operators, *Comm. Pure Appl. Math.* **13** (1960), 469–472.

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https://doi.org/10.1017/S0013091500016436 Published online by Cambridge University Press