Chaos and the Explanatory Significance of Equilibrium: Strange Attractors in Evolutionary Game Dynamics[1]

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1. Introduction

The classical game theory of von Neumann and Morgenstern (1947) is built on the concept of equilibrium. I will begin this essay with two more or less controversial philosophical claims regarding that equilibrium concept:

(1) The explanatory significance of the equilibrium concept depends on the underlying dynamics.

(2) When the underlying dynamics is taken seriously, it becomes apparent that equilibrium is not the central explanatory concept.

With regard to the first thesis, let me emphasize a point first made by von Neumann and Morgenstern themselves. Their theory is a static theory which discusses the nature and existence of equilibrium, but which does not address the question: "How is equilibrium reached?" The explanatory significance of the equilibrium concept, however, depends on the plausibility of the underlying dynamics which is supposed to bring players to equilibrium. One sort of story supposes that the decision makers involved reach equilibrium by an idealized reasoning process which requires a great deal of common knowledge, godlike calculational powers, and perhaps allegiance to the recommendations of a particular theory of strategic interaction. Another kind of story—deriving from evolutionary biology—views game theoretic equilibria as fixed points of evolutionary adaptation, with none of the rational idealization of the first story. The power of game theory to explain a state of affairs as an equilibrium thus depends on the viability of a dynamical scenario appropriate to the situation in question, which shows how such an equilibrium would be reached.

It is well-known that the problem is especially pressing in an area of game theory which von Neumann and Morgenstern did not emphasize: the theory of non-zero sum, non-cooperative games. Here, unlike the zero-sum case, many non-equivalent equilibria are possible. If different decision makers aim for different equilibria, then the joint result of their actions may not be an equilibrium at all. Thus the dynamics must bear the burden of accounting for *equilibrium selection* by the players, because without an account of equilibrium selection the equilibrium concept itself loses its plausibility.

Once one has asked the first dynamical question: "How is equilibrium reached?" it becomes impossible not to ask the more radical question: "Is equilibrium reached?" Perhaps it is not. If not, then it is important to canvass the ways in which may not be reached and explore complex non-convergent behavior permitted by the underlying dynamics. This essay will take a small step in that direction.

In particular, I will present numerical evidence for extremely complicated behavior in the evolutionary game dynamics introduced by Taylor and Jonker (1978). This dynamics, which is based on the process of replication, is found at various levels of chemical and biological organization (Hofbauer and Sigmund 1988). For a taste of what is possible in this dynamics with only four strategies, see the "strange attractor" in figure 1. This is a projection of a single orbit for a four strategy evolutionary game onto the three simplex of the probabilities of the first three strategies. A strange attractor cannot occur in the Taylor-Jonker flow in three strategy evolutionary games because the dynamics takes place on a two dimensional simplex. Zeeman (1980) leaves it open as to whether strange attractors are possible in higher dimensions or not. This paper presents strong numerical evidence for the existence of strange attractors in the lowest dimension in which they could possibly occur.
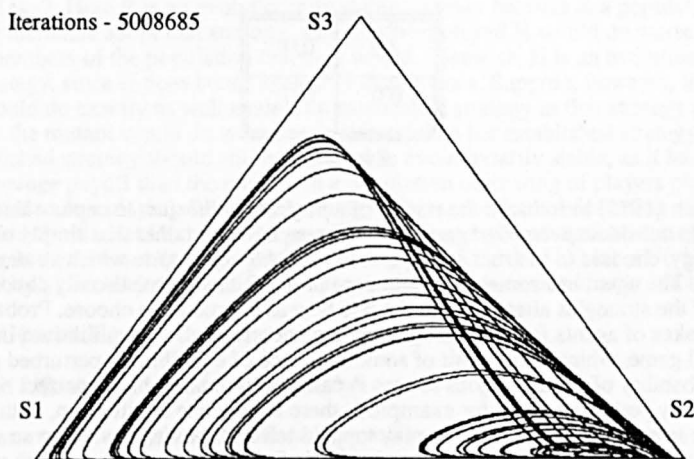

Figure 1: Parameter = 5

The plan of the paper is as follows: Sections 2, 3, and 4 introduce key concepts of games, dynamics, and evolutionary game dynamics. Section 5 will describe the four-strategy evolutionary game which gives rise to chaotic dynamics, and the bifurcations which lead to chaos as the parameters of the model are varied. Section 6 will give a stability analysis of the equilibria encountered along the road to chaos described in section 5. Section 7 describes the numerical calculation of Liapunov exponents. Section 8 indicates some related literature, and discusses the relation to Lotka-Volterra ecological models. My second philosophical claim will be discussed in Section 9.

2. Games

We will be concerned with finite, non-cooperative, normal form games. There are a finite number of players and each player has a finite number of possible strategies.

Each player has only one choice to make and makes it without being informed of the choices of any other players. The games are to be thought of as non-cooperative. There is no communication or precommitment before the players make their choices. Each possible combination of strategies determines the payoffs for each of the players.

A specification of the number of players, the number of strategies for each player and the payoff function determines the game. A *Nash equilibrium* of the game is a strategy combination such that no player does better on any unilateral deviation. We extend players' possible acts to include randomized choices at specified probabilities over the originally available acts. The new randomized acts are called *mixed strategies*, and the original acts are called *pure strategies*. The payoffs for mixed strategies are defined as their expected values using the probabilities in the mixed acts to define the expectation (and assuming independence between different players acts.) We will assume that mixed acts are always available. Then every finite non-cooperative normal form game has a Nash equilibrium.

The game in example 1 has two Nash equilibria in pure strategies, one at <bottom, right> and one at <top, left>. Intuitively, the former equilibrium is—in some sense - highly instable, and the latter equilibrium is the only sensible one.

1,1	0,0
0,0	0,0

Example 1

Selten (1975) introduced the notion of a *perfect equilibrium* to capture this intuition. He considers *perturbed* games wherein each player rather that simply choosing a strategy, chooses to instruct a not perfectly reliable agent as to which strategy to choose. The agent has some small non-zero probabilities for mistakenly choosing each of the strategies alternative to the one he was instructed to choose. Probabilities of mistakes of agents for different players are uncorrelated. An equilibrium in the original game, which is the limit of some sequence of equilibria in perturbed games as the probability of mistakes goes to zero is called a (trembling-hand) perfect equilibrium. In any perturbed game for example 1, there is only one equilibrium, with row and column instructing their agents to play top and left and their agents doing so with probability of one minus the small probability of a mistake.

Classical game theory is intended as a theory of strategic interaction between rational human payoff-maximizers. It has sometimes been criticized as incorporating an unrealistically idealized model of human rationality. Maynard-Smith and Price (1973) found a way to apply game theory to model conflicts between animals of the same species. The rationale obviously cannot be that snakes or mule deer are hyperrational, but rather that evolution is a process with a tendency in the direction of increased payoff where payoff is reckoned in terms of evolutionary fitness. A rest point of such a process must be an optimal point. The insight that just such a tendency may be enough to make rational choice theory and game theory relevant can be carried back to the human realm, and accounts for much of the current interest in dynamic models of learning and deliberation in game theoretic contexts.

Maynard Smith and Price are interested in providing an evolutionary explanation of "limited war" type conflicts between members of the same species, without re-

course to group selection. The key notion that they introduce is that of a strategy that would be a stable equilibrium under natural selection, an *evolutionarily stable strategy*. If all members of the population adopt that strategy, then no mutant can invade. Suppose that there is a large population, that contests are pairwise and that pairing is random. Then the relevant payoff is the average change in evolutionary fitness of an individual, and it is determined by its strategy and the strategy against which it is paired. These numbers can be conveniently presented in a *fitness matrix* and can be thought of as defining the evolutionary game. The fitness matrix is read as giving row's payoff when playing against column.

	R	H
R	2	-3
H	-1	-2

Example 2

Thus in example 2, the payoff to R when playing against R is 2 but when playing against H is -3. The payoff to H when playing against R is -1 and when playing against H is -2. Here R is an evolutionarily stable strategy because in a population where all members adopt that strategy, a mutant who played H would do worse against members of the population that they would. Likewise, H is an evolutionarily stable strategy, since H does better against H than R does. Suppose, however, that a mutant could do exactly as well against an established strategy as that strategy against itself, but the mutant would do worse against itself than the established strategy. Then the established strategy should still be counted as evolutionarily stable, as it has greater average payoff than the mutant, in a population consisting of players playing it together with a few playing the mutant strategy. This is the formal definition adopted by Maynard-Smith. Let $U(x|y)$ be the payoff to strategy x played against strategy y. Strategy x is *evolutionarily stable* just in case $U(x|x) > U(y|x)$ or $U(x|x) = U(y|x)$ and $U(x|y) > U(y|y)$ for all y different from x. Equivalently, x is evolutionarily stable if:

1: $U(x|x) \geq U(y|x)$

2: If $U(x|x)=U(y|x)$ then $U(x|y)>U(y|y)$

The fitness matrix determines a symmetric payoff matrix for a two person game—the symmetry deriving from the fact that only the strategies matter, not whether they are played by row or column—as is shown in example 3.

2,2	-3,-1
-1,-3	-2,-2

Example 3

An evolutionarily stable strategy is—by condition 1 above—a symmetric Nash equilibrium of the two-person non-cooperative game. Condition 2 adds a kind of stability requirement.

The formal definition of evolutionarily stable strategy applies to mixed strategies as well as pure ones, and some fitness matrices will have the consequence that the only evolutionarily stable strategy is a mixed one. This is illustrated in example 4.

	H	D
H	-2	2
D	0	1

Example 4

Neither H nor D is an evolutionarily stable strategy, but a mixed strategy, M, of (1/3) H, (2/3) D is. This illustrates condition 2 in the definition of evolutionarily stable strategy. U(x|M)=2/3 if x is H or D or any mixture of H and D. But an invader who plays H or D or a different mixture of H and D will do worse against herself that M does against her. For example, consider H as an invader. U(H|H)=-2 while U(M|H)=-2/3. The interpretation of mixed strategies as strategies adopted by each member of the population is the only one which makes sense of the characterization: if all members of the population adopt that strategy, then no mutant can invade. There is an alternative interpretation of mixed strategies in terms of proportions of a polymorphic population, all of whose members play pure strategies. The formal definition of evolutionarily stable strategy in terms of 1 and 2 still makes sense on this reinterpretation of mixed strategies.

If we consider the two person non-cooperative normal form game associated with a fitness matrix, an evolutionarily stable strategy, x, induces a symmetric Nash equilibrium <x,x> of the game which has certain stability properties. Earlier, we considered Selten's concept of perfect equilibrium, which rules out certain instabilities. Evolutionary stability is a stronger requirement than perfection. If x is an evolutionarily stable strategy, then <x,x> is a perfect symmetric Nash equilibrium of the associated game, but the converse does not hold. In the game associated with the fitness matrix in example 5, <S2,S2> is a perfect equilibrium.[2]

	S1	S2	S3
S1	1	0	-9
S2	0	0	-4
S3	-9	-4	-4

Example 5

S2, however, is not an evolutionarily stable strategy because U(S1|S2)=U(S2|S2) and U(S1|S1)>U(S2|S1).

The concepts of equilibrium and stability in game theory are quasi-dynamical notions. How do they relate to their full dynamical counterparts when game theory is embedded in a dynamical theory of equilibration?

3. Dynamics

The state of a system is characterized by a state vector, x, which specifies the values of relevant variables. (In the case of prime interest here, the relevant variables will be the probabilities of strategies in a game.) The dynamics of the system specifies how the state vector evolves in time. The path that a state vector describes in state space as it evolves according to the dynamics is called a trajectory, or orbit. Time can either be modeled as discrete or as continuous. For the former case, a deterministic dynamics consists of a map which may be specified by a system of difference equations:

$$x(t + 1) = f(x(t))$$

In the latter case, a deterministic dynamics is a flow which may be specified by a system of differential equations:

$$dx/dt = f(x(t))$$

An *equilibrium point* is a fixed point of the dynamics. In the case of discrete time, it is a point, x of the state space such that $f(x) = x$. For continuous time, it is a state, $x = <x_1,...,x_i,...>$ such that $dx_i/dt = 0$, for all i. An equilibrium x is *stable* if points near to it remain near to it. More precisely, x is stable if for every neighborhood, V of x, there is a neighborhood, V', of x such that if the state y is in V' at time t=0, it remains in V for all time t > 0. A equilibrium, x, is *strongly stable* (or asymptotically stable) if nearby points tend towards it. That is, to the definition of stability we add the clause that the limit as t goes to infinity of $y(t) = x$.

An *invariant set* is a set, S, of points of the state space such that if the system starts at a point in S, then at any subsequent time the state of the system is still in S. A unit set is an invariant set just in case it's member is an equilibrium. A closed invariant set, S, is an *attracting set* if nearby points tend towards it; that is, if there is a neighborhood, V, of S such that the orbit of any point in V remains in V and converges to S. An *attractor* is an indecomposable attracting set. [Sometimes other conditions are added to the definition.]

A dynamical system displays *sensitive dependence on initial conditions* at a point if the distance between the orbits of that point and one infinitesimally close to it increases exponentially with time. This sensitivity can be quantified by the *Lyapunov exponent*(s) of an orbit. For a one-dimensional map, $x(t+1) = f(x(t))$, this is defined as follows[3]:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log_2\left|\frac{df}{dx} \, at \, x_i\right|$$

A positive Lyapunov exponent may be taken as the mark of a *chaotic* orbit. For example, consider the "tent" map:

Tent:

$$x(t+1) = 1 - 2\left|\frac{1}{2} - x(t)\right|$$

The derivative is defined and its absolute value is 2 at all points except x=1/2. Thus, for almost all orbits the Lyapunov exponent is equal to one.

An attractor for which the orbit of almost every point is chaotic is a *strange attractor*. For most known "strange attractors"—like the Lorenz attractor and the Rössler attractor—there is no mathematical proof that they are strange attractors, although the computer experiments strongly suggest that they are. The "strange attractor" in game dynamics which appears in figure 1 and which will be discussed in sections 5-7 has the same status.

4. Game Dynamics

A number of different dynamical models of equilibration processes have been studied in economics and biology. Perhaps the oldest is the dynamics considered by

Cournot (1897) in his studies of oligopoly. There is a series of production quantity setting by the oligopolists, at each time period of which each oligopolist makes her optimal decision on the assumption that the others will do what they did in the last round. The dynamics of the system of oligopolists is thus defined by a *best response map*. A Nash equilibrium is a fixed point of this map. It may be dynamically stable or unstable, depending on the parameters of the Cournot model.

A somewhat more conservative adaptive strategy has been suggested by evolutionary game theory. Here we will suppose that there is a large population, all of whose members play pure strategies. The interpretation of a mixed strategy is now as a polymorphism of the population. Asexual reproduction is assumed for simplicity. We assume that individuals are paired at random, that each individual engages in one contest (per unit time), and that the payoff in terms of expected number of offspring to an individual playing strategy S_i against strategy S_j is U_{ij}—given in the ith row and jth column of the fitness matrix, U. The proportion of the population playing strategy S_j will be denoted by $Pr(S_j)$. The expected payoff to strategy i is:

$$U(S_i) = \sum_j pr(S_j) \, U_{ij}$$

The average fitness of the population is:

$$U(StatusQuo) = \sum_i pr(S_i) \, U(S_i)$$

The interpretation of payoff in terms of Darwinian fitness then gives us a map for the dynamics of evolutionary games in discrete time:

$$pr'(S_i) = pr(S_i) \frac{U(S_i)}{U(Status\ Quo)}$$

(where pr' is the proportion in the next time period.}
The corresponding flow is given by:

$$\frac{pr'(S_i)}{dt} = pr(S_i) \frac{U(S_i) - (Status\ Quo)}{U(Status\ Quo)}$$

As long as we are concerned—as we are here—only with symmetric evolutionary games, the same orbits are given by a simpler differential equation:

$$\frac{pr'(S_i)}{dt} = pr(S_i)[U(S_i) - U(StatusQuo)]$$

This equation was introduced by Taylor and Jonker (1978) to provide a dynamical foundation for the quasi-dynamical notion of evolutionarily stable strategy of Maynard Smith and Price (1973). It has subsequently been studied by Zeeman (1980), Hofbauer (1981), Bomze (1985), van Damme (1987), Hofbauer and Sigmund (1988), Samuelson (1988), Crawford (1989) and Nachbar (1990). It will be the dynamics considered in our example in the next section. It worth noting that even though the Taylor-Jonker dynamics is motivated by context where the payoffs are measured on an absolute scale of evolutionary fitness, nevertheless the orbits in phase space (although not the velocity along these orbits) is invariant under a linear transformation of the payoffs. Thus the Taylor-Jonker dynamics may be of some interest in contexts for which it was not intended, where the payoffs are given in von Neumann-Morgenstern utilities.

Relying on the foregoing studies, I will briefly summarize some of the known relations between quasi-dynamical equilibrium concepts and dynamical equilibrium concepts for this dynamics. If [M,M] is a Nash equilibrium of the two-person non-cooperative game associated with an evolutionary game, then M is a dynamic equilibrium of the Taylor-Jonker flow. The converse is not true, since every pure strategy is an equilibrium of the flow. However, if an orbit starts at a completely mixed point and converges to a pure strategy then that strategy is a Nash equilibrium. Furthermore, if M is a stable dynamic equilibrium in the Taylor-Jonker flow, then [M,M] must be a Nash equilibrium of the associated game. However if M is dynamically stable, [M,M] need not be perfect, and if [M,M] is perfect, then M need not be dynamically stable. If M is dynamically strongly stable (asymptotically stable) then [M,M] must be perfect, but the converse does not hold. If M is an evolutionarily stable strategy in the sense of Maynard-Smith and Price then it is perfect, but the converse does not hold. We do have equivalence between evolutionarily stable strategy and strongly dynamically stable strategy in the special case of two strategy evolutionary games, but already in the case of three strategies there can be a strongly dynamically stable polymorphic population which is not a mixed evolutionarily stable strategy. Thus, although their are important relations here between the quasi-dynamical and dynamical equilibrium concepts, they tend to draw the line at somewhat different places.

As an example of a third kind of dynamics, we mention the fictitious play of Brown (1951). Like the Cournot dynamics, there is a process in discrete time, at each stage of which each player plays a strategy which maximizes expected utility, according to her beliefs. But these beliefs are not quite so naive as those of the Cournot player. Rather than proceeding on the assumption that all other players will do just what they did last time, Brown's players form their probabilities of another player's next act according to the proportion of times that player has played that strategy in the past.[4] Brown interpreted his as fictitious play, and Cournot interpreted his as real play, but either could just as well be interpreted the other way. Thorlund-Peterson (1990) studies a dynamics closely related to Brown's in the context of a Cournot oligopoly, where it is shown to have convergence properties superior to those of the Cournot dynamics. Brown's dynamics is driven by a simple inductive rule: Use the observed relative frequency as your probability. The basic scheme could be implemented using modified inductive rules. A class of simple Bayesian inductive rules which share the same asymptotic properties as Brown's rule are investigated in Skyrms (1991). For these models, if the dynamics converges it converges to a Nash equilibrium in undominated strategies. For two-person games, such an equilibrium must be *perfect*. This contrasts with the Taylor-Jonker dynamics where an orbit can converge to a dynamically stable equilibrium, M, where [M,M] is an imperfect equilibrium of the corresponding two-person non-cooperative game.

5. The Road to Chaos

In this section we will focus on the Taylor-Jonker flow. Flows are usually better behaved than the corresponding maps, but we will see that this dynamics is capable of quite complicated behavior. Taylor and Jonker already note the possibility of non-convergence because of oscillations in three strategy evolutionary games. They consider the game whose fitness matrix, U, is given in example 6 (where a is a parameter to be varied):

	S1	S2	S3
S1	2	1	5
S2	5	a	0
S3	1	4	3

Example 6

For a=1 the completely mixed equilibrium serves as an example of an equilibrium which is dynamically strongly stable but not an evolutionarily stable strategy. For a<3 the equilibrium is strongly stable, but at a=3 a qualitative change takes place. Now the mixed equilibrium is stable but not strongly stable. It is surrounded by closed orbits. At a>3 the mixed equilibrium is unstable and the trajectories spiral outward to the boundary of the space. The change that takes place at a=3 is a *degenerate* Hopf bifurcation.[See Guckenheimer and Holmes 1986, pp. 73 and 150 ff.] It is degenerate because the situation at a=3 is not structurally stable. Any small perturbation of the value of a destroys the closed orbits. This is just about as wild as the dynamical behavior can get with three strategies. In particular, *generic* Hopf bifurcations are impossible here. [See Zeeman (1980) and Hofbauer (1981). Zeeman proves that a generic Hopf bifurcation is impossible for 3-strategy games, and describes the structurally stable flows for such games under the assumption that is discharged in Hofbauer.] And chaotic strange attractors are not possible, because the flow takes place on a two-dimensional simplex.

However, with four strategies we get the strange attractor pictured in figure 1. (This is a projection of of the three dimensional simplex of probabilities for four strategies onto the two dimensional simplex of the first three strategies. The three dimensional structure, however, is fairly easy to see in the figure.) There is a route to this strange attractor that leads through a generic Hopf bifurcation. Consider the fitness matrix, U, of example 7 (where a is the parameter to be varied):

-1	-1	-10	1,000
-1.5	-1	-1	1,000
a	.5	0	-1,000
0	0	0	0

Example 7

Figures 1 through 6 are snapshots taken along the path to chaos as this parameter is varied. At a=2.4 there is convergence to a mixed equilibrium as shown in figure 2. The orbit spirals in towards the mixed equilibrium which is visible as the white dot in the center of the orbit. As the value of a is raised there is a generic Hopf bifurcation giving rise to a limit cycle around the mixed equilibrium. This closed orbit is structurally stable; it persists for small variations in the parameter. It is also an attracting set. This closed orbit is shown for a=2.55 in figure 3. As the value of the parameter is raised further, the limit cycle expands and then undergoes a period doubling bifurcation. Figure 4 shows the cycle of period 2 at a=3.885. This is followed by another period doubling bifurcation, leading to a cycle of period 4 at a = 4.0, as shown in figure 5. There are very long transients before the orbit settles down to this cycle. At a=5, we get a transition to chaotic dynamics on the strange attractor shown in figure 1. Raising the parameter to a=6 leads to further geometrical complications in the strange attractor as shown in figure 6.
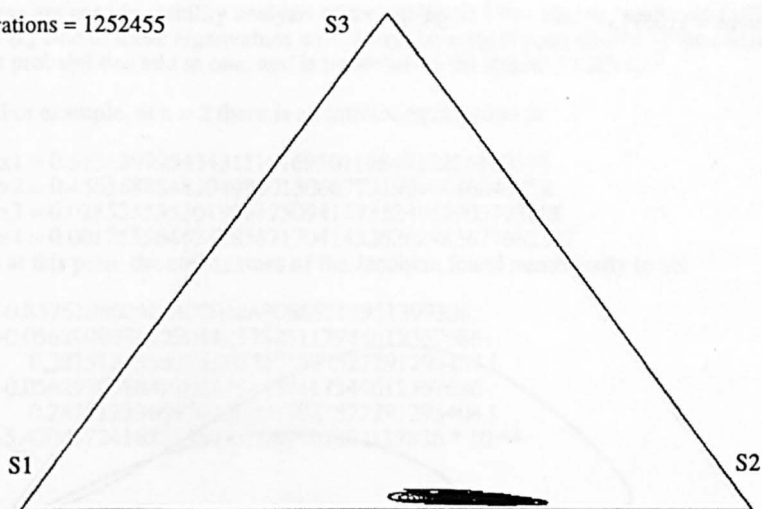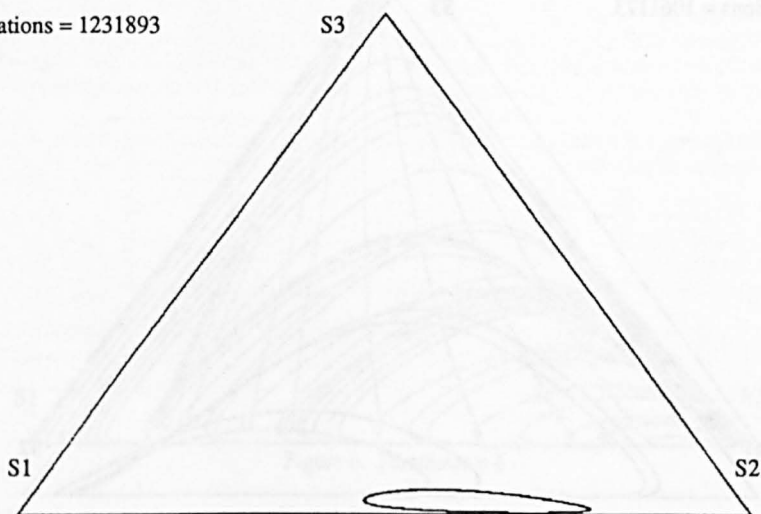
Figure 2. Parameter = 2.4

Figure 3. Parameter = 2.55

Differential equations were numerically integrated in double precision using fourth-order Runge-Kutta method [Press, et al (1989)].For figures 1 through 4 and 6 a fixed step size of .001 was used. For figure 5 a fixed step size of .01 was used. This was done on an IBM model 70 personal computer with a 387 math coprocessor. The projection of the orbit on the simplex of probabilities of the first three strategies was plotted to the screen in vga graphics mode. For figures 1 through 4, the first 50,000 steps (= 50 time units) were not plotted to eliminate transients. For figure 5, the first 100,000 (=1,000 time units) steps were omitted to eliminate very long transients. For
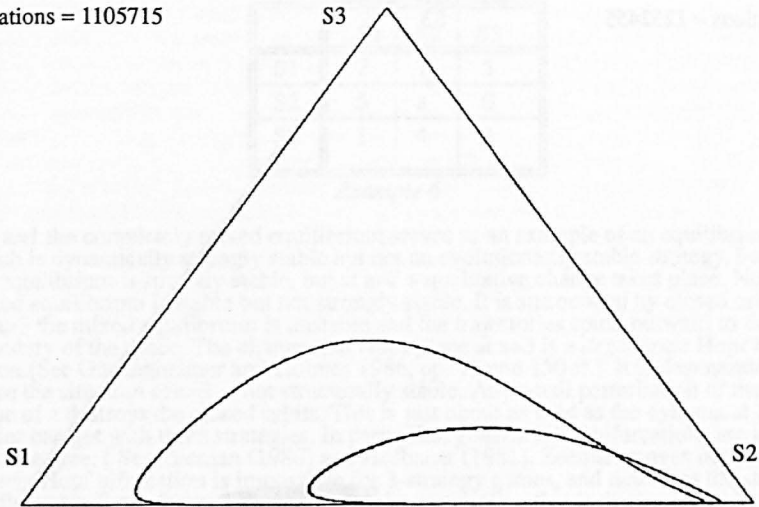
Iterations = 1105715　　　　　　　　　　S3

Figure 4. Parameter = 3.885
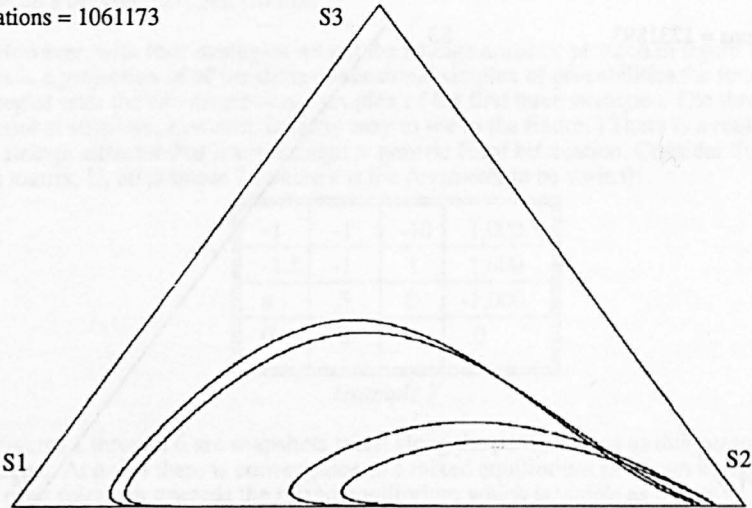
Iterations = 1061173　　　　　　　　　　S3

Figure 5. Parameter = 4

figure 6, only the first 1,000 steps were omitted. In each case, the total number of steps run is shown in the top left corner of the illustration. The screen was captured using the WordPerfect 5.1 GRAB utility and printed on a Hewlett Packard LaserJet II.

6. Stability Analysis of Equilibria

As a supplement and a check on the graphical information presented in the previous section, the interior equilibrium points along the route to chaos were calculated in high precision (40 decimal places) using <u>Mathematica</u>. The Jacobian matrix of partial derivatives was then evaluated at the equilibrium point, and its eigenvalues found.

These are used in stability analysis of the equilibria.[See Hirsch and Smale (1974) Ch. 6.] One of these eigenvalues will always be zero; it is an artifact of the constraint that probabilities add to one, and is irrelevant to the stability analysis.[5]

For example, at a = 2 there is an interior equilibrium at:

x1 = 0.513639995434311151695011984933226800594
x2 = 0.456568884830498801506677319940646044972
x3 = 0.0285355553019061750941673324962903778108
x4 = 0.0012555644332838717041433626298367766237

and at this point the eigenvalues of the Jacobian found numerically to be:

-0.857610802580407062636665715951399308,
-0.0562990388422014452825117944612367686+
    0.287512336647416098919275272912954404 I
-0.0562990388422014452825117944612367686-
    0.287512336647416098919275272912954404 I
-5.42045724163219646523489178011112836 * 10^{-42}
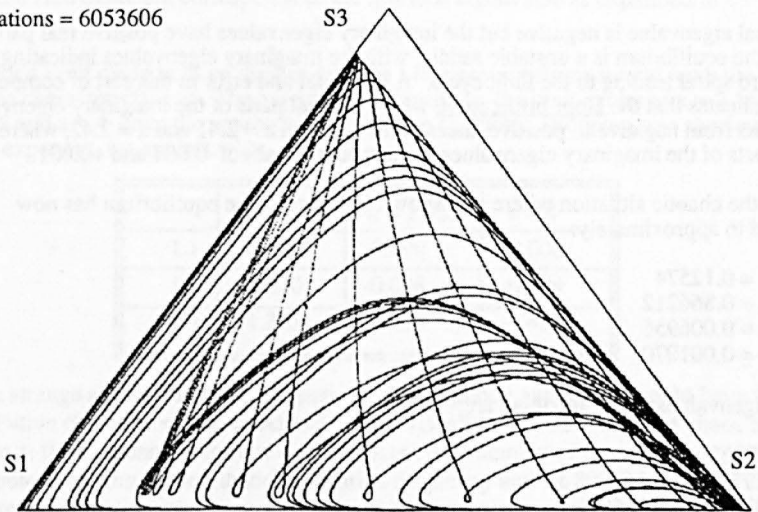
Iterations = 6053606

Figure 6. Parameter = 6

The last eigenvalue is the insignificant zero eigenvalue. The significant eigenvalues all have negative real parts, indicating a strongly stable equilibrium, which attracts much in the way illustrated in figure 2. Indeed at a = 2.4—the situation actually illustrated in figure 2—the situation is qualitatively much the same. The equilibrium has moves to about:

x1 = 0.363942
x2 = 0.614658
x3 = 0.020219
x4 = 0.001181

(henceforth I suppress the full precision in reporting the results). The non-significant

zero eigenvalue of the Jacobian is numerically calculated at the order of 10^{-39}. The significant eigenvalues are approximately:

-0.9752593,
-0.001670447 + 0.26020784 I
-0.001670447 - 0.26020784 I

However, when we move to the limit cycle illustrated in figure 3 at a = 2.55, the situation changes drastically. The equilibrium is now at approximately:

x1 = 0.328467
x2 = 0.653285
x3 = 0.018248
x4 = 0.001164

and the significant eigenvalues of the Jacobian are:

-.993192,
0.00572715 + 0.250703 I,
0.00572715 - 0.250703 I

The real eigenvalue is negative but the imaginary eigenvalues have positive real parts. Thus the equilibrium is a unstable saddle, with the imaginary eigenvalues indicating the outward spiral leading to the limit cycle. A little trial and error in this sort of computation indicates that the Hopf bifurcation, where the real parts of the imaginary eigenvalues pass from negative to positive, takes place between a = 2.41 and a = 2.42, where the real parts of the imaginary eigenvalues are respectively about -0.001 and +0.001.

In the chaotic situation where a=5 shown in figure 1, the equilibrium has now moved to approximately:

x1 = 0.12574
x2 = 0.866212
x3 = 0.006956
x4 = 0.001070

The eigenvalues of the Jacobian are:

-1.0267,
0.173705 + 0.166908 I
0.173705 + 0.166908 I

This still indicates a saddle point equilibrium but here—as shown in figure 1—the orbit passes very close to this unstable equilibrium point.

7. Numerical Calculation of Liapunov Exponents

Liapunov exponents were calculated numerically using the algorithm presented in Wolf et. al. (1985) Appendix A. This integrates the differential equations of the dynamical system to obtain a fiducial trajectory, and simultaneously integrates four copies of the linearized differential equations of the system with coefficients determined by the location on the fiducial trajectory, to calculate the Liapunov spectrum. The latter are started at points representing a set of orthonormal vectors in the tangent space, and are periodically reorthonormalized during the process. In the calculation,

logarithms are taken to the base 2. The code was implemented for the replicator dynamics by Linda Palmer. Differential equations were integrated in double precision using the IMSL Library integrator DIVPRK. The program was tested running it at a=2, starting it on the attracting equilibrium. In this case, the spectrum of Liapunov exponents (when converted to natural logarithms) should just consist of the real parts of the eigenvalues of the Jacobian evaluated at the equilibrium, which were discussed in the last section. The experimental results of a run from t=0 to t=110,000 were in agreement with the theoretical results up to four or five decimal places:

Experimental Results	Theoretical Results
-0.85761	-0.85761
-0.0563	-0.0563
-0.0563	-0.0563
$-3.4 * 10^{-6}$	0

The three negative exponents indicate the attracting nature of the equilibrium point, and the zero exponent corresponds to the spurious eigenvalue as explained in the last section.

For a limit cycle in three dimensions, the Liapunov spectrum should have the qualitative character <0, -, ->. The experimental results on the limit cycles at a = 2.55, a = 3.885 and a = 4 have the appropriate qualitative character. Dropping one spurious zero exponent, we are left with:

	a = 2.55	a = 3.885	a = 4
L1	0.000	0.000	0.000
L2	-0.020	-0.008	-0.004
L3	-1.395	-1.419	-1.423

For a strange attractor in three dimensions, the Liapunov exponents should have the qualitative character <+, 0, ->. At a=5, where visually we see the onset of chaos in figure 1, the Liapunov spectrum was calculated on a number of runs on a number of computers varying the reorthonormalization frequency and various parameters of the differential equation integrator. Dropping one spurious zero exponent, the following results are very robust:

L1: 0.010
L2: 0.000
L3: -1.44

For a "gold standard run" the equations were integrated from t=0 to t= 1,000,000 with an error tolerance of 10^{-11}. On this run the zeros (both L2 and the spurious exponent) are zeros to six decimal places. Details of the convergence are shown graphically in figures 7—10 (where one unit on the x axis represents 10,000 units of time). The positive value of the largest Liapunov exponent, L1, indicates that there has indeed been a transition to chaos.[6,7]
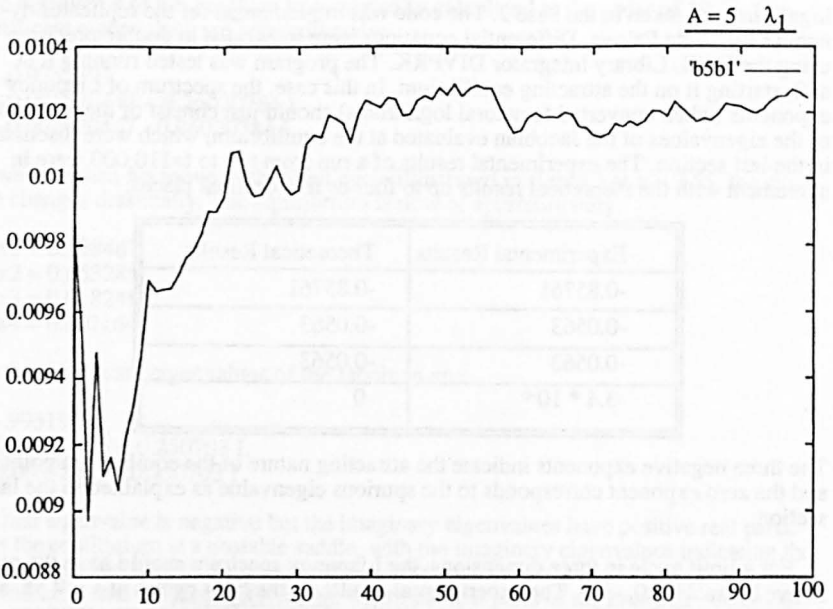
$A = 5 \quad \lambda_1$

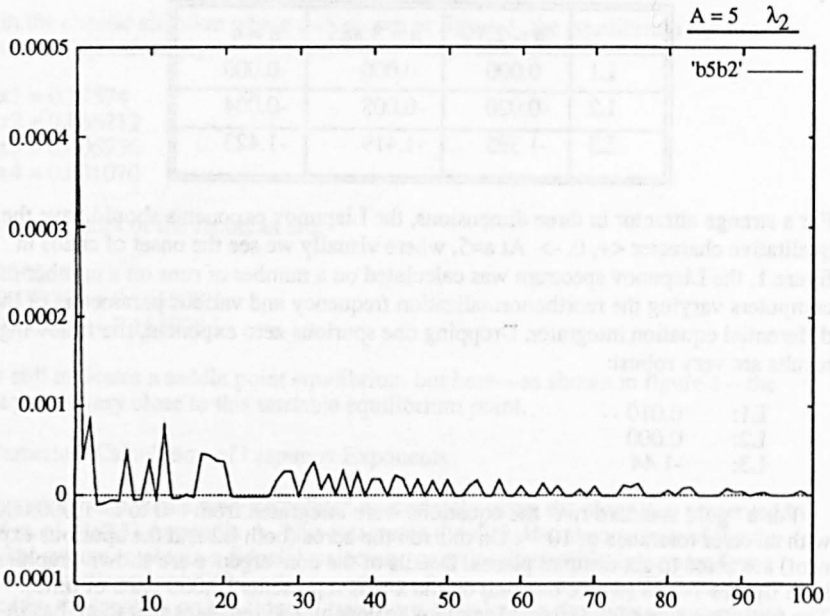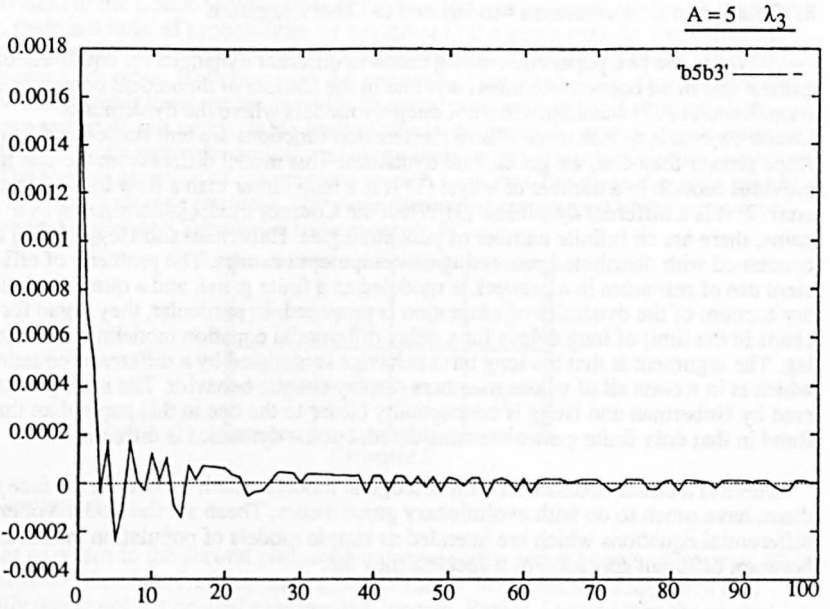Figure 7

$A = 5 \quad \lambda_2$
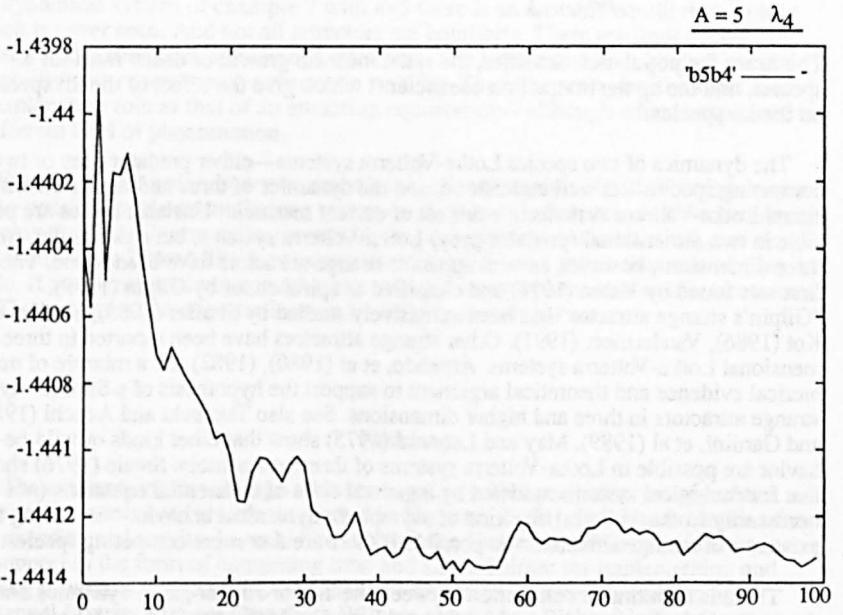

Figure 8

Figure 9

Figure 10

8. Relation to Lotka-Volterra Models and to Other Literature

There are two papers discussing chaos in different dynamics for equilibration in games: one in an economic context and one in the context of theoretical computer science. Rand (1978) considers Cournot duopoly models where the dynamics is Cournot's best response map. Where the reaction functions are tent shaped and have slope greater than one, we get chaotic dynamics. This model differs from the one in the previous section in a number of ways: (1) It is a map rather than a flow that is considered (2) it is a different dynamics (3) When the Cournot model is considered as a game, there are an infinite number of pure strategies. Huberman and Hogg (1988) are concerned with distributed processing on computer networks. The problem of efficient use of resources in a network is modeled as a finite game, and a quasi-evolutionary account of the dynamics of adaptation is proposed. In particular, they argue for chaos in the limit of long delays for a delay differential equation modeling information lag. The argument is that the long term behavior is modeled by a difference equation which is in a class all of whose members display chaotic behavior. The setting considered by Huberman and Hogg is conceptually closer to the one in this paper than that of Rand in that only finite games are considered, but the dynamics is different.

There is a closer connection with ecological models which do not, on the face of them, have much to do with evolutionary game theory. These are the Lotka-Volterra differential equations which are intended as simple models of population interactions between different species. For n species, they are:

$$\frac{dx_i}{dt} = x_i\left[r_i + \sum_{j=1}^{n} a_{ij}x_j \right]$$

The x_i are the population densities, the r_i the intrinsic growth or decay rates for a species, and the a_{ij} the interaction coefficients which give the effect of the jth species on the ith species.

The dynamics of two species Lotka-Volterra systems—either predator prey or two competing species—is well understood, and the dynamics of three and higher dimensional Lotka-Volterra systems is a subject of current research. Unstable cycles are possible in two dimensional (predator-prey) Lotka-Volterra systems, but chaos is not. In three dimensions, however, several apparent strange attractors have been found. The first was found by Vance (1978) and classified as spiral chaos by Gilpin (1979). "Gilpin's strange attractor" has been extensively studied by Shaffer (1985), Shaffer and Kot (1986), Vandermeer (1991). Other strange attractors have been reported in three-dimensional Lotka-Volterra systems. Arneodo, et al (1980), (1982) use a mixture of numerical evidence and theoretical argument to support the hypothesis of a Silnikov-type strange attractors in three and higher dimensions. See also Takeuchi and Adachi (1984) and Gardini, et al (1989). May and Leonard (1975) show that other kinds of wild behavior are possible in Lotka-Volterra systems of three competitors. Smale (1976) shows that for ecological systems modeled by a general class of differential equations (not necessarily Lotka-Volterra) any kind of asymptotic dynamical behavior—including the existence of strange attractors—is possible if there are 5 or more competing species.

There is an intimate connection between the Taylor-Jonker game dynamics and the Lotka-Volterra dynamics, which is established by Hofbauer (1981). A Lotka-Volterra system with n species corresponds to an evolutionary game with n+1 strategies such that the game dynamics on the evolutionary game is topologically orbital

equivalent to the Lotka-Volterra dynamics. To each species in the Lotka-Volterra system, there is a ratio of probabilities of strategies in the game with the same dynamics. Thus it is possible to use known facts about one kind of dynamical system to establish facts about the other. Hofbauer uses the known fact the 2 species Lotka-Volterra systems do not admit limit cycles to verify Zeeman's conjecture that 3 strategy evolutionary games do not admit stable limit cycles in the game dynamics. It is thus possible to investigate game dynamical pathology with an eye towards ecological pathology. The strange attractor of the previous section is, in fact, the game theoretic counterpart to Gilpin's strange attractor. For game dynamical counterpart of the attractor of Arneodo, Coullet and Tresser we have example 8:

0	-.6	0	1
1	0	0	-.5
-1.05	-.2	0	1.75
.5	-.1	.1	0

Example 8

9 Conclusion

Let us return to the second philosophical thesis with which I began this essay. That is that *When the underlying dynamics is taken seriously, it becomes apparent that equilibrium is not the central explanatory concept*. Rather, I would take the central dynamical explanatory concept to be that of an *attractor* (or attracting set). Not all dynamical equilibria are attractors. Some are unstable fixed points of the dynamics. In the dynamical system of example 7 with a=5 there is an unstable equilibrium point which is never seen. And not all attractors are equilibria. There are limit cycles, quasiperiodic attractors, and strange attractors. The latter combine a kind of internal instability with macroscopic asymptotic stability. Thus, they can play the same kind of explanatory role as that of an attracting equilibrium—although what is explained is a different kind of phenomonon.

Even this latter point, however, must be taken with a grain of salt. That is because of the possibility of extremely long transients. In example 7 with a=4, if we had omitted only the first 50 time units, we would not have eliminated the transient, and the plot would have looked like the strange attractor of figure 1 rather than one of a limit cycle. If transients are long enough, they may govern the phenomonae of interest to us. The concept of an attractor lives at t = infinity, but we do not.

Notes

[1]The existence of this strange attractor together with a preliminary study of the route to chaos involved was first reported in Skyrms (1992a). This paper contains further experimental results. I would like to thank the University of California at Irvine for support in the form of computing time and Linda Palmer for implementing and running programs to determine the Liapunov spectrum. I would also like to thank Immanuel Bomze, Vincent Crawford, William Harper and Richard Jeffrey for comments on an earlier version of this paper.

[2]But it is not a proper equilibrium. See van Damme (1987) for a definition of proper equilibrium, a proof that if S is an evolutionarily stable strategy, then <S,S> is a perfect and proper equilibrium of the associated game, and a great deal of other information about relations between various stability concepts.

[3]For flows the sum is replaced with an integral. For 3 dimensions, there is a spectrum of three Lyapunov exponents, each quantifying divergence of the orbit in a different direction.

[4]To make the dynamics autonomous expand the concept of state of the system to include a "memory" of frequencies of past plays.

[5]See Bomze (1986) p. 48 or van Damme (1987) p. 222 and note that in the example given, the expected utility of the status quo (= the average population fitness) at a completely mixed equilibrium point must be equal to zero, since for this fitness matrix, the expected utility of strategy 4 is identically zero.

[6]For purposes of comparison, the largest Liapunov exponent here is roughly an order of magnitude smaller than that of the Rössler attractor. But the mean orbital period of the attractor is roughly an order of magnitude larger. If we measured time in terms of mean orbital periods, L1 would here be of the same order of magnitude as L1 for the Rössler attractor. Data on the Rössler attractor was obtained from Wolf et. al. (1985).

[7]At a=6, although the attractor appears to become geometrically more complex, the Liapunov spectrum is little changed:

L1:	0.009
L2:	0.000
L3:	-1.44

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