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ON COMPUTING THE NON-ABELIAN TENSOR SQUARES OF THE FREE 2-ENGEL GROUPS

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Abstract In this paper we compute the non-abelian tensor square for the free 2-Engel group of rank $n > 3$. The non-abelian tensor square for this group is a direct product of a free abelian group and a nilpotent group of class 2 whose derived subgroup has exponent 3. We also compute the non-abelian tensor square for one of the group's finite homomorphic images, namely, the Burnside group of rank n and exponent 3.

Keywords: non-abelian tensor square; free 2-Engel groups; Burnside groups

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1. Introduction

The non-abelian tensor square $G \otimes G$ of a group G is the group generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h'),$$

for all $g, g', h, h' \in G$, where ${}^{g}g'$ denotes the conjugation action $gg'g^{-1}$.

A group is 2-Engel if it satisfies the left normed commutator law $[x, y, y]$. The purpose of this paper is to compute the non-abelian tensor square of the free 2-Engel group of rank n for each $n > 3$. By computing the tensor square of a group G, we mean finding a simplified and standard presentation for $G \otimes G$.

The non-abelian tensor square has its roots in algebraic K-theory [**9**] as well as in topology [**6**, **7**]. This group construction was first investigated from a purely group theoretic perspective in the seminal paper by Brown *et al*. [**8**], in which they compute $G \otimes G$

for all groups of order up to thirty. Subsequent papers investigate explicit descriptions of the non-abelian tensor square for particular groups; for nilpotent of class 2 groups see [**1**–**3**,**13**,**16**], for metacyclic groups see [**5**,**14**], and for linear groups see [**12**]. A recent survey article on the non-abelian tensor squares and the more general non-abelian tensor products of groups can be found in [**15**].

In [**1**,**2**], the non-abelian tensor squares of nilpotent of class 2 groups were investigated. The non-abelian tensor square of the free nilpotent group of class 2 and rank 2, which is exactly the free 2-Engel group of rank 2, was shown to be free abelian of rank 6. The non-abelian tensor square of the free 2-Engel group of rank 3 was computed in [**4**]; it is the direct product of a free abelian group of rank 11 and a nilpotent group of class 2.

Since 2-Engel groups are metabelian, the following proposition from [**4**] shows that their tensor squares are abelian or nilpotent of class 2.

Proposition 1.1. *Let G be a group. If the derived subgroup G' is nilpotent of class $\mathrm{cl}(G')$, then $G \otimes G$ is nilpotent with class equal to $\mathrm{cl}(G')$ or $\mathrm{cl}(G') + 1$.*

In light of Proposition 1.1 and the fact that the tensor square of the free 2-Engel group of rank 3 is not abelian [**4**], we conclude that the tensor squares of the free 2-Engel groups of rank $n > 3$ are nilpotent of class 2.

Our method for computing non-abelian tensor squares uses the concept of a crossed pairing.

Definition 1.2. Let G and L be groups. A function $\Phi : G \times G \to L$ is called a *crossed pairing* if

$$\Phi(gg', g'') = \Phi({}^{g}g', {}^{g}g'')\Phi(g, g''), \tag{1.1}$$

$$\Phi(g, g'g'') = \Phi(g, g')\Phi({}^{g'}g, {}^{g'}g'') \tag{1.2}$$

for all $g, g', g'' \in G$.

The following proposition allows us to determine homomorphic images of $G \otimes G$.

Proposition 1.3 (see [7]). *A crossed pairing $\Phi : G \times G \to L$ determines a unique homomorphism of groups $\Phi^* : G \otimes G \to L$ such that $\Phi^*(g \otimes g') = \Phi(g, g')$ for all g, g' in G.*

To compute the non-abelian tensor square of a group G, we first conjecture a group L and find a crossed pairing $\Phi : G \times G \to L$. We then show that the homomorphism induced by the crossed pairing is actually an isomorphism and hence $G \otimes G \cong L$.

In the case of computing the tensor square for the free 2-Engel groups, we were guided in our conjecture of L by computing the tensor square of a finite homomorphic image of these groups, namely, the Burnside groups of rank n and exponent 3 for $n = 4$, 5 and 6. These computations were performed using methods developed in [**10**] and implemented in Gap [**11**]. We provide a more detailed outline of our computational method in § 3.

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Our main result is the following.

Theorem 1.4. *The non-abelian tensor square of the free 2-Engel group of rank $n > 2$ is a direct product of a free abelian group of rank $\frac{1}{3}n(n^2+2)$ and an $n(n-1)$-generated nilpotent group of class 2 whose derived subgroup has exponent 3.*

Corollary 1.5. *The non-abelian tensor square of the Burnside group of rank $n > 2$ and exponent 3 is a direct product of an elementary abelian 3-group of rank $\frac{1}{3}n(n^2+2)$ and an $n(n-1)$-generated nilpotent group of class 2 having exponent 3.*

In the next section we record various results needed to compute the non-abelian tensor square for 2-Engel groups. We prove Theorem 1.4 and Corollary 1.5 in § 3.

2. Preliminary results

The following familiar identities for 2-Engel groups are stated here without proof (see, for example, [**18**]). Recall that any nilpotent group of class 2 is a 2-Engel group.

Lemma 2.1. *Let G be a 2-Engel group. For $x, y, z, w \in G$ and $n \in \mathbb{Z}$ we have*

$$[x, y, z, w] = [x, y, z]^3 = 1, \tag{2.1}$$

$$[x, y, z] = [z, x, y] = [x, z, y]^{-1}, \tag{2.2}$$

$$[x, y^n] = [x^n, y] = [x, y]^n. \tag{2.3}$$

Throughout the paper, let $\mathcal{E} = \mathcal{E}(n, 2)$ denote the free 2-Engel group of rank $n > 2$ with a fixed ordering on its generators, which are labelled $g_1, g_2, \dots, g_n$. It follows from Lemma 2.1 that any element g in $\mathcal{E}$ can be written uniquely (with respect to the ordering of the generators) as the product

$$g = \prod_{i=1}^{n} g_i^{\alpha_i} \prod_{1 \leqslant j < k \leqslant n} [g_j, g_k]^{\beta_{j,k}} \prod_{1 \leqslant r < s < t \leqslant n} [g_r, g_s, g_t]^{\gamma_{r,s,t}}, \tag{2.4}$$

where each α_i and $\beta_{j,k}$ is an integer and (by Equation (2.1)) each $\gamma_{r,s,t}$ is an integer modulo 3. Since $\mathcal{E}/\mathcal{E}^3$ is isomorphic to $\mathcal{B}(n, 3)$, the Burnside group of rank n and exponent 3, we can also express each element of $\mathcal{B}(n, 3)$ similarly, where each α_i and $\beta_{j,k}$ is now also an integer modulo 3.

The following lemma provides formulae for multiplication and conjugation of arbitrary elements in $\mathcal{E}$.

Lemma 2.2. *Let g and g' be elements of $\mathcal{E}$, where g is defined in (2.4) and*

$$g' = \prod_{i=1}^{n} g_i^{\alpha'_i} \prod_{1 \leqslant j < k \leqslant n} [g_j, g_k]^{\beta'_{j,k}} \prod_{1 \leqslant r < s < t \leqslant n} [g_r, g_s, g_t]^{\gamma'_{r,s,t}}, \tag{2.5}$$

where each α'_i and $\beta'_{j,k}$ is an integer and each $\gamma'_{r,s,t}$ is an integer modulo 3. Then the product $g \cdot g'$ can be written in the form

$$g \cdot g' = \prod_{i=1}^{n} g_i^{\alpha^*_i} \prod_{1 \leqslant j < k \leqslant n} [g_j, g_k]^{\beta^*_{j,k}} \prod_{1 \leqslant r < s < t \leqslant n} [g_r, g_s, g_t]^{\gamma^*_{r,s,t}}, \tag{2.6}$$

where $\alpha_i^* = \alpha_i + \alpha_i'$, $\beta_{j,k}^* = \beta_{j,k} + \beta_{j,k}' - \alpha_j'\alpha_k$ *and*

$$\gamma_{r,s,t}^* \equiv \gamma_{r,s,t} + \gamma_{r,s,t}' + \beta_{s,t}\alpha_r' - \beta_{r,t}\alpha_s' + \beta_{r,s}\alpha_t' - \alpha_r'\alpha_s\alpha_t + \alpha_r'\alpha_s'\alpha_t - \alpha_r'\alpha_s\alpha_t' \mod 3.$$

Left conjugation of g' *by* g *is the product*

$$^{g}g' = \prod_{i=1}^{n} g_i^{\alpha_i^\dagger} \prod_{1\leqslant j<k\leqslant n} [g_j, g_k]^{\beta_{j,k}^\dagger} \prod_{1\leqslant r<s<t\leqslant n} [g_r, g_s, g_t]^{\gamma_{r,s,t}^\dagger}, \tag{2.7}$$

where

$$\begin{aligned}
\alpha_i^\dagger &= \alpha_i', \\
\beta_{j,k}^\dagger &= \beta_{j,k}' - \alpha_j'\alpha_k + \alpha_j\alpha_k', \\
\gamma_{r,s,t}^\dagger &\equiv \gamma_{r,s,t}' + \beta_{s,t}\alpha_r' - \beta_{r,t}\alpha_s' + \beta_{r,s}\alpha_t' - \alpha_r'\alpha_s\alpha_t + \alpha_r'\alpha_s'\alpha_t \\
&\quad - \alpha_r'\alpha_s\alpha_t' - \beta_{s,t}'\alpha_r + \beta_{r,t}'\alpha_s - \beta_{r,s}'\alpha_t - \alpha_r\alpha_s\alpha_t' + \alpha_r\alpha_s'\alpha_t + \alpha_r\alpha_s'\alpha_t' \mod 3.
\end{aligned}$$

The following lemma states general properties for the non-abelian tensor square of any group.

Lemma 2.3 (see [8]). *Let* G *be any group and* $x, v, y, z \in G$*. Then in* $G \otimes G$ *we have*

$$x \otimes 1 = 1 \otimes x = 1_\otimes, \tag{2.8}$$

$$[x \otimes v, y \otimes z] = [x, v] \otimes [y, z], \tag{2.9}$$

where $1_\otimes$ *denotes the identity element of* $G \otimes G$*.*

Let each of g and g' be either a generator or commutator of $\mathcal{E}$. We define the weight of an element $g \otimes g'$ of $\mathcal{E} \otimes \mathcal{E}$ as the sum of the commutator weights of g and g', where a generator has weight one. For example, the three variable element $x \otimes [y, z]$ has weight three while the three variable element $x \otimes [x, y, z]$ has weight four.

We list several identities used to express the elements of $\mathcal{E}\otimes\mathcal{E}$ in terms of the generators $g_1, g_2, \dots, g_n$ of $\mathcal{E}$. We start with two- and three-variable identities that hold in the tensor square of a 2-Engel group and its subgroups. The proofs of these identities can be found in [**3**, **4**].

Lemma 2.4. *Let* G *be a nilpotent group of class at most 2. Then, for* $x, y \in G$ *and* $m, n \in \mathbb{Z}$*, the following identity holds in* $G \otimes G$*:*

$$x^m \otimes y^n = (x \otimes y)^{mn} (y \otimes [x, y])^{m\binom{n}{2}} (x \otimes [x, y])^{n\binom{m}{2}}. \tag{2.10}$$

Lemma 2.5. *Let* G *be a 2-Engel group. Then, for any* $x, y, z \in G$*, the following identities hold in* $G \otimes G$*:*

$$(z \otimes [y, x])(y \otimes [x, z])(x \otimes [z, y]) = ([y, z] \otimes [x, z])([y, z] \otimes [x, y])([x, z] \otimes [x, y]), \tag{2.11}$$

$$([x, y] \otimes z)^{-1} = ([y, x] \otimes z) = (z \otimes [y, x])^{-1} = (z \otimes [x, y]). \tag{2.12}$$

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The following lemma lists four-variable identities found in [**4**].

Lemma 2.6. *Let G be a 2-Engel group. For $x, y, z, v \in G$, the following identities hold in $G \otimes G$:*

$$[x, v] \otimes [y, z] = ([x, v, y] \otimes z)(y \otimes [x, v, z]), \tag{2.13}$$

$$([x, v, y] \otimes z)(y \otimes [x, v, z])([y, z, x] \otimes v)(x \otimes [y, z, v]) = 1_{\otimes}. \tag{2.14}$$

The next lemma explicitly shows that all generators of $\mathcal{E} \otimes \mathcal{E}$ involving five or more variables equal the identity. Moreover, all generators of $\mathcal{E} \otimes \mathcal{E}$ expressed in four variables can be written as a product of weight four generators of a common form and have exponent 3.

Lemma 2.7. *Let G be a 2-Engel group. Then, for any $u, v, x, y, z \in G$, the following hold in $G \otimes G$:*

$$[u, v] \otimes [x, y, z] = [x, y, z] \otimes [u, v] = 1_{\otimes}, \tag{2.15}$$

$$([x, v] \otimes [y, z])^3 = 1_{\otimes}, \tag{2.16}$$

$$v \otimes [x, y, z] = ([v, x] \otimes [y, z])([v, y] \otimes [x, z])^{-1}([v, z] \otimes [x, y]). \tag{2.17}$$

Proof. Substituting $[u, v]$ for x, x for y, y for z and z for v in (2.14) yields (2.15).

We note that, by (2.10) and (2.1), for any v, x, y, z in G,

$$(v \otimes [x, y, z])^3 = v \otimes [x, y, z]^3 = v \otimes 1 = 1_{\otimes}.$$

Hence, by (2.13), we have

$$([x, v] \otimes [y, z])^3 = ([x, v, y] \otimes z)^3 (y \otimes [x, v, z])^3 = 1_{\otimes},$$

and thus (2.16) holds.

To show (2.17) we first rearrange (2.13) using (2.12) as follows:

$$z \otimes [x, v, y] = (y \otimes [x, v, z])([v, x] \otimes [y, z]). \tag{2.18}$$

By interchanging v with y in (2.18), and using (2.2) and (2.12), we obtain

$$z \otimes [v, x, y] = (v \otimes [x, y, z])([v, z] \otimes [x, y]). \tag{2.19}$$

Now interchange (respectively) y with z and x with z in (2.19) to get

$$y \otimes [v, x, z] = (v \otimes [x, y, z])^{-1}([v, y] \otimes [x, z]) \tag{2.20}$$

and

$$x \otimes [v, y, z] = (v \otimes [x, y, z])([v, x] \otimes [y, z]). \tag{2.21}$$

Also, from (2.13), we have

$$[x, v] \otimes [y, z] = (z \otimes [v, x, y])(y \otimes [x, v, z]). \tag{2.22}$$

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Interchange x with z and y with v in (2.22), and use (2.2) and (2.12) to obtain

$$[x,v]\otimes[y,z]=(x\otimes[z,y,v])(v\otimes[y,z,x]). \tag{2.23}$$

Note that the left-hand sides of (2.22) and (2.23) are equal, so we may equate the right-hand sides, and, after applying (2.2) as needed, substitute the expressions on the right-hand sides of (2.19), (2.20) and (2.21). This gives

$$\begin{aligned}(v\otimes[x,y,z])([v,z]\otimes[x,y])(v\otimes[x,y,z])([v,y]\otimes[x,z])^{-1}\\ =(v\otimes[x,y,z])^{-1}([v,x]\otimes[y,z])^{-1}(v\otimes[x,y,z]).\end{aligned}$$

Hence

$$(v\otimes[x,y,z])^2=([v,x]\otimes[y,z])^{-1}([v,y]\otimes[x,z])([v,z]\otimes[x,y])^{-1}.$$

Since $(v\otimes[x,y,z])^3=1_\otimes$, we have $(v\otimes[x,y,z])^2=(v\otimes[x,y,z])^{-1}$ and we arrive at Equation (2.17). □

The following proposition is from [**4**].

Proposition 2.8. *For a 2-Engel group G, the defining relations of $G\otimes G$ reduce to*

$$xy\otimes z=(x\otimes[y,z])(y\otimes z)(x\otimes z), \tag{2.24}$$

$$x\otimes yz=([z,x]\otimes y)(x\otimes y)(x\otimes z). \tag{2.25}$$

We generalize Proposition 2.8 in two steps. We first show the following.

Proposition 2.9. *Let G be a 2-Engel group. Let $x_1,\dots,x_n\in G$. Let b be an element of the derived subgroup of G. Then, for $n\geqslant 2$,*

$$\left(\prod_{i=1}^{n}x_i\right)\otimes b=\prod_{k=2}^{n}\prod_{j=1}^{k-1}(x_j\otimes[x_k,b])\prod_{i=1}^{n}(x_i\otimes b) \tag{2.26}$$

and

$$b\otimes\left(\prod_{i=1}^{n}x_i\right)=\prod_{k=2}^{n}\prod_{j=1}^{k-1}([x_k,b]\otimes x_j)\prod_{i=1}^{n}(b\otimes x_i). \tag{2.27}$$

Proof. We induct on n. Consider Equation (2.26). Let $x_1,x_2\in G$ and let b lie in the derived subgroup of G. By Proposition 2.8 and Equation (2.9) we have

$$x_1x_2\otimes b=(x_1\otimes[x_2,b])(x_2\otimes b)(x_1\otimes b)=(x_1\otimes[x_2,b])(x_1\otimes b)(x_2\otimes b),$$

which has the form of (2.26) for $n=2$.

Suppose now that Equation (2.26) holds for some $n\geqslant 2$. For $i=1,\dots,n+1$, let $x_i\in G$. Let b be an element of the derived subgroup of G. Then, by Proposition 2.8, we have

$$\left(\prod_{i=1}^{n+1}x_i\right)\otimes b=\left(\left(\prod_{i=1}^{n}x_i\right)\otimes[x_{n+1},b]\right)(x_{n+1}\otimes b)\left(\left(\prod_{i=1}^{n}x_i\right)\otimes b\right). \tag{2.28}$$

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Since $[x_{n+1}, b]$ is in the centre of G, the factor $(\prod_{i=1}^{n} x_i) \otimes [x_{n+1}, b]$ of (2.28) expands linearly (by Proposition 2.8). By the inductive hypothesis, the last factor on the right-hand side of (2.28) is

$$\left(\prod_{i=1}^{n} x_i\right) \otimes b = \prod_{k=2}^{n} \prod_{j=1}^{k-1} (x_j \otimes [x_k, b]) \prod_{i=1}^{n} (x_i \otimes b).$$

These facts put together give us the following:

$$\begin{aligned}\left(\prod_{i=1}^{n+1} a x_i\right) \otimes b &= \left(\left(\prod_{i=1}^{n} x_i\right) \otimes [x_{n+1}, b]\right)(x_{n+1} \otimes b)\left(\left(\prod_{i=1}^{n} x_i\right) \otimes b\right) \\ &= \left(\prod_{i=1}^{n} (x_i \otimes [x_{n+1}, b])\right)(x_{n+1} \otimes b) \prod_{k=2}^{n} \prod_{j=1}^{k-1} (x_j \otimes [x_k, b]) \prod_{i=1}^{n} (x_i \otimes b) \\ &= \prod_{k=2}^{n+1} \prod_{j=1}^{k-1} (x_j \otimes [x_k, b]) \prod_{i=1}^{n+1} (x_i \otimes b).\end{aligned}$$

The proof of Equation (2.27) is similar. □

Proposition 2.9 is used to show the following result, which has a similar inductive proof. The proof can be found in [**17**].

Proposition 2.10. *Let G be a 2-Engel group. For $u = 1, \dots, n$, let x_u, y_u, x and y be elements of G. Then, for $n \geqslant 3$,*

$$\left(\prod_{u=1}^{n} x_u\right) \otimes y = \prod_{l=3}^{n} \prod_{k=2}^{l-1} \prod_{j=1}^{k-1} (x_j \otimes [x_k, [x_l, y]]) \prod_{s=2}^{n} \prod_{r=1}^{s-1} (x_r \otimes [x_s, y]) \prod_{m=0}^{n-1} (x_{n-m} \otimes y)$$

and

$$x \otimes \left(\prod_{u=1}^{n} y_u\right) = \prod_{l=3}^{n} \prod_{k=2}^{l-1} \prod_{j=1}^{k-1} ([y_k, [y_l, x]] \otimes y_j) \prod_{s=2}^{n} \prod_{r=1}^{s-1} ([y_s, x] \otimes y_r) \prod_{u=1}^{n} (x \otimes y_u).$$

Appropriate substitutions in Proposition 2.10 show that some expansions are linear, such as in the following corollary.

Corollary 2.11. *Let G be a 2-Engel group. Let $x \in G$ and suppose that b_i is an element of the derived subgroup of G for $i = 1, \dots, n$. Then, for $n \geqslant 2$,*

$$\left(\prod_{i=1}^{n} b_i\right) \otimes x = \prod_{i=1}^{n} (b_i \otimes x) \quad \text{and} \quad x \otimes \left(\prod_{i=1}^{n} b_i\right) = \prod_{i=1}^{n} (x \otimes b_i).$$

3. Computing the tensor square

In this section we prove Theorem 1.4, which provides a simplified presentation for $\mathcal{E}\otimes\mathcal{E}$. The following is an outline of our method for making the computations needed in proving the theorem.

Using the identities for $\mathcal{E}\otimes\mathcal{E}$ developed in § 2, we first express an arbitrary generator $g\otimes g'$ of $\mathcal{E}\otimes\mathcal{E}$ as a product of a fixed set of elements of $\mathcal{E}\otimes\mathcal{E}$ whose exponent expressions depend on the exponents in the representations of g and g' found in (2.4) and (2.5). Then we construct a group L_n and define a multiplication formula for it in terms of the exponents of its generators (see Example 3.3).

To prove Theorem 1.4, we define a function $\Phi:\mathcal{E}\times\mathcal{E}\to L_n$ that we show is a crossed pairing. Verifying that Φ is a crossed pairing involves multiplying and conjugating elements of $\mathcal{E}$ using Lemma 2.2 and multiplying elements in L_n as defined in Example 3.3. These operations are described by formulae on the exponents of the generators of these groups, respectively. We consider these formulae as functions. Verification of the identities (1.1) and (1.2) for Φ is completed by composing these exponent functions. Since some of these compositions involve manipulating hundreds of terms, the computations were performed using Maple [**19**]. We complete the proof of the theorem by showing the homomorphism induced by the crossed pairing Φ is an isomorphism.

We express the element g of $\mathcal{E}$, defined in (2.4), as the product abc, where

$$a=\prod_{i=1}^{n}g_i^{\alpha_i},\qquad b=\prod_{1\leqslant j<k\leqslant n}[g_j,g_k]^{\beta_{j,k}},\qquad c=\prod_{1\leqslant r<s<t\leqslant n}[g_r,g_s,g_t]^{\gamma_{r,s,t}}. \tag{3.1}$$

Similarly, we express $g'\in\mathcal{E}$, defined in (2.5), as the product $a'b'c'$, where

$$a'=\prod_{i=1}^{n}g_i^{\alpha'_i},\qquad b'=\prod_{1\leqslant j<k\leqslant n}[g_j,g_k]^{\beta'_{j,k}},\qquad c'=\prod_{1\leqslant r<s<t\leqslant n}[g_r,g_s,g_t]^{\gamma'_{r,s,t}}. \tag{3.2}$$

As a first step, we use Proposition 2.8 repeatedly to compute an expansion formula for $g\otimes g'$ in terms of the factors a, b, c, a', b' and c'. We use the fact that $\mathcal{E}$ is nilpotent of class 3 to simplify the expansion,

$$\begin{aligned}
g\otimes g' &= a\cdot b\cdot c\otimes a'\cdot b'\cdot c'\\
&=(a\cdot b\otimes[c,a'\cdot b'\cdot c'])(c\otimes a'\cdot b'\cdot c')(a\cdot b\otimes a'\cdot b'\cdot c')\\
&=(c\otimes a'\cdot b'\cdot c')(a\cdot b\otimes a'\cdot b'\cdot c')\\
&=([c',c]\otimes a'\cdot b')(c\otimes a'\cdot b')(c\otimes c')([c',a\cdot b]\otimes a'\cdot b')(a\cdot b\otimes a'\cdot b')(a\cdot b\otimes c')\\
&=([b',c]\otimes a')(c\otimes a')(c\otimes b')(c\otimes c')(a\cdot b\otimes a'\cdot b')(a\otimes[b,c'])(b\otimes c')(a\otimes c')\\
&=(c\otimes a')(c\otimes b')(c\otimes c')(a\cdot b\otimes a'\cdot b')(b\otimes c')(a\otimes c').
\end{aligned}\tag{3.3}$$

By Corollary 2.11, the factors $(c\otimes c')$, $(c\otimes b')$ and $(b\otimes c')$ in Equation (3.3) expand linearly into products of elements of $\mathcal{E}\otimes\mathcal{E}$ of weight at least five, which are all equal to the identity by (2.15). Hence our expansion becomes

$$a\cdot b\cdot c\otimes a'\cdot b'\cdot c'=(c\otimes a')(a\cdot b\otimes a'\cdot b')(a\otimes c'). \tag{3.4}$$

We expand $(a \cdot b \otimes a' \cdot b')$ using Proposition 2.8, obtaining

$$\begin{aligned}(a \cdot b \otimes a' \cdot b') &= (a \otimes [b, a' \cdot b'])(b \otimes a' \cdot b')(a \otimes a' \cdot b') \\ &= (a \otimes [b, a'])([b', b] \otimes a')(b \otimes a')(b \otimes b')([b', a] \otimes a')(a \otimes a')(a \otimes b') \\ &= (a \otimes [b, a'])(b \otimes a')(b \otimes b')([b', a] \otimes a')(a \otimes a')(a \otimes b'). \end{aligned} \tag{3.5}$$

Putting the expansions (3.4) and (3.5) together we get

$$\begin{aligned} &a \cdot b \cdot c \otimes a' \cdot b' \cdot c' \\ &\quad = (c \otimes a')(a \otimes [b, a'])(b \otimes a')(b \otimes b')([b', a] \otimes a')(a \otimes a')(a \otimes b')(a \otimes c'). \end{aligned} \tag{3.6}$$

We next show that every generator of $\mathcal{E} \otimes \mathcal{E}$ can be expressed as a product of a prescribed set of elements in $\mathcal{E} \otimes \mathcal{E}$ in a fixed ordering. We accomplish this goal by expanding each of the eight factors in (3.6) as products involving the generators $g_1, \dots, g_n$ of $\mathcal{E}$. For the sake of notational convenience in the sequel, we set

$$I_1 = \{(i, j, k) \mid 1 \leqslant i, j, k \leqslant n;\ i \leqslant \max\{j, k\};\ j < k\},$$
$$I_2 = \{(i, j, k, l) \mid 1 \leqslant i < j \leqslant n;\ i \leqslant k < l \leqslant n;\ (i, j) < (k, l) \text{ lexicographically}\}.$$

Lemma 3.1. *Let g and g' be arbitrary elements of $\mathcal{E}$. Then*

$$\begin{aligned} g \otimes g' = \prod_{i=1}^{n} (g_i \otimes g_i)^{\rho_i} \cdot \prod_{(i,j,k) \in I_1} (g_i \otimes [g_j, g_k])^{\sigma_{i,j,k}} \\ \cdot \prod_{(i,j,k,l) \in I_2} ([g_i, g_j] \otimes [g_k, g_l])^{\tau_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1 \\ j \neq n-i}}^{n} (g_{n-i} \otimes g_j)^{\nu_{n-i,j}}, \end{aligned} \tag{3.7}$$

where each ρ_i, $\sigma_{i,j,k}$ and $\nu_{i,j}$ is an integer and each $\tau_{i,j,k,l}$ is an integer modulo 3.

Proof. We continue to consider g and g' as written in (2.4) and (2.5), respectively. The eight sets of forms listed below in (3.8)–(3.15) are the possible element forms that arise when the eight terms of (3.6) are expanded using Propositions 2.8–2.10:

$$[g_r, g_s, g_t]^{\gamma_{r,s,t}} \otimes g_u^{\alpha'_u}, \tag{3.8}$$

$$g_u^{\alpha_u} \otimes [[g_r, g_s]^{\beta_{r,s}}, g_v^{\alpha'_v}], \tag{3.9}$$

$$[g_r, g_s]^{\beta_{r,s}} \otimes g_u^{\alpha'_u}, \qquad [g_q^{\alpha'_q}, [g_r, g_s]^{\beta_{r,s}}] \otimes g_p^{\alpha'_p}, \tag{3.10}$$

$$[g_r, g_s]^{\beta_{r,s}} \otimes [g_p, g_q]^{\beta'_{p,q}}, \tag{3.11}$$

$$[[g_r, g_s]^{\beta'_{r,s}}, g_v^{\alpha_v}] \otimes g_u^{\alpha'_u}, \tag{3.12}$$

$$\left.\begin{aligned} &g_u^{\alpha_u} \otimes g_v^{\alpha'_v}, \qquad g_r^{\alpha_r} \otimes [g_s^{\alpha_s}, g_u^{\alpha'_u}], \qquad g_r^{\alpha_r} \otimes [g_s^{\alpha_s}, g_t^{\alpha_t}, g_u^{\alpha'_u}], \\ &g_r^{\alpha_r} \otimes [g_s^{\alpha_s}, g_p^{\alpha'_p}, g_q^{\alpha'_q}], \qquad g_r^{\alpha'_r} \otimes [g_t^{\alpha'_t}, g_u^{\alpha_u}, g_s^{\alpha'_s}], \qquad g_r^{\alpha'_r} \otimes [g_s^{\alpha'_s}, g_u^{\alpha_u}], \end{aligned}\right\} \tag{3.13}$$

$$g_u^{\alpha_u} \otimes [g_r, g_s]^{\beta'_{r,s}}, \qquad g_p^{\alpha_p} \otimes [g_q^{\alpha_q}, [g_r, g_s]^{\beta'_{r,s}}], \tag{3.14}$$

$$g_u^{\alpha_u} \otimes [g_r, g_s, g_t]^{\gamma'_{r,s,t}}, \tag{3.15}$$

where $1 \leqslant r < s < t \leqslant n$, $1 \leqslant p < q \leqslant n$ and $1 \leqslant u, v \leqslant n$.

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For example, by Corollary 2.11 and Proposition 2.9,

$$\begin{aligned} b \otimes a' &= \prod_{1 \leqslant r<s \leqslant n} [g_r, g_s]^{\beta_{r,s}} \otimes \prod_{u=1}^{n} g_u^{\alpha'_u} \\ &= \prod_{1 \leqslant r<s \leqslant n} \biggl([g_r, g_s]^{\beta_{r,s}} \otimes \prod_{u=1}^{n} g_u^{\alpha'_u}\biggr) \\ &= \prod_{1 \leqslant r<s \leqslant n} \biggl(\prod_{q=2}^{n} \prod_{p=1}^{q-1} ([g_q^{\alpha'_q}, [g_r, g_s]^{\beta_{r,s}}] \otimes g_p^{\alpha'_p}) \prod_{u=1}^{n} ([g_r, g_s]^{\beta_{r,s}} \otimes g_u^{\alpha'_u})\biggr). \end{aligned}$$

Hence we have the element forms shown in (3.10).

We extract the exponents for each of the forms found in (3.8)–(3.15) using (2.3) and (2.10):

$$([g_r, g_s, g_t] \otimes g_u)^{\alpha'_u \gamma_{r,s,t}}, \tag{3.16}$$

$$(g_u \otimes [g_r, g_s, g_v])^{\alpha_u \beta_{r,s} \alpha'_v}, \tag{3.17}$$

$$([g_r, g_s] \otimes g_u)^{\beta_{r,s} \alpha'_u}, \qquad (g_u \otimes [g_r, g_s, g_u])^{\beta_{r,s} \binom{\alpha'_u}{2}}, \qquad ([g_q, [g_r, g_s]] \otimes g_p)^{\alpha'_q \beta_{r,s} \alpha'_p}, \tag{3.18}$$

$$([g_r, g_s] \otimes [g_p, g_q])^{\beta'_{p,q} \beta_{r,s}}, \tag{3.19}$$

$$([g_r, g_s, g_v] \otimes g_u)^{\beta'_{r,s} \alpha_v \alpha'_u}, , \tag{3.20}$$

$$\left.\begin{array}{cc} (g_u \otimes g_v)^{\alpha_u \alpha'_v}, & (g_v \otimes [g_u, g_v])^{\alpha_u \binom{\alpha'_v}{2}}, \\ (g_u \otimes [g_u, g_v])^{\alpha'_v \binom{\alpha_u}{2}}, & (g_r \otimes [g_s, g_u])^{\alpha_r \alpha_s \alpha'_u}, \\ (g_r \otimes [g_r, [g_s, g_u]])^{\alpha_s \alpha'_u \binom{\alpha_r}{2}}, & (g_r \otimes [g_s, [g_t, g_u]])^{\alpha_r \alpha_s \alpha_t \alpha'_u}, \\ (g_r \otimes [g_s, g_p, g_q])^{\alpha_r \alpha_s \alpha'_p \alpha'_q}, & (g_r \otimes [g_t, g_u, g_s])^{\alpha'_r \alpha'_t \alpha_u \alpha'_s}, \\ (g_r \otimes [g_u, g_s])^{\alpha'_r \alpha'_s \alpha_u}, & (g_r \otimes [g_r, [g_s, g_u]])^{\alpha'_s \alpha_u \binom{\alpha'_r}{2}}, \end{array}\right\} \tag{3.21}$$

$$(g_u \otimes [g_r, g_s])^{\alpha_u \beta'_{r,s}}, \qquad (g_u \otimes [g_u, [g_r, g_s]])^{\beta'_{r,s} \binom{\alpha_i}{2}}, \qquad (g_p \otimes [g_q, [g_r, g_s]])^{\alpha_p \alpha_q \beta'_{r,s}}, \tag{3.22}$$

$$(g_u \otimes [g_r, g_s, g_t])^{\gamma'_{r,s,t} \alpha_u}, \tag{3.23}$$

where $1 \leqslant r < s < t \leqslant n$, $1 \leqslant p < q \leqslant n$ and $1 \leqslant u, v \leqslant n$.

Hence an arbitrary generator of $\mathcal{E} \otimes \mathcal{E}$ can be written as a product of the element forms listed in (3.16)–(3.23). The only non-central elements in this list are the powers of elements of the form $g_i \otimes g_j$ for $i \neq j$. Hence the exponents have no effect in rewriting the other element forms to match the factors listed in the statement of the lemma. There are only two basic form types that need to be rewritten. For (3.16), we have, using Equations (2.12) and (2.17),

$$\begin{aligned} ([g_r, g_s, g_t] \otimes g_u) &= (g_u \otimes [g_r, g_s, g_t])^{-1} \\ &= ([g_u, g_r] \otimes [g_s, g_t])^{-1} ([g_u, g_s] \otimes [g_u, g_t]) ([g_u, g_t] \otimes [g_s, g_r])^{-1}. \end{aligned}$$

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We then reorder the subscripts as needed by repeatedly using (2.12) and keeping track of the sign changes of the exponents as necessary. We rewrite the other element forms that involve a weight three commutator in $\mathcal{E}$ similarly.

Elements of the form $g_k \otimes [g_i, g_j]$ for which $k > \max\{i, j\}$ can be rewritten using (2.11),

$$\begin{aligned}(g_k \otimes [g_i, g_j]) \\ = (g_i \otimes [g_j, g_i])^{-1}(g_j \otimes [g_j, g_k])^{-1}([g_i, g_k] \otimes [g_j, g_k])([g_i, g_k] \otimes [g_j, g_i])([g_j, g_k] \otimes [g_j, g_i]).\end{aligned}$$

Again, we reorder the subscripts, as needed, using (2.12).

We can order the non-central factors $g_i \otimes g_j$, for $i \neq j$, as specified in the lemma, noting that the commutators $[g_i \otimes g_j, g_k \otimes g_l] = [g_i, g_j] \otimes [g_k, g_l]$ (where $k \neq l$) lie in the centre of $\mathcal{E} \otimes \mathcal{E}$. Hence the reordering of the non-central factors does not add any new element forms. □

An arbitrary element of $\mathcal{E} \otimes \mathcal{E}$ is a product of generators of the form $g \otimes g'$. By Lemma 3.1, each of these generators can be written in a common form (3.7). Hence $\mathcal{E} \otimes \mathcal{E}$ is finitely generated by the factors in the product (3.7).

Remark 3.2. We note that Lemma 3.1 also holds for the generators of

$$\mathcal{B}(n, 3) \otimes \mathcal{B}(n, 3),$$

where the exponents are taken to be integers modulo 3. Moreover, an arbitrary element of $\mathcal{B}(n, 3) \otimes \mathcal{B}(n, 3)$ can be expressed as the product (3.7), where all the exponents are again considered to be integers modulo 3.

The exponents ρ_i, $\sigma_{i,j,k}$, $\nu_{i,j}$ and $\tau_{i,j,k,l}$ specified in Lemma 3.1 are used later in the proof of Theorem 1.4 to define a mapping $\Phi : \mathcal{E} \times \mathcal{E} \to L_n$ (where L_n is defined in Example 3.3 below). An explicit expression for these exponents is determined by a careful analysis of how the exponents arise during the expansion and collection process of $g \otimes g'$ as a product of the generators indicated in Lemma 3.1. This analysis, which was performed completely in [**17**], relies on tracking the indices of each of the products in the expansion process described in the proof of the lemma to form the exponent expressions listed below. When all of the forms in the expansion are collected, we note that each basic form must be considered in several cases dependent only on the relative ordering of its indices. We arrive at the following descriptions of the exponents of the generators of $\mathcal{E} \otimes \mathcal{E}$.

For generators of weight two, the exponents are

$$\rho_i = \alpha_i \alpha'_i \quad \text{and} \quad \nu_{i,j} = \alpha_i \alpha'_j \quad (i \neq j). \tag{3.24}$$

For generators of weight three, we have four cases to consider, with indices ordered $i < j < k$. The exponent for each generator $g_i \otimes [g_j, g_k]$ is

$$\sigma_{i,j,k} = -\alpha'_i \beta_{j,k} + \alpha_i \beta'_{j,k} - \alpha_i \alpha'_j \alpha_k + \alpha'_i \alpha_j \alpha'_k + \alpha_i \alpha_j \alpha'_k - \alpha'_i \alpha'_j \alpha_k + \alpha'_k \beta_{i,j} - \alpha_k \beta'_{i,j}. \tag{3.25}$$

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For each generator $g_i \otimes [g_i, g_j]$, the exponent is

$$\sigma_{i,i,j} = -\alpha_i'\beta_{i,j} + \alpha_i\beta_{i,j}' - \alpha_i\alpha_i'\alpha_j + \alpha_i\alpha_i'\alpha_j' - \alpha_j\binom{\alpha_i'}{2} + \alpha_j'\binom{\alpha_i}{2}.$$

The exponent for each generator $(g_j \otimes [g_i, g_k])$ is

$$\sigma_{j,i,k} = -\alpha_j'\beta_{i,k} + \alpha_j\beta_{i,k}' - \alpha_i'\alpha_j\alpha_k + \alpha_i\alpha_j'\alpha_k' - \alpha_k'\beta_{i,j} + \alpha_k\beta_{i,j}'.$$

For each generator $(g_j \otimes [g_i, g_j])$, the exponent is

$$\sigma_{j,i,j} = -\alpha_j'\beta_{i,j} + \alpha_j\beta_{i,j}' + \alpha_i\binom{\alpha_j'}{2} - \alpha_i'\binom{\alpha_j}{2}.$$

For generators of weight four, we have six cases to consider, with indices ordered $i < j < k < l$. The exponent for each generator $[g_i, g_j] \otimes [g_i, g_k]$ is

$$\begin{aligned}
\tau_{i,j,i,k} = {} & -\alpha_i'\gamma_{i,j,k} + \alpha_i\gamma_{i,j,k}' + \alpha_i\alpha_i'\beta_{j,k} - \alpha_i\alpha_j'\beta_{i,k} + \alpha_i\alpha_k'\beta_{i,j} - \alpha_i\alpha_i'\beta_{j,k}' + \alpha_i'\alpha_j\beta_{i,k}' \\
& - \alpha_i'\alpha_k\beta_{i,j}' + \beta_{i,j}\beta_{i,k}' - \beta_{i,j}'\beta_{i,k} + \beta_{j,k}\binom{\alpha_i'}{2} - \alpha_i'\alpha_j'\beta_{i,k} + \alpha_i'\alpha_k'\beta_{i,j} + \alpha_i\alpha_j\beta_{i,k}' \\
& - \alpha_i\alpha_k\beta_{i,j}' - \beta_{j,k}'\binom{\alpha_i}{2} - \alpha_i\alpha_i'\alpha_j\alpha_k + \alpha_i\alpha_i'\alpha_j'\alpha_k' - \alpha_i\alpha_i'\alpha_j\alpha_k' + \alpha_i\alpha_i'\alpha_j'\alpha_k \\
& + \alpha_j'\alpha_k\binom{\alpha_i}{2} - \alpha_j\alpha_k'\binom{\alpha_i'}{2} - \alpha_j\alpha_k'\binom{\alpha_i}{2} + \alpha_j'\alpha_k\binom{\alpha_i'}{2} - \alpha_k'\beta_{i,j} + \alpha_k\beta_{i,j}'.
\end{aligned}$$

For each generator $[g_i, g_j] \otimes [g_j, g_k]$, the exponent is

$$\begin{aligned}
\tau_{i,j,j,k} = {} & -\alpha_j'\gamma_{i,j,k} + \alpha_j\gamma_{i,j,k}' + \alpha_i'\alpha_j\beta_{j,k} - \alpha_j\alpha_j'\beta_{i,k} + \alpha_j\alpha_k'\beta_{i,j} - \alpha_i\alpha_j'\beta_{j,k}' + \alpha_j\alpha_j'\beta_{i,k}' \\
& - \alpha_j'\alpha_k\beta_{i,j}' + \beta_{i,j}\beta_{j,k}' - \beta_{i,j}'\beta_{j,k} - \beta_{i,k}\binom{\alpha_j'}{2} + \alpha_j'\alpha_k'\beta_{i,j} - \alpha_j\alpha_k\beta_{i,j}' \\
& + \beta_{i,k}'\binom{\alpha_j}{2} + \alpha_i'\alpha_j\alpha_j'\alpha_k - \alpha_i'\alpha_k\binom{\alpha_j}{2} + \alpha_i\alpha_k'\binom{\alpha_j'}{2} - \alpha_k'\beta_{i,j} + \alpha_k\beta_{i,j}'.
\end{aligned}$$

The exponent for each generator $[g_i, g_k] \otimes [g_j, g_k]$ is

$$\begin{aligned}
\tau_{i,k,j,k} = {} & -\alpha_k'\gamma_{i,j,k} + \alpha_k\gamma_{i,j,k}' + \alpha_i'\alpha_k\beta_{j,k} - \alpha_j'\alpha_k\beta_{i,k} + \alpha_k\alpha_k'\beta_{i,j} - \alpha_i\alpha_k'\beta_{j,k}' + \alpha_j\alpha_k'\beta_{i,k}' \\
& - \alpha_k\alpha_k'\beta_{i,j}' + \beta_{i,k}\beta_{j,k}' - \beta_{i,k}'\beta_{j,k} + \beta_{i,j}\binom{\alpha_k'}{2} - \beta_{i,j}'\binom{\alpha_k}{2} - \alpha_k'\beta_{i,j} + \alpha_k\beta_{i,j}'.
\end{aligned}$$

For each generator $[g_i, g_j] \otimes [g_k, g_l]$, the exponent is

$$\begin{aligned}
\tau_{i,j,k,l} = {} & -\alpha_i'\gamma_{j,k,l} + \alpha_j'\gamma_{i,k,l} + \alpha_k'\gamma_{i,j,l} - \alpha_l'\gamma_{i,j,k} + \alpha_i\gamma_{j,k,l}' - \alpha_j\gamma_{i,k,l}' - \alpha_k\gamma_{i,j,l}' \\
& + \alpha_l\gamma_{i,j,k}' + \alpha_i\alpha_j'\beta_{k,l} - \alpha_i'\alpha_j\beta_{k,l} - \alpha_i'\alpha_k\beta_{j,l} + \alpha_i'\alpha_l\beta_{j,k} - \alpha_i\alpha_k'\beta_{j,l} + \alpha_j\alpha_k'\beta_{i,l} \\
& + \alpha_j'\alpha_k\beta_{i,l} - \alpha_j'\alpha_l\beta_{i,k} + \alpha_i\alpha_l'\beta_{j,k} - \alpha_j\alpha_l'\beta_{i,k} - \alpha_k\alpha_l'\beta_{i,j} + \alpha_k'\alpha_l\beta_{i,j} - \alpha_i'\alpha_j\beta_{k,l}' \\
& + \alpha_i\alpha_j'\beta_{k,l}' + \alpha_i\alpha_k'\beta_{j,l}' - \alpha_i\alpha_l'\beta_{j,k}' + \alpha_i'\alpha_k\beta_{j,l}' - \alpha_j'\alpha_k\beta_{i,l}' - \alpha_j\alpha_k'\beta_{i,l}' + \alpha_j\alpha_l'\beta_{i,k}'
\end{aligned}$$

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$$- \alpha'_i\alpha_l\beta'_{j,k} + \alpha'_j\alpha_l\beta'_{i,k} + \alpha'_k\alpha_l\beta'_{i,j} - \alpha_k\alpha'_l\beta'_{i,j} + \beta_{i,j}\beta'_{k,l} - \beta'_{i,j}\beta_{k,l} + \alpha'_i\alpha'_j\beta_{k,l}$$
$$- \alpha'_i\alpha'_k\beta_{j,l} + \alpha'_j\alpha'_k\beta_{i,l} + \alpha'_i\alpha'_l\beta_{j,k} - \alpha'_j\alpha'_l\beta_{i,k} - \alpha'_k\alpha'_l\beta_{i,j} - \alpha_i\alpha_j\beta'_{k,l} + \alpha_i\alpha_k\beta'_{j,l}$$
$$- \alpha_j\alpha_k\beta'_{i,l} - \alpha_i\alpha_l\beta'_{j,k} + \alpha_j\alpha_l\beta'_{i,k} + \alpha_k\alpha_l\beta'_{i,j} + \alpha'_i\alpha_j\alpha_k\alpha_l - \alpha_i\alpha'_j\alpha'_k\alpha'_l$$
$$- \alpha_i\alpha'_j\alpha_k\alpha_l + \alpha'_i\alpha_j\alpha'_k\alpha'_l + \alpha_i\alpha_j\alpha'_k\alpha_l - \alpha'_i\alpha'_j\alpha_k\alpha'_l - \alpha_i\alpha_j\alpha_k\alpha'_l + \alpha'_i\alpha'_j\alpha'_k\alpha_l$$
$$+ \alpha_i\alpha_j\alpha'_k\alpha'_l - \alpha_i\alpha'_j\alpha_k\alpha'_l + \alpha'_i\alpha_j\alpha_k\alpha'_l + \alpha_i\alpha'_j\alpha'_k\alpha_l - \alpha'_i\alpha_j\alpha'_k\alpha_l - \alpha'_i\alpha'_j\alpha_k\alpha_l.$$

The exponent for each generator $[g_i, g_k] \otimes [g_j, g_l]$ is

$$\begin{aligned}
\tau_{i,k,j,l} = {} & \alpha'_i\gamma_{j,k,l} + \alpha'_j\gamma_{i,k,l} + \alpha'_k\gamma_{i,j,l} + \alpha'_l\gamma_{i,j,k} - \alpha_i\gamma'_{j,k,l} - \alpha_j\gamma'_{i,k,l} - \alpha_k\gamma'_{i,j,l} - \alpha_l\gamma'_{i,j,k} \\
& - \alpha_i\alpha'_j\beta_{k,l} - \alpha'_i\alpha_j\beta_{k,l} - \alpha'_i\alpha_k\beta_{j,l} - \alpha'_i\alpha_l\beta_{j,k} + \alpha_i\alpha'_k\beta_{j,l} + \alpha_j\alpha'_k\beta_{i,l} + \alpha'_j\alpha_k\beta_{i,l} \\
& + \alpha'_j\alpha_l\beta_{i,k} - \alpha_i\alpha'_l\beta_{j,k} - \alpha_j\alpha'_l\beta_{i,k} - \alpha_k\alpha'_l\beta_{i,j} - \alpha'_k\alpha_l\beta_{i,j} + \alpha'_i\alpha_j\beta'_{k,l} + \alpha_i\alpha'_j\beta'_{k,l} \\
& + \alpha_i\alpha'_k\beta'_{j,l} + \alpha_i\alpha'_l\beta'_{j,k} - \alpha'_i\alpha_k\beta'_{j,l} - \alpha'_j\alpha_k\beta'_{i,l} - \alpha_j\alpha'_k\beta'_{i,l} - \alpha_j\alpha'_l\beta'_{i,k} + \alpha'_i\alpha_l\beta'_{j,k} \\
& + \alpha'_j\alpha_l\beta'_{i,k} + \alpha'_k\alpha_l\beta'_{i,j} + \alpha_k\alpha'_l\beta'_{i,j} + \beta_{i,k}\beta'_{j,l} - \beta'_{i,k}\beta_{j,l} - \alpha'_i\alpha'_j\beta_{k,l} + \alpha'_i\alpha'_k\beta_{j,l} \\
& + \alpha'_j\alpha'_k\beta_{i,l} - \alpha'_i\alpha'_l\beta_{j,k} - \alpha'_j\alpha'_l\beta_{i,k} - \alpha'_k\alpha'_l\beta_{i,j} + \alpha_i\alpha_j\beta'_{k,l} - \alpha_i\alpha_k\beta'_{j,l} \\
& - \alpha_j\alpha_k\beta'_{i,l} + \alpha_i\alpha_l\beta'_{j,k} + \alpha_j\alpha_l\beta'_{i,k} + \alpha_k\alpha_l\beta'_{i,j} + \alpha'_i\alpha_j\alpha_k\alpha_l - \alpha_i\alpha'_j\alpha'_k\alpha'_l \\
& + \alpha_i\alpha'_j\alpha_k\alpha_l - \alpha'_i\alpha_j\alpha'_k\alpha'_l - \alpha_i\alpha_j\alpha'_k\alpha_l + \alpha'_i\alpha'_j\alpha_k\alpha'_l + \alpha_i\alpha_j\alpha_k\alpha'_l - \alpha'_i\alpha'_j\alpha'_k\alpha_l \\
& - \alpha_i\alpha_j\alpha'_k\alpha'_l + \alpha_i\alpha'_j\alpha_k\alpha'_l + \alpha'_i\alpha_j\alpha_k\alpha'_l - \alpha_i\alpha'_j\alpha'_k\alpha_l - \alpha'_i\alpha_j\alpha'_k\alpha_l - \alpha'_i\alpha'_j\alpha_k\alpha_l.
\end{aligned}$$

Finally, the exponent of each generator $[g_i, g_l] \otimes [g_j, g_k]$ is

$$\begin{aligned}
\tau_{i,l,j,k} = {} & -\alpha'_i\gamma_{j,k,l} - \alpha'_j\gamma_{i,k,l} + \alpha'_k\gamma_{i,j,l} + \alpha'_l\gamma_{i,j,k} + \alpha_i\gamma'_{j,k,l} + \alpha_j\gamma'_{i,k,l} - \alpha_k\gamma'_{i,j,l} \\
& - \alpha_l\gamma'_{i,j,k} + \alpha_i\alpha'_j\beta_{k,l} + \alpha'_i\alpha_j\beta_{k,l} - \alpha'_i\alpha_k\beta_{j,l} - \alpha'_i\alpha_l\beta_{j,k} - \alpha_i\alpha'_k\beta_{j,l} - \alpha_j\alpha'_k\beta_{i,l} \\
& + \alpha'_j\alpha_k\beta_{i,l} + \alpha'_j\alpha_l\beta_{i,k} + \alpha_i\alpha'_l\beta_{j,k} + \alpha_j\alpha'_l\beta_{i,k} - \alpha_k\alpha'_l\beta_{i,j} - \alpha'_k\alpha_l\beta_{i,j} - \alpha'_i\alpha_j\beta'_{k,l} \\
& - \alpha_i\alpha'_j\beta'_{k,l} + \alpha_i\alpha'_k\beta'_{j,l} + \alpha_i\alpha'_l\beta'_{j,k} + \alpha'_i\alpha_k\beta'_{j,l} + \alpha'_j\alpha_k\beta'_{i,l} - \alpha_j\alpha'_k\beta'_{i,l} - \alpha_j\alpha'_l\beta'_{i,k} \\
& - \alpha'_i\alpha_l\beta'_{j,k} - \alpha'_j\alpha_l\beta'_{i,k} + \alpha'_k\alpha_l\beta'_{i,j} + \alpha_k\alpha'_l\beta'_{i,j} + \beta_{i,l}\beta'_{j,k} - \beta'_{i,l}\beta_{j,k} + \alpha'_i\alpha'_j\beta_{k,l} \\
& - \alpha'_i\alpha'_k\beta_{j,l} - \alpha'_j\alpha'_k\beta_{i,l} + \alpha'_i\alpha'_l\beta_{j,k} + \alpha'_j\alpha'_l\beta_{i,k} - \alpha'_k\alpha'_l\beta_{i,j} - \alpha_i\alpha_j\beta'_{k,l} + \alpha_i\alpha_k\beta'_{j,l} \\
& + \alpha_j\alpha_k\beta'_{i,l} - \alpha_i\alpha_l\beta'_{j,k} - \alpha_j\alpha_l\beta'_{i,k} + \alpha_k\alpha_l\beta'_{i,j} - \alpha'_i\alpha_j\alpha_k\alpha_l + \alpha_i\alpha'_j\alpha'_k\alpha'_l \\
& - \alpha_i\alpha'_j\alpha_k\alpha_l + \alpha'_i\alpha_j\alpha'_k\alpha'_l + \alpha_i\alpha_j\alpha'_k\alpha_l - \alpha'_i\alpha'_j\alpha_k\alpha'_l - \alpha_i\alpha_j\alpha_k\alpha'_l + \alpha'_i\alpha'_j\alpha'_k\alpha_l \\
& + \alpha_i\alpha_j\alpha'_k\alpha'_l - \alpha_i\alpha'_j\alpha_k\alpha'_l - \alpha'_i\alpha_j\alpha_k\alpha'_l + \alpha_i\alpha'_j\alpha'_k\alpha_l + \alpha'_i\alpha_j\alpha'_k\alpha_l - \alpha'_i\alpha'_j\alpha_k\alpha_l.
\end{aligned}$$

We now construct the group L_n that we show is isomorphic to $\mathcal{E} \otimes \mathcal{E}$.

Example 3.3. Let F be the free group of rank $n(n-1)$ and set $\mathcal{N}$ to be $F/\gamma_3(F) = \langle y_{i,j} \mid 1 \leqslant i,\ j \leqslant n;\ i \neq j\rangle$, the free nilpotent group of class 2 and rank $n(n-1)$. Set

$$N = \langle [y_{i,j}, y_{j,i}], [y_{i,j}, y_{k,l}][y_{i,j}, y_{l,k}], [y_{i,j}, y_{k,l}][y_{j,i}, y_{k,l}], [y_{i,j}, y_{k,l}][y_{l,k}, y_{j,i}], [y_{i,j}, y_{k,l}]^3 \mid 1 \leqslant i,j,k,l \leqslant n;\ i \neq j;\ k \neq l\rangle.$$

Since $N \leqslant \mathcal{N}' \leqslant Z(\mathcal{N})$, the subgroup N is normal in $\mathcal{N}$. Set

$$W_n = \mathcal{N}/N = \langle w_{i,j} \mid 1 \leqslant i,\ j \leqslant n;\ i \neq j\rangle,$$

where $w_{i,j} = y_{i,j}N$ for $1 \leqslant i,\ j \leqslant n$ and $i \neq j$. Set U_n to be the free abelian group on the generating set

$$\{x_i \mid 1 \leqslant i \leqslant n\} \cup \{u_{i,j,k} \mid 1 \leqslant i,j,k \leqslant n;\ i \leqslant \max(j,k);\ j < k\}.$$

The allowable subscript triples for $u_{i,j,k}$ match the four cases of subscript triples that arise in the generators of the tensor square $\mathcal{E} \otimes \mathcal{E}$ for factors of the form $g_i \otimes [g_j, g_k]$. There are $2\binom{n}{2} + 2\binom{n}{3} = 2\binom{n+1}{3}$ such triples, and thus U_n has rank $n + 2\binom{n+1}{3} = \frac{1}{3}n(n^2+2)$. We set L_n to be the direct product $U_n \times W_n$.

Denote the commutator $[w_{i,j}, w_{k,l}]$ by $z_{i,j,k,l}$. We represent an arbitrary element h of L_n as

$$h = \prod_{i=1}^{n} x_i^{\kappa_i} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\lambda_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\mu_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1\\ j\neq n-i}}^{n} w_{n-i,j}^{\eta_{n-i,j}}, \tag{3.26}$$

where each κ_i, $\lambda_{i,j,k}$ and $\eta_{n-i,j}$ is an integer and each $\mu_{i,j,k,l}$ is an integer modulo 3. Let

$$h' = \prod_{i=1}^{n} x_i^{\kappa'_i} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\lambda'_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\mu'_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1\\ j\neq n-i}}^{n} w_{n-i,j}^{\eta'_{n-i,j}} \tag{3.27}$$

be another element of L_n, where each κ'_i, $\lambda'_{i,j,k}$ and $\eta'_{n-i,j}$ is an integer and each $\mu'_{i,j,k,l}$ is an integer modulo 3. Then the product hh' is

$$hh' = \prod_{i=1}^{n} x_i^{\kappa^*_i} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\lambda^*_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\mu^*_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1\\ j\neq n-i}}^{n} w_{n-i,j}^{\eta^*_{n-i,j}},$$

where

$$\kappa^*_i = \kappa_i + \kappa'_i, \qquad \eta^*_{i,j} = \eta_{i,j} + \eta'_{i,j}, \qquad \lambda^*_{i,j,k} = \lambda_{i,j,k} + \lambda'_{i,j,k} \tag{3.28}$$

and (computing modulo 3)

$$\begin{aligned}
\mu^*_{i,j,i,k} &= \mu_{i,j,i,k} + \mu'_{i,j,i,k} + \eta_{j,i}\eta'_{k,i} - \eta_{i,j}\eta'_{k,i} + \eta_{i,k}\eta'_{j,i} - \eta_{i,k}\eta'_{i,j}, && i<j<k,\\
\mu^*_{i,j,j,k} &= \mu_{i,j,j,k} + \mu'_{i,j,j,k} + \eta_{j,i}\eta'_{k,j} - \eta_{i,j}\eta'_{k,j} + \eta_{j,k}\eta'_{j,i} + \eta_{i,j}\eta'_{j,k}, && i<j<k,\\
\mu^*_{i,k,j,k} &= \mu_{i,k,j,k} + \mu'_{i,k,j,k} - \eta_{k,j}\eta'_{k,i} + \eta_{j,k}\eta'_{k,i} - \eta_{i,k}\eta'_{k,j} + \eta_{i,k}\eta'_{j,k}, && i<j<k,\\
\mu^*_{i,j,k,l} &= \mu_{i,j,k,l} + \mu'_{i,j,k,l} + \eta_{j,i}\eta'_{l,k} - \eta_{i,j}\eta'_{l,k} - \eta_{j,i}\eta'_{k,l} + \eta_{i,j}\eta'_{k,l}, && i<j<k<l,\\
\mu^*_{i,k,j,l} &= \mu_{i,k,j,l} + \mu'_{i,k,j,l} + \eta_{k,i}\eta'_{l,j} - \eta_{i,k}\eta'_{l,j} + \eta_{j,l}\eta'_{k,i} + \eta_{i,k}\eta'_{j,l}, && i<j<k<l,\\
\mu^*_{i,l,j,k} &= \mu_{i,l,j,k} + \mu'_{i,l,j,k} - \eta_{k,j}\eta'_{l,i} + \eta_{j,k}\eta'_{l,i} - \eta_{i,l}\eta'_{k,j} + \eta_{i,l}\eta'_{j,k}, && i<j<k<l.
\end{aligned}$$

Proof of Theorem 1.4. Let g, g', g'' be arbitrary elements of $\mathcal{E}$, where g and g' are defined in (2.4) and (2.5), respectively, and

$$g'' = \prod_{i=1}^{n} g_i^{\alpha''_i} \cdot \prod_{1\leqslant j<k\leqslant n} [g_j, g_k]^{\beta''_{j,k}} \cdot \prod_{1\leqslant r<s<t\leqslant n} [g_r, g_s, g_t]^{\gamma''_{r,s,t}}, \tag{3.29}$$

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where each α''_i and $\beta''_{j,k}$ is an integer and each $\gamma''_{r,s,t}$ is an integer modulo 3. We define the mapping $\Phi : \mathcal{E} \times \mathcal{E} \to L_n$ by

$$\Phi(g, g') = \prod_{i=1}^{n} x_i^{\rho_i} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\sigma_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\tau_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1 \\ j\neq n-i}}^{n} w_{n-i,j}^{\nu_{n-i,j}},$$

where the exponents ρ_i, $\sigma_{i,j,k}$, $\tau_{i,j,k,l}$ and $\nu_{n-i,j}$ are as specified in Lemma 3.1.

To show that Φ is a crossed pairing, we must show that Equations (1.1) and (1.2) hold. Consider Equation (1.1) and let

$$\Phi(gg', g'') = \prod_{i=1}^{n} x_i^{\rho_i^\dagger} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\sigma^\dagger_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\tau^\dagger_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1 \\ j\neq n-i}}^{n} w_{n-i,j}^{\nu^\dagger_{n-i,j}} \quad (3.30)$$

and

$$\Phi({}^g g', {}^g g'')\Phi(g, g'') = \prod_{i=1}^{n} x_i^{\rho_i^\ddagger} \cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\sigma^\ddagger_{i,j,k}} \cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\tau^\ddagger_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1 \\ j\neq n-i}}^{n} w_{n-i,j}^{\nu^\ddagger_{n-i,j}}. \quad (3.31)$$

The expressions (3.30) and (3.31) are equal if the exponents of the corresponding generators are equal. Hence we must show that $\rho_i^\dagger = \rho_i^\ddagger$, $\sigma^\dagger_{i,j,k} = \sigma^\ddagger_{i,j,k}$, $\tau^\dagger_{i,j,k,l} \equiv \tau^\ddagger_{i,j,k,l} \bmod 3$, and $\nu^\dagger_{i,j} = \nu^\ddagger_{i,j}$.

We start by showing that $\rho_i^\dagger = \rho_i^\ddagger$. For each i, where $1 \leqslant i \leqslant n$, consider α_i^*, $\alpha_i^\dagger$, ρ_i, κ_i^* (defined in (2.6), (2.7), (3.24) and (3.28), respectively) as functions of two variables. Thus

$$\alpha_i^*(\alpha_i, \alpha'_i) = \alpha_i + \alpha'_i, \quad \alpha_i^\dagger(\alpha_i, \alpha'_i) = \alpha'_i, \quad \rho_i(\alpha_i, \alpha'_i) = \alpha_i \cdot \alpha'_i \quad \text{and} \quad \kappa_i^*(\kappa_i, \kappa'_i) = \kappa_i + \kappa'_i.$$

By the definition of Φ and the formulae for computing in $\mathcal{E}$ and in L_n, we obtain

$$\begin{aligned} \rho_i^\dagger &= \rho_i(\alpha_i^*(\alpha_i, \alpha'_i), \alpha''_i) \\ &= \rho_i(\alpha_i + \alpha'_i, \alpha''_i) \\ &= (\alpha_i + \alpha'_i)\alpha''_i \\ &= \alpha_i\alpha''_i + \alpha'_i\alpha''_i \end{aligned}$$

and

$$\begin{aligned} \rho_i^\ddagger &= \kappa_i^*(\rho_i(\alpha_i^\dagger(\alpha_i, \alpha'_i), \alpha_i^\dagger(\alpha_i, \alpha''_i)), \rho_i(\alpha_i, \alpha''_i)) \\ &= \kappa_i^*(\rho_i(\alpha'_i, \alpha''_i), \rho_i(\alpha_i, \alpha''_i)) \\ &= \rho_i(\alpha'_i, \alpha''_i) + \rho_i(\alpha_i, \alpha''_i) \\ &= \alpha'_i\alpha''_i + \alpha_i\alpha''_i. \end{aligned}$$

By inspection, $\rho_i^\dagger = \rho_i^\ddagger$, as needed. A similar argument shows that $\nu^\dagger_{i,j} = \nu^\ddagger_{i,j}$ for each pair (i, j), where $1 \leqslant i$, $j \leqslant n$ and $i \neq j$.

We next show

$$\sigma^{\dagger}_{i,j,k} = \sigma^{\ddagger}_{i,j,k}$$

for two of the four possible orderings of the indices i, j and k. First we write $\beta^*_{i,j}$, $\beta^{\dagger}_{i,j}$ and $\lambda^*_{i,j,k}$ (defined in (2.6), (2.7) and (3.28), respectively) as functions:

$$\begin{aligned}\beta^*_{i,j}(\alpha_i,\alpha_j,\beta_{i,j},\alpha'_i,\alpha'_j,\beta'_{i,j}) &= \beta_{i,j}+\beta'_{i,j}-\alpha'_i\alpha_j,\\ \beta^{\dagger}_{i,j}(\alpha_i,\alpha_j,\beta_{i,j},\alpha'_i,\alpha'_j,\beta'_{i,j}) &= \beta'_{i,j}-\alpha'_i\alpha_k+\alpha_i\alpha'_j,\\ \lambda^*_{i,j,k}(\lambda_{i,j,k},\lambda'_{i,j,k}) &= \lambda_{i,j,k}+\lambda'_{i,j,k}.\end{aligned}$$

Consider the case when the indices are ordered $i < j < k$. In functional notation, Equation (3.25) becomes

$$\begin{aligned}&\sigma_{i,j,k}(\alpha_i,\alpha_j,\alpha_k,\beta_{i,j},\beta_{i,k},\beta_{j,k},\alpha'_i,\alpha'_j,\alpha'_k,\beta'_{i,j},\beta'_{i,k},\beta'_{j,k})\\ &\quad = -\alpha'_i\beta_{j,k}+\alpha_i\beta'_{j,k}-\alpha_i\alpha'_j\alpha_k+\alpha'_i\alpha_j\alpha'_k+\alpha_i\alpha_j\alpha'_k-\alpha'_i\alpha'_j\alpha_k+\alpha'_k\beta_{i,j}-\alpha_k\beta'_{i,j}.\end{aligned}$$

Using the definition of Φ and the formulae for computing in $\mathcal{E}$ and in L_n, we express

$$\begin{aligned}\sigma^{\dagger}_{i,j,k} = \sigma_{i,j,k}(&\alpha^*_i(\alpha_i,\alpha'_i),\alpha^*_j(\alpha_j,\alpha'_j),\alpha^*_k(\alpha_k,\alpha'_k),\beta^*_{i,j}(\alpha_i,\alpha_j,\beta_{i,j},\alpha'_i,\alpha'_j,\beta'_{i,j}),\\ &\beta^*_{i,k}(\alpha_i,\alpha_k,\beta_{i,k},\alpha'_i,\alpha'_k,\beta'_{i,k}),\beta^*_{j,k}(\alpha_j,\alpha_k,\beta_{j,k},\alpha'_j,\alpha'_k,\beta'_{j,k}),\\ &\alpha''_i,\alpha''_j,\alpha''_k,\beta''_{i,j},\beta''_{i,k},\beta''_{j,k})\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\sigma^{\ddagger}_{i,j,k} = \lambda^*_{i,j,k}(\sigma_{i,j,k}(&\alpha^{\dagger}_i(\alpha_i,\alpha'_i),\alpha^{\dagger}_j(\alpha_j,\alpha'_j),\alpha^{\dagger}_k(\alpha_k,\alpha'_k),\beta^{\dagger}_{i,j}(\alpha_i,\alpha_j,\beta_{i,j},\alpha'_i,\alpha'_j,\beta'_{i,j}),\\ &\beta^{\dagger}_{i,k}(\alpha_i,\alpha_k,\beta_{i,k},\alpha'_i,\alpha'_k,\beta'_{i,k}),\beta^{\dagger}_{j,k}(\alpha_j,\alpha_k,\beta_{j,k},\alpha'_j,\alpha'_k,\beta'_{j,k}),\\ &\alpha^{\dagger}_i(\alpha_i,\alpha''_i),\alpha^{\dagger}_j(\alpha_j,\alpha''_j),\alpha^{\dagger}_k(\alpha_k,\alpha''_k),\beta^{\dagger}_{i,j}(\alpha_i,\alpha_j,\beta_{i,j},\alpha''_i,\alpha''_j,\beta''_{i,j}),\\ &\beta^{\dagger}_{i,k}(\alpha_i,\alpha_k,\beta_{i,k},\alpha''_i,\alpha''_k,\beta''_{i,k}),\beta^{\dagger}_{j,k}(\alpha_j,\alpha_k,\beta_{j,k},\alpha''_j,\alpha''_k,\beta''_{j,k})),\\ &\sigma_{i,j,k}(\alpha_i,\alpha_j,\alpha_k,\beta_{i,j},\beta_{i,k},\beta_{j,k},\alpha''_i,\alpha''_j,\alpha''_k,\beta''_{i,j},\beta''_{i,k},\beta''_{j,k})).\end{aligned} \tag{3.33}$$

MAPLE [**19**] was then used to compose these functions and to simplify the resulting expressions. All subsequent calculations were performed with MAPLE and independently checked using GAP [**11**]. In particular, these computations immediately show that $\sigma^{\dagger}_{i,j,k} - \sigma^{\ddagger}_{i,j,k} = 0$.

The corresponding computation of the case $\sigma^{\dagger}_{i,i,j} - \sigma^{\ddagger}_{i,i,j}$, where $i < j$, results in the expression

$$-\alpha''_j\binom{\alpha'_i}{2} - \alpha''_j\binom{\alpha_i}{2} - \alpha_i\alpha'_i\alpha''_j + \alpha''_j\binom{\alpha_i+\alpha'_i}{2}.$$

Since, for any two integers m and n, the identity

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn \tag{3.34}$$

holds, this expression also evaluates to zero. By similar calculations and applications of (3.34), for each of the remaining two cases, $\sigma^{\dagger}_{i,j,k} - \sigma^{\ddagger}_{i,j,k}$ is also zero.

The exponent of $z_{i,j,k,l}$ has six possible orderings of the indices i, j, k and l. The case $\tau^{\dagger}_{i,j,i,k} - \tau^{\ddagger}_{i,j,i,k} \bmod 3$ for $i < j < k$ yields the expression

$$\begin{aligned}\beta''_{j,k}&\left(\alpha_i\alpha'_i + \binom{\alpha_i}{2} + \binom{\alpha'_i}{2} - \binom{\alpha_i+\alpha'_i}{2}\right)\\ &+ \alpha'_j\alpha''_k\left(\alpha_i - 2\alpha_i\alpha'_i - \alpha_i^2 + \binom{\alpha'_i}{2} - \binom{\alpha_i+\alpha'_i}{2}\right)\\ &\quad + \alpha_j\alpha'_k\left(\alpha''_i - \alpha''^2_i - \binom{\alpha''_i}{2}\right) + \alpha_j\alpha''_k\left(\alpha_i\alpha'_i + \binom{\alpha_i}{2} + \binom{\alpha'_i}{2} - \binom{\alpha_i+\alpha'_i}{2}\right)\\ &\qquad + \alpha''_j\alpha_k\left(\binom{\alpha_i+\alpha'_i}{2} - \alpha_i\alpha'_i - \binom{\alpha_i}{2} - \binom{\alpha'_i}{2}\right)\\ &\qquad\quad + \alpha''_j\alpha'_k\left(\alpha_i^2 - \alpha_i + 2\alpha_i\alpha'_i - \binom{\alpha'_i}{2} + \binom{\alpha_i+\alpha'_i}{2}\right).\end{aligned}$$

By the identity (3.34) and the fact that $\binom{m}{2} \equiv m - m^2 \bmod 3$, we conclude that

$$\tau^{\dagger}_{i,j,i,k} - \tau^{\ddagger}_{i,j,i,k} \equiv 0 \mod 3.$$

For each of the other five possible orderings of the indices i, j, k and l, the expression $\tau^{\dagger}_{i,j,k,l} - \tau^{\ddagger}_{i,j,k,l} \bmod 3$ was similarly computed using MAPLE and verified to be congruent to zero modulo three.

These computations show that Equation (1.1) holds for Φ. Equation (1.2) holds by a similar analysis. Thus Φ is a crossed pairing.

By Proposition 1.3, Φ determines a unique homomorphism $\Phi^* : \mathcal{E} \otimes \mathcal{E} \to L_n$ with $\Phi^*(g \otimes g') = \Phi(g, g')$ for all $g, g' \in \mathcal{E}$. It follows that

$$\begin{aligned}\Phi^*(g_i \otimes g_i) &= \Phi(g_i, g_i) = x_i,\\ \Phi^*(g_i \otimes g_j) &= \Phi(g_i, g_j) = w_{i,j},\\ \Phi^*(g_i \otimes [g_j, g_k]) &= \Phi(g_i, [g_j, g_k]) = u_{i,j,k},\\ \Phi^*([g_i, g_j] \otimes [g_k, g_l]) &= \Phi([g_i, g_j], [g_k, g_l]) = z_{i,j,k,l}.\end{aligned}$$

It remains to be shown that Φ^* is one-to-one and onto. Let h, represented as in (3.26), be an arbitrary element of L_n and define V in $\mathcal{E} \otimes \mathcal{E}$ to be

$$\begin{aligned}V = \prod_{1\leqslant i\leqslant n} (g_i \otimes g_i)^{\kappa_i} \cdot \prod_{(i,j,k)\in I_1} (g_i \otimes [g_j, g_k])^{\lambda_{i,j,k}}\\ \cdot \prod_{(i,j,k,l)\in I_2} ([g_i, g_j] \otimes [g_k, g_l])^{\mu_{i,j,k,l}} \cdot \prod_{i=0}^{n-1} \prod_{\substack{j=1\\ j\neq n-i}}^{n} (g_{n-i} \otimes g_j)^{\eta_{n-i,j}}. \quad (3.35)\end{aligned}$$

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Then

$$\begin{aligned}\Phi^*(V) &= \prod_{1\leqslant i\leqslant n} \Phi^*(g_i\otimes g_i)^{\kappa_i}\cdot \prod_{(i,j,k)\in I_1} \Phi^*(g_i\otimes [g_j,g_k])^{\lambda_{i,j,k}} \\ &\qquad \cdot \prod_{(i,j,k,l)\in I_2} \Phi^*([g_i,g_j]\otimes[g_k,g_l])^{\mu_{i,j,k,l}}\cdot \prod_{i=0}^{n-1}\prod_{\substack{j=1\\ j\neq n-i}}^{n} \Phi^*(g_{n-i}\otimes g_j)^{\eta_{n-i,j}} \\ &= \prod_{i=1}^{n} x_i^{\kappa_i}\cdot \prod_{(i,j,k)\in I_1} u_{i,j,k}^{\lambda_{i,j,k}}\cdot \prod_{(i,j,k,l)\in I_2} z_{i,j,k,l}^{\mu_{i,j,k,l}}\cdot \prod_{i=0}^{n-1}\prod_{\substack{j=1\\ j\neq n-i}}^{n} w_{n-i,j}^{\eta_{n-i,j}} \\ &= h.\end{aligned}$$

Hence Φ^* is onto.

Suppose $\Phi^*(V) = 1_{L_n}$, where V is defined as in (3.35). It is immediate from the definitions of L_n and Φ that $V = 1_{\mathcal{E}\otimes\mathcal{E}}$. We conclude that $\Phi^* : \mathcal{E}\otimes\mathcal{E} \to L_n$ is one-to-one, and therefore is an isomorphism. □

Proof of Corollary 1.5. Let $\mathcal{B}$ denote the Burnside group of rank n and exponent 3 and let

$$f(n) = n + 2\binom{n+1}{3} + 3\binom{n+1}{4} + n(n-1),$$

which enumerates both the generators listed in Equation (3.7) and the generators x_i, $u_{i,j,k}$, $z_{i,j,k,l}$ and $w_{n-i,j}$ of L_n. As we noted in Remark 3.2, our analysis holds for $\mathcal{B}\otimes\mathcal{B}$ by replacing 'integer' with 'integer modulo 3' throughout. It follows that $|\mathcal{B}\otimes\mathcal{B}| \leqslant 3^{f(n)}$. Let $\bar{g}, \bar{g}', \bar{g}'' \in \mathcal{B}$ have representations analogous to (2.4), (2.5) and (3.29), where all of the exponents are taken modulo 3. We define $\Psi : \mathcal{B}\times\mathcal{B} \to L_n/L_n^3$ by

$$\Psi(\bar{g}, \bar{g}') = \Psi(g\mathcal{E}^3, g'\mathcal{E}^3) = \Phi(g, g')L_n^3.$$

Since Φ is a crossed pairing, a similar argument shows that Ψ is a crossed pairing. Thus, by Proposition 1.3 and the fact that Ψ is onto, L_n/L_n^3 is a homomorphic image of $\mathcal{B}\otimes\mathcal{B}$. Thus $3^{f(n)} \geqslant |\mathcal{B}\otimes\mathcal{B}| \geqslant |L_n/L_n^3| = 3^{f(n)}$, proving our claim. □

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