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Part 7. Stochastic geometry

## CONTINUUM AB PERCOLATION AND AB RANDOM GEOMETRIC GRAPHS

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# CONTINUUM AB PERCOLATION AND AB RANDOM GEOMETRIC GRAPHS

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**Abstract**

Consider a bipartite random geometric graph on the union of two independent homogeneous Poisson point processes in $d$-space, with distance parameter $r$ and intensities $\lambda$ and $\mu$. We show for $d \geq 2$ that if $\lambda$ is supercritical for the one-type random geometric graph with distance parameter $2r$, there exists $\mu$ such that $(\lambda, \mu)$ is supercritical (this was previously known for $d = 2$). For $d = 2$, we also consider the restriction of this graph to points in the unit square. Taking $\mu = \tau\lambda$ for fixed $\tau$, we give a strong law of large numbers as $\lambda \to \infty$ for the connectivity threshold of this graph.

*Keywords:* Bipartite geometric graph; continuum percolation; connectivity threshold

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## 1. Introduction and statement of results

The continuum AB percolation model, introduced by Iyer and Yogeshwaran [3], goes as follows. Particles of two types, A and B, are scattered randomly in Euclidean space as two independent Poisson processes, and edges are added between particles of opposite type that are sufficiently close together. This provides a continuum analogue of lattice AB percolation which is discussed in, e.g. [2]. Motivation for considering continuum AB percolation is discussed in detail in [3]; the main motivation comes from wireless communications networks with two types of transmitter.

Another type of continuum percolation model with two types of particle is the *secrecy random graph* [9] in which the type-B particles (representing eavesdroppers) inhibit percolation; each type-A particle may send a message to every other type-A particle lying closer than its nearest neighbour of type B. See also [7]. Such models are not considered here; they are complementary to ours.

To describe continuum AB percolation more precisely, we make some definitions. Let $d \in \mathbb{N}$. Given any two locally finite sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$, and given $r > 0$, let $G(\mathcal{X}, \mathcal{Y}, r)$ be the bipartite graph with vertex sets $\mathcal{X}$ and $\mathcal{Y}$, and with an undirected edge $\{X, Y\}$ included for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with $\|X - Y\| \leq r$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$ (our parameter $r$ would be denoted $2r$ in the notation of [3]). Also, let $G(\mathcal{X}, r)$ be the graph with vertex set $\mathcal{X}$ and with an undirected edge $\{X, X'\}$ included for each $X, X' \in \mathcal{X}$ with $\|X - X'\| \leq r$.

For $\lambda, \mu > 0$, let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\mu$ be independent homogeneous Poisson point processes in $\mathbb{R}^d$ of intensity $\lambda$ and $\mu$, respectively, where we view each point process as a random subset of $\mathbb{R}^d$. Our first results are concerned with the bipartite graph $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$.

Let $\mathcal{I}$ be the class of graphs having at least one infinite component. By a version of the Kolmogorov zero–one law, given parameters $r, \lambda, \mu$ (and $d$), we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] \in \{0, 1\}$. Provided $r$, $\lambda$, and $\mu$ are sufficiently large, we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1$; see [3],





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or the discussion below. Set

$$\mu_c(r,\lambda) := \inf\{\mu\colon \mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1\},$$

with the infimum of the empty set interpreted as $+\infty$. Also, for the more standard one-type continuum percolation graph $G(\mathcal{P}_\lambda, r)$, define

$$\lambda_c(2r) := \inf\{\lambda\colon \mathbb{P}[G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}] = 1\},$$

which is well known to be finite for $d \geq 2$ [2, 5], but is not known analytically. By scaling (see Proposition 2.11 of [5]), $\lambda_c(2r) = r^{-d}\lambda_c(2)$, and explicit bounds for $\lambda_c(2)$ are provided in [5]. Simulation studies indicate that $1 - \mathrm{e}^{-\pi\lambda_c(2)} \approx 0.676\,35$ for $d = 2$ [8] and $1 - \mathrm{e}^{-(4\pi/3)\lambda_c(2)} \approx 0.289\,57$ for $d = 3$ [4].

Obviously, if $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$ then also $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$, and, therefore, a necessary condition for $\mu_c(r,\lambda)$ to be finite is that $\lambda \geq \lambda_c(2r)$. In other words, for any $r > 0$, we have

$$\lambda_c^{\mathrm{AB}}(r) := \inf\{\lambda\colon \mu_c(r,\lambda) < \infty\} \geq \lambda_c(2r). \tag{1.1}$$

For $d = 2$ only, Iyer and Yogeshwaran [3] showed that the inequality in (1.1) is in fact an equality. For general $d \geq 2$, they also provided an explicit finite upper bound, here denoted by $\tilde{\lambda}_c^{\mathrm{AB}}$, for $\lambda_c^{\mathrm{AB}}(r)$, and established explicit upper bounds on $\mu_c(r,\lambda)$ for $\lambda > \tilde{\lambda}_c^{\mathrm{AB}}(r)$. Note that even for $d = 2$, their explicit upper bounds for $\mu_c(r,\lambda)$ are given only when $\lambda > \tilde{\lambda}_c^{\mathrm{AB}}(r)$, with $\tilde{\lambda}_c^{\mathrm{AB}}(r) > \lambda_c(2r)$ for all $d \geq 2$; for the case with $d = 2$ and $\lambda_c(2r) < \lambda \leq \tilde{\lambda}_c^{\mathrm{AB}}(r)$, their proof that $\mu_c(r,\lambda) < \infty$ does not provide an explicit upper bound on $\mu_c(r,\lambda)$.

In our first result, proved in Section 2, we establish for all dimensions (and all $r > 0$) that the inequality in (1.1) is an equality, and provide explicit asymptotic upper bounds on $\mu_c(r,\lambda)$ as $\lambda$ approaches $\lambda_c(2r)$ from above. Let $\pi_d$ denote the volume of the ball in $d$ dimensions with unit radius.

**Theorem 1.1.** *Let $d \geq 2$ and $r > 0$. Then*

(i) $\lambda_c^{\mathrm{AB}}(r) = \lambda_c(2r)$, *and*

(ii) *with* $\lambda_c = \lambda_c(2r)$,

$$\limsup_{\delta\downarrow 0}\left(\frac{\mu_c(r,\lambda_c+\delta)}{\delta^{-2d}|\log\delta|}\right) \leq \left(\frac{4\lambda_c^2}{r}\right)^d d^{3d}(d+1)\pi_d. \tag{1.2}$$

Our proof (see Section 2) is based on the classic elementary continuum percolation techniques of discretization, coupling, and scaling. We also indicate how, for any given $\lambda > \lambda_c(2r)$, we can compute an explicit upper bound for $\mu_c(r,\lambda)$ (see (2.7) below).

It would be interesting to try to find complementary *lower* bounds for $\mu_c(r,\lambda)$. An analogous problem in the lattice is mixed bond-site percolation, which similarly has two parameters. For that model, similar questions have been studied by Chayes and Schonman [1], but it is not clear to what extent their methods can be adapted to the continuum.

Our second result concerns full connectivity for the *AB random geometric graph*, i.e. the restriction of the AB percolation model to points in a bounded region of $\mathbb{R}^d$. For $\lambda > 0$, let $\mathcal{P}_\lambda^{\mathrm{F}} := \mathcal{P}_\lambda \cap [0,1]^d$ and $\mathcal{Q}_\lambda^{\mathrm{F}} := \mathcal{Q}_\lambda \cap [0,1]^d$ (these are finite Poisson processes of intensity $\lambda$; hence, the superscript F). Given also $\tau > 0$ and $r > 0$, let $\mathcal{G}^1(\lambda,\tau,r)$ be the graph on the vertex set $\mathcal{P}_\lambda^{\mathrm{F}}$, with an edge between each pair of vertices sharing at least one common neighbour in $G(\mathcal{P}_\lambda^{\mathrm{F}}, \mathcal{Q}_{\tau\lambda}^{\mathrm{F}}, r)$.

Let $\mathcal{G}^2(\lambda, \tau, r)$ be the graph on the vertex set $\mathcal{Q}^{\mathrm{F}}_{\tau\lambda}$, with an edge between each pair of vertices sharing at least one common neighbour in $G(\mathcal{P}^{\mathrm{F}}_{\lambda}, \mathcal{Q}^{\mathrm{F}}_{\tau\lambda}, r)$. Then $G(\mathcal{P}^{\mathrm{F}}_{\lambda}, \mathcal{Q}^{\mathrm{F}}_{\tau\lambda}, r)$ is connected, if and only if both $\mathcal{G}^1(\lambda, \tau, r)$ and $\mathcal{G}^2(\lambda, \tau, r)$ are connected.

Let $\mathcal{K}$ be the class of connected graphs, and let

$$\rho_n(\tau) = \min\{r : \mathcal{G}^1(n, \tau, r) \in \mathcal{K}\},$$

which is a random variable determined by the configuration of $(\mathcal{P}_n, \mathcal{Q}_{\tau n})$. It is a *connectivity threshold* for the AB random geometric graph. Let us assume that $\mathcal{P}^{\mathrm{F}}_{\lambda}$ and $\mathcal{Q}^{\mathrm{F}}_{\mu}$ are coupled for all $\lambda, \mu > 0$ as follows. Let $(X_1, Y_1, X_2, Y_2, \ldots)$ be a sequence of independent uniform random $d$-vectors uniformly distributed over $[0, 1]^d$. Independently, let $(N_t,\ t \geq 0)$ and $(N'_t,\ t \geq 0)$ be independent Poisson counting processes of rate 1. Let $\mathcal{P}^{\mathrm{F}}_{\lambda} = \{X_1, \ldots, X_{N_\lambda}\}$ and $\mathcal{Q}^{\mathrm{F}}_{\mu} = \{Y_1, \ldots, Y_{N'_\mu}\}$.

In Section 3 we prove the following result, with '$\xrightarrow{\text{a.s.}}$' denoting almost-sure convergence as $n \to \infty$ (with $n \in \mathbb{N}$).

**Theorem 1.2.** *Assume that $d = 2$. Let $\tau > 0$. Then*

$$\frac{n\pi(\rho_n(\tau))^2}{\log n} \xrightarrow{\text{a.s.}} \max\left(\frac{1}{\tau}, \frac{1}{4}\right). \tag{1.3}$$

**Remark 1.1.** The restriction to $d = 2$ arises because boundary effects become more important in higher dimensions (and $d = 1$ is a different case). It should be possible to adapt the proof to obtain a similar result to (1.3) in the unit *torus* in arbitrary dimensions $d \geq 2$, namely, $n\pi_d(\rho_n(\tau))^d / \log n \xrightarrow{\text{a.s.}} \max(1/\tau, 2^{-d})$, although we have not checked the details.

**Remark 1.2.** Iyer and Yogeshwaran [3, Theorem 3.1] gave a.s. lower and upper bounds for $\rho_n(\tau)$ in the torus. The extension of our result mentioned in Remark 1.1 would show that the lower bound of [3] is sharp for $\tau \leq 2^d$, and improve on their upper bound.

*Notation.* Given a countable set $\mathcal{X}$, we write $|\mathcal{X}|$ for the number of elements of $\mathcal{X}$ and if also $\mathcal{X} \subset \mathbb{R}^d$, given $A \subset \mathbb{R}^d$, we write $\mathcal{X}(A)$ for $|\mathcal{X} \cap A|$. Also, for $a > 0$, we write $aA$ for $\{ay : y \in A\}$. Let '$\oplus$' denote the Minkowski addition of sets (see, e.g. [6]).

## 2. Percolation: proof of Theorem 1.1

Fix $r > 0$, and let $\lambda > \lambda_c(2r)$. We first prove that $\mu_c(r, \lambda) < \infty$; combined with (1.1) this shows that $\lambda^{\mathrm{AB}}_c(r) = \lambda_c(2r)$, which is part (i) of the theorem. Later we shall quantify the estimates in our argument, thereby establishing part (ii).

Choose $s < r$ and $\nu < \lambda$ such that $\mathbb{P}[G(\mathcal{P}_\nu, 2s) \in \mathcal{I}] = 1$. This is possible because decreasing the radius slightly is equivalent to decreasing the Poisson intensity slightly, by scaling (see [5]; also the first equality of (2.5) below). Set $t = (r + s)/2$, and let $\varepsilon > 0$ be chosen small enough so that any cube of side length $\varepsilon$ has Euclidean diameter at most $t - s = \frac{1}{2}(r - s)$. For $a > 0$, let $p_a := 1 - \exp(-\varepsilon^d a)$, the probability that a given cube of side length $\varepsilon$ contains at least one point of $\mathcal{P}_a$.

Consider Bernoulli site percolation on the graph $(\varepsilon\mathbb{Z}^d, \sim)$, where, for $u$ and $v \in \varepsilon\mathbb{Z}^d$, $u \sim v$ if and only if there exists $w \in \varepsilon\mathbb{Z}^d$ with $\|w - u\| \leq t$ and $\|w - v\| \leq t$. Given $p > 0$, suppose that each site $u \in \varepsilon\mathbb{Z}^d$ is independently occupied with probability $p$. Let $D_1$ be the event that there is an infinite path of occupied sites in the graph, and let $\mathbb{P}_p[D_1]$ be the probability that this event occurs.

Divide $\mathbb{R}^d$ into cubes $Q_u$, $u \in \varepsilon\mathbb{Z}^d$, defined by $Q_u := \{u\} \oplus [0, \varepsilon)^d$. For $x \in \mathbb{R}^d$, let $z_x \in \varepsilon\mathbb{Z}^d$ be such that $x \in Q_{z_x}$. The Poisson process $\mathcal{P}_\nu$ may be coupled to a realization of the site percolation process with parameter $p_\nu$, by deeming each $z \in \varepsilon\mathbb{Z}^d$ to be occupied if and only if $\mathcal{P}_\nu(Q_z) \geq 1$. By the choice of $\varepsilon$, for $X, Y \in \mathcal{P}_\nu$, if $\|X - Y\| \leq 2s$ then $\|z_X - z_{(X+Y)/2}\| \leq t$ and $\|z_Y - z_{(X+Y)/2}\| \leq t$, and, hence, $z_X \sim z_Y$. Therefore, with this coupling, if $G(\mathcal{P}_\nu, 2s) \in \mathcal{I}$ then there is an infinite path of occupied sites in $(\varepsilon\mathbb{Z}^d, \sim)$. Because we chose $\nu$ and $s$ in such a way that $\mathbb{P}[G(\mathcal{P}_\nu, 2s) \in \mathcal{I}] = 1$, we must have $\mathbb{P}_{p_\nu}[D_1] = 1$.

Now consider a form of lattice AB percolation on $\varepsilon\mathbb{Z}^d$ with parameter pair $(p, q) \in [0, 1]^2$ (not necessarily the same as any of the lattice AB percolation models in the literature). Let each of $\{V_u,\ u \in \varepsilon\mathbb{Z}^d\}$ and $\{W_u,\ u \in \varepsilon\mathbb{Z}^d\}$ be a family of independent Bernoulli random variables, with parameters $p$ and $q$, respectively. Let $D_2$ be the event that there is an infinite sequence $u_1, u_2, \dots$ of distinct elements of $\varepsilon\mathbb{Z}^d$ and an infinite sequence $v_1, v_2, \dots$ of elements of $\varepsilon\mathbb{Z}^d$ such that, for each $i \in \mathbb{N}$, we have $V_{u_i} W_{v_i} = 1$ and $\max\{\|u_i - v_i\|, \|v_i - u_{i+1}\|\} \leq t$. Let $\widetilde{\mathbb{P}}_{p,q}[D_2]$ be the probability that event $D_2$ occurs, given the parameter pair $(p, q)$.

Since $\mathbb{P}_{p_\nu}[D_1] = 1$, clearly, $\widetilde{\mathbb{P}}_{p_\nu,1}[D_2] = 1$. Increasing $p$ slightly and decreasing $q$ slightly, we shall show that there exists $q < 1$ such that

$$\widetilde{\mathbb{P}}_{p_\lambda,q}[D_2] = 1. \tag{2.1}$$

This is enough to demonstrate that $\mu_c(r, \lambda) < \infty$. Indeed, suppose that such a $q$ exists and choose $\mu$ such that $p_\mu = q$. Then, for $u \in \varepsilon\mathbb{Z}^d$, set $V_u = 1$ if and only if $\mathcal{P}_\lambda(Q_u) \geq 1$ and $W_u = 1$ if and only if $\mathcal{Q}_\mu(Q_u) \geq 1$. Suppose that $D_2$ occurs, and let $u_1, v_1, u_2, v_2, \dots$ be as in the definition of the event $D_2$. Then, for each $i \in \mathbb{N}$, we have $V_{u_i} = 1$, so we can pick a point $X_i \in \mathcal{P}_\lambda \cap Q_{u_i}$, and $W_{v_i} = 1$, so we can pick a point $Y_i \in \mathcal{Q}_\mu \cap Q_{v_i}$. Then, by the choice of $\varepsilon$, for each $i \in \mathbb{N}$, we have

$$\max\{\|X_i - Y_i\|, \|Y_i - X_{i+1}\|\} \leq t + (t - s) = r,$$

and, hence, $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$. Hence, by (2.1) we have $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1$. Therefore, $\mu_c(r, \lambda) \leq \mu < \infty$, as asserted.

To complete the proof of part (i), it remains to prove that (2.1) holds for some $q < 1$. Let $\{T_u,\ u \in \varepsilon\mathbb{Z}^d\}$ be independent Bernoulli variables with parameter $p_\lambda$. For each ordered pair $(u, v) \in (\varepsilon\mathbb{Z}^d)^2$ with $0 < \|u - v\| \leq t$, let $U_{u,v}$ be independent Bernoulli random variables with parameter $(p_\nu / p_\lambda)^{1/\Delta}$, where we set

$$\Delta := |\{u \in \varepsilon\mathbb{Z}^d : 0 < \|u\| \leq t\}|. \tag{2.2}$$

Assume that the variables $U_{u,v}$ and $T_u$ are all mutually independent, and, for $u, v \in \varepsilon\mathbb{Z}^d$, define the Bernoulli variables

$$V_u := T_u \prod_{\{v \in \varepsilon\mathbb{Z}^d :\ 0 < \|v - u\| \leq t\}} U_{u,v},$$

$$W_v := 1 - \prod_{\{u \in \varepsilon\mathbb{Z}^d :\ 0 < \|v - u\| \leq t\}} (1 - U_{u,v}).$$

Then each of $\{V_u\}_{u \in \varepsilon\mathbb{Z}^d}$ and $\{W_v\}_{v \in \varepsilon\mathbb{Z}^d}$ is a family of independent Bernoulli variables, with respective parameters $p_\nu$ and

$$q := 1 - \left(1 - \left(\frac{p_\nu}{p_\lambda}\right)^{1/\Delta}\right)^{\Delta} < 1, \tag{2.3}$$

and each is independent of $\{T_u,\ u \in \varepsilon\mathbb{Z}^d\}$.

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Since $\mathbb{P}_{p_\nu}[D_1] = 1$, with probability 1, there exists an infinite sequence $u_1, u_2, \ldots$ of distinct elements of $\varepsilon\mathbb{Z}^d$ with $u_i \sim u_{i+1}$ for all $i \in \mathbb{N}$, and with $V_{u_i} = 1$ for each $i \in \mathbb{N}$. By the definition of the relation '$\sim$', we can choose a sequence $v_1, v_2, \ldots$ of elements of $\varepsilon\mathbb{Z}^d$ such that, for each $i \in \mathbb{N}$, we have $\max(\|v_i - u_i\|, \|v_i - u_{i+1}\|) \leq t$. Then, for each $i$, since $V_{u_i} = 1$, we have $U_{u_i, v_i} = 1$, and, therefore, $W_{v_i} = 1$; also, $T_{u_i} = 1$. Hence, (2.1) holds as required, establishing that $\mu_c(r, \lambda) < \infty$. We have proved part (i).

To prove part (ii), we need to quantify the preceding argument. First note that the value of $\mu$ associated with $q$ given by (2.3) (i.e. with $p_\mu = q$) has $\exp(-\mu\varepsilon^d) = (1 - (p_\nu/p_\lambda)^{1/\Delta})^\Delta$, so that, since $\varepsilon^d\Delta \leq \pi_d r^d$ by (2.2), we have

$$\mu_c(r, \lambda) \leq \mu = \varepsilon^{-d}\Delta \log\bigg(\frac{1}{1 - (p_\nu/p_\lambda)^{1/\Delta}}\bigg) \leq \varepsilon^{-2d}\pi_d r^d \log\bigg(\frac{1}{1 - (p_\nu/p_\lambda)^{(\varepsilon/r)^d/\pi_d}}\bigg). \tag{2.4}$$

From now on, set $\lambda_c := \lambda_c(2r)$ and $\lambda = \lambda_c + \delta$ for some $\delta > 0$. We need to choose $s < r$ and $\nu < \lambda$ such that $\mathbb{P}[G(\mathcal{P}_\nu, 2s) \in \mathcal{I}] = 1$. Choose $\alpha, \beta > 0$ with $\alpha + \beta < 1$, and also let $\alpha' \in (0, \alpha)$ and $\beta' \in (0, \beta)$. Set

$$s := r\bigg(1 + \frac{\alpha\delta}{\lambda_c}\bigg)^{-1/d} \quad \text{and} \quad \nu := \lambda_c + (1 - \beta)\delta.$$

By scaling (see [5, Proposition 2.11]) and our choice of $s$, we have

$$\lambda_c(2s) = \bigg(\frac{r}{s}\bigg)^d \lambda_c(2r) = \lambda_c + \alpha\delta, \tag{2.5}$$

and, hence, $\nu > \lambda_c(2s)$, so $\mathbb{P}[G(\mathcal{P}_\nu, 2s) \in \mathcal{I}] = 1$, as required.

Our choice of $\varepsilon$ in the discretization needs to satisfy

$$\varepsilon \leq \frac{r - s}{2\sqrt{d}} = \frac{r}{2\sqrt{d}}\bigg(1 - \bigg(1 + \frac{\alpha\delta}{\lambda_c}\bigg)^{-1/d}\bigg), \tag{2.6}$$

and the right-hand side of (2.6) is asymptotic to $\alpha r\delta/(2d^{3/2}\lambda_c)$ as $\delta \to 0$. Hence, taking $\varepsilon = \alpha' r\delta/(2d^{3/2}\lambda_c)$, we have (2.6) provided $\delta \leq \delta_1$ for some fixed $\delta_1 > 0$. Also,

$$\frac{p_\nu}{p_\lambda} \leq \frac{\varepsilon^d\nu}{\varepsilon^d\lambda\exp(-\varepsilon^d\lambda)} = \bigg(\frac{\lambda_c + (1 - \beta)\delta}{\lambda_c + \delta}\bigg)\exp(\varepsilon^d\lambda),$$

and so, by Taylor's expansion, there is some $\delta_2 > 0$ such that, provided $0 < \delta \leq \delta_2$, taking $\varepsilon = \alpha' r\delta/(2d^{3/2}\lambda_c)$ we have

$$\bigg(\frac{p_\nu}{p_\lambda}\bigg)^{(\varepsilon/r)^d/\pi_d} \leq 1 - \frac{\beta'\delta\varepsilon^d}{\pi_d r^d \lambda_c} = 1 - \frac{\beta'\delta^{d+1}}{\pi_d\lambda_c(2d^{3/2}\lambda_c/\alpha')^d}.$$

Therefore, by (2.4), for $0 < \delta \leq \min\{\delta_1, \delta_2\}$, we have

$$\mu_c(r, \lambda) \leq \bigg(\frac{2d^{3/2}\lambda_c}{r\delta\alpha'}\bigg)^{2d} \pi_d r^d \log\bigg(\frac{\pi_d\lambda_c(2d^{3/2}\lambda_c/\alpha')^d}{\beta'\delta^{d+1}}\bigg)$$

and since we can take $\alpha'$ arbitrarily close to 1, (1.2) follows, completing the proof.

For a given value of $\lambda$ with $\lambda = \lambda_c(2r) + \delta$ for some $\delta > 0$, an explicit upper bound for $\mu_c(r, \lambda)$ could be computed as follows. Choose $\alpha, \beta > 0$ with $\alpha + \beta < 1$, and let $\varepsilon$ be given by the right-hand side of (2.6). Then a numerical upper bound for $\mu_c(r, \lambda)$ can be obtained by computing the right-hand side of (2.4). To make this bound as small as possible (given $\alpha$), we make $\nu$ as small as we can, i.e. make $\beta$ approach $1 - \alpha$ and $\nu$ approach $\lambda_c + \alpha\delta$. Taking this limit and then optimizing further over $\alpha$ gives us the upper bound

$$\mu_c(r, \lambda) \leq \inf_{\alpha \in (0,1)} \varepsilon(\alpha)^{-2d} \pi_d r^d \log\left(\frac{1}{1 - (p_{\lambda_c + \alpha\delta}/p_\lambda)^{(\varepsilon(\alpha)/r)^d/\pi_d}}\right), \tag{2.7}$$

with $\varepsilon = \varepsilon(\alpha)$ given by the right-hand side of (2.6).

## 3. Connectivity: proof of Theorem 1.2

Throughout this section, we assume that $d = 2$. All asymptotics are as $n \to \infty$. Given $a, b \in \mathbb{R}$, we sometimes write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Fix $\tau > 0$. Given $\tau$ and $r_n$, let $\delta_n$ denote the minimum degree of $\mathcal{G}^1(n, \tau, r_n)$.

**Lemma 3.1.** *Let $\alpha \in (0, 1/\tau)$. If $n\pi r_n^2/\log n = \alpha$ for $n \geq 2$ then, almost surely, $\delta_n = 0$ for all but finitely many $n$.*

*Proof.* See [3, Proposition 5.1].

**Lemma 3.2.** *Let $\alpha \in (0, \frac{1}{4})$. If $n\pi r_n^2/\log n = \alpha$ for $n \geq 2$ then, almost surely, $\delta_n = 0$ for all but finitely many $n$.*

*Proof.* By [6, Theorem 7.8], for this choice of $r_n$, almost surely, the minimum degree of the (one-type) geometric graph $G(\mathcal{P}_n^{\mathrm{F}}, 2r_n)$ is 0 for all but finitely many $n$, and, therefore, so is the minimum degree of $\mathcal{G}^1(n, \tau, r_n)$.

**Corollary 3.1.** *Let $d = 2$. Given $\varepsilon > 0$, almost surely, $n\pi(\rho_n(\tau))^2/\log n > (1 - \varepsilon)\max\{\frac{1}{4}, 1/\tau\}$ for all but finitely many $n$.*

*Proof.* Assume that $\varepsilon < 1$. For $n \geq 2$, set $r_n = [(1 - \varepsilon)(\frac{1}{4} \vee 1/\tau)\log n/(n\pi)]^{1/2}$, so $n\pi r_n^2/\log n = (1 - \varepsilon)(\frac{1}{4} \vee 1/\tau)$. Let $\delta_n$ be the minimum degree of $\mathcal{G}^1(n, \tau, r_n)$. If the minimum degree of a graph of order greater than 1 is zero, then it is not connected; hence,

$$\begin{aligned}\left\{\frac{n\pi(\rho_n(\tau))^2}{\log n} \leq (1 - \varepsilon)\left(\frac{1}{4} \vee \frac{1}{\tau}\right)\right\} &= \{\mathcal{G}^1(n, \tau, r_n) \in \mathcal{K}\} \\ &\subset \{\delta_n > 0\} \cup \{\mathcal{P}_n^{\mathrm{F}}([0, 1]^2) \leq 1\},\end{aligned}$$

and, by Lemmas 3.1 and 3.2, this occurs for only finitely many $n$ almost surely.

To complete the proof of Theorem 1.2, it suffices to prove the following result.

**Theorem 3.1.** *Suppose for some fixed $\alpha$ that $\{r_n\}_{n \in \mathbb{N}}$ is such that, for all $n \geq 2$,*

$$\frac{n\pi r_n^2}{\log n} = \alpha > \max\left\{\frac{1}{\tau}, \frac{1}{4}\right\}. \tag{3.1}$$

*Then, almost surely, $\mathcal{G}^1(n, \tau, r_n) \in \mathcal{K}$ for all but finitely many $n$.*

Our proof of this theorem requires a series of lemmas and proceeds by discretization of space. Assume that $\alpha$ and $r_n$ are given, satisfying (3.1). Let $\varepsilon_0 \in (0, \frac{1}{99})$ be chosen in such a way that, for $\varepsilon = \varepsilon_0$, we have both

$$\alpha\tau(1 - 12\varepsilon) > 1 + \varepsilon \tag{3.2}$$

$$\text{and} \quad \alpha(4 - 12\varepsilon(3 + \tau)) > 1 + \varepsilon. \tag{3.3}$$

Given $n$, partition $[0, 1]^2$ into squares of side $\varepsilon_n r_n$ with $\varepsilon_n$ chosen so that $\varepsilon_0 \le \varepsilon_n < \frac{1}{99}$ and $1/(\varepsilon_n r_n) \in \mathbb{N}$, and $\varepsilon = \varepsilon_n$ satisfies (3.2) and (3.3); this is possible for all large enough $n$, $n \ge n_0$ say. In the sequel we assume that $n \ge n_0$ and often write just $\varepsilon$ for $\varepsilon_n$.

Let $\mathcal{L}_n$ be the set of centres of the squares in this partition (a finite lattice). Then $|\mathcal{L}_n| = \Theta(n/\log n)$. List the squares as $Q_i$, $1 \le i \le |\mathcal{L}_n|$, and the corresponding centres of squares (i.e. the elements of $\mathcal{L}_n$) as $q_i$, $1 \le i \le |\mathcal{L}_n|$.

Given a set $\mathcal{X} \subset [0, 1]^2$, define the *projection of* $\mathcal{X}$ *onto* $\mathcal{L}_n$ to be the set of $q_i \in \mathcal{L}_n$ such that $\mathcal{X} \cap Q_i \ne \varnothing$. Given also $\mathcal{Y} \subset [0, 1]^2$, define the projection of $(\mathcal{X}, \mathcal{Y})$ onto $\mathcal{L}_n$ to be the pair $(\mathcal{X}', \mathcal{Y}')$, where $\mathcal{X}'$ is the projection of $\mathcal{X}$ onto $\mathcal{L}_n$ and $\mathcal{Y}'$ is the projection of $\mathcal{Y}$ onto $\mathcal{L}_n$. We refer to $|\mathcal{X}'| + |\mathcal{Y}'|$ (respectively $|\mathcal{X}'|$, $|\mathcal{Y}'|$) as the *order* of the projection of $(\mathcal{X}, \mathcal{Y})$ (respectively of $\mathcal{X}$, of $\mathcal{Y}$) onto $\mathcal{L}_n$.

**Lemma 3.3.** *Let $n \in \mathbb{N}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are finite subsets of $[0, 1]^2$ such that $G(\mathcal{X}, \mathcal{Y}, r_n)$ is connected. Let $(\mathcal{X}', \mathcal{Y}')$ be the projection of $(\mathcal{X}, \mathcal{Y})$ onto $\mathcal{L}_n$. Then the bipartite geometric graph $G(\mathcal{X}', \mathcal{Y}', r_n(1 + 2\varepsilon_n))$ is connected.*

*Proof.* If $q_i, q_j \in \mathcal{L}_n$, and $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ with $\|X - Y\| \le r_n$, then by the triangle inequality we have

$$\|q_i - q_j\| \le \|X - q_i\| + \|X - Y\| + \|Y - q_j\| \le r_n(1 + 2\varepsilon),$$

and, therefore, since $G(\mathcal{X}, \mathcal{Y}, r_n)$ is connected, so is $G(\mathcal{X}', \mathcal{Y}', r_n(1 + 2\varepsilon))$.

Given $n, m \in \mathbb{N}$, let $\mathcal{A}_{n,m}$ denote the set of pairs $(\sigma_1, \sigma_2)$ with each $\sigma_j \subset \mathcal{L}_n$, with $|\sigma_1| + |\sigma_2| = m$ and $|\sigma_1| \ge 1$, such that $G(\sigma_1, \sigma_2, r_n(1 + 2\varepsilon_n))$ is connected; these may be viewed as 'bipartite lattice animals'.

Let $\mathcal{A}^2_{n,m}$ be the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n,m}$ such that all elements of $\sigma_1 \cup \sigma_2$ are distances at least $2r_n$ from the boundary of $[0, 1]^2$.

Let $\mathcal{A}^1_{n,m}$ be the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n,m}$ such that $\sigma_1 \cup \sigma_2$ is a distance less than $2r_n$ from *just one edge* of $[0, 1]^2$.

Let $\mathcal{A}^0_{n,m} := \mathcal{A}_{n,m} \setminus (\mathcal{A}^2_{n,m} \cup \mathcal{A}^1_{n,m})$, the set of $(\sigma_1, \sigma_2) \in \mathcal{A}_{n,m}$ such that $\sigma_1 \cup \sigma_2$ is a distance less than $2r_n$ from *two edges* of $[0, 1]^2$ (i.e. near a corner of $[0, 1]^2$).

**Lemma 3.4.** *Given $m \in \mathbb{N}$, there exists a constant $C = C(m)$ such that, for all $n \ge n_0$,*

$$|\mathcal{A}_{n,m}| \le C\left(\frac{n}{\log n}\right), \qquad |\mathcal{A}^1_{n,m}| \le C\left(\frac{n}{\log n}\right)^{1/2}, \qquad |\mathcal{A}^0_{n,m}| \le C.$$

*Proof.* Fix $m$. Consider how many ways there are to choose $\sigma \in \mathcal{A}_{n,m}$.

For the first element of $\sigma_1$ in the lexicographic ordering, there are at most $|\mathcal{L}_n|$ choices, and, hence, $O(n/\log n)$ choices. Having chosen the first element of $\sigma_1$, there are a bounded number of ways to choose the rest of $\sigma$.

We now consider how many ways there are to choose $\sigma \in \mathcal{A}^1_{n,m}$. There are $O(r_n^{-1}) = O((n/\log n)^{1/2})$ ways to choose the first element of $\sigma_1$ (a distance at most $2r_n$ from the boundary of $[0, 1]^2$), and then a bounded number of ways to choose the rest of $\sigma$.

Finally, consider how many ways there are to choose $\sigma \in \mathcal{A}^0_{n,m}$. There are $O(1)$ ways to choose the first element of $\sigma_1$, and then a bounded number of ways to choose the rest of $\sigma$.

For $n \in \mathbb{N}$, set $\nu(n) := n^{\lceil 4/\varepsilon_0 \rceil}$. Note that $\nu(n+1) \sim \nu(n)$ and $r_{\nu(n+1)} \sim r_{\nu(n)}$ as $n \to \infty$, and that $r_n$ is monotone decreasing in $n$ for $n \geq 3$.

Given $n \in \mathbb{N}$ with $\nu(n) \geq n_0$, and given $\sigma_1 \subset \mathcal{L}_{\nu(n)}$ and $\sigma_2 \subset \mathcal{L}_{\nu(n)}$, let $E_{(\sigma_1,\sigma_2)}$ be the event that there exists some $n' \in \mathbb{N} \cap [\nu(n), \nu(n+1))$ such that there is a component $(U, V)$ of $G(\mathcal{P}^{\mathrm{F}}_{n'}, \mathcal{Q}^{\mathrm{F}}_{\tau n'}, r_{n'})$ such that $(\sigma_1, \sigma_2)$ is the projection of $(U, V)$ onto $\mathcal{L}_{\nu(n)}$.

For $x \in \mathbb{R}^2$ and $r > 0$, let $B(x, r) := \{y \in \mathbb{R}^2 \colon \|y - x\| \leq r\}$. Also, let $B_+(r)$ be the right half of $B((0, 0), r)$, and let $B_-(r)$ be the left half of $B((0, 0), r)$. Let $\upsilon_2(\cdot)$ denote the Lebesgue measure, defined on Borel subsets of $\mathbb{R}^2$.

**Lemma 3.5.** *There exists $n_1 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$ and $n \geq n_1$,*

$$\sup_{\sigma \in \mathcal{A}^2_{\nu(n),m}} (\mathbb{P}[E_\sigma]) \leq \nu(n)^{-(1+\varepsilon)}, \tag{3.4}$$

$$\sup_{\sigma \in \mathcal{A}^1_{\nu(n),m}} (\mathbb{P}[E_\sigma]) \leq \nu(n)^{-(1+\varepsilon)/2}, \tag{3.5}$$

$$\sup_{\sigma \in \mathcal{A}^0_{\nu(n),m}} (\mathbb{P}[E_\sigma]) \leq \nu(n)^{-1/20}. \tag{3.6}$$

*Proof.* Choose $n_1$ so that $\nu(n_1) \geq n_0$ and also $(1 - \varepsilon_0) r_{\nu(n)} < r_{\nu(n+1)}$ for $n \geq n_1$. Assume from now on that $n \geq n_1$.

Given $\sigma = (\sigma_1, \sigma_2) \in \mathcal{A}_{\nu(n),m}$, let $q_i$ and $q_j$ respectively be the lexicographically first and last elements of $\sigma_1$. Let $\sigma_2^-$ be the set of $q_k \in \sigma_2 \cap B(q_i, r_{\nu(n)}(1-4\varepsilon))$ lying strictly to the left of $q_i$ (in this proof, $\varepsilon := \varepsilon_{\nu(n)}$). Let $\sigma_2^+$ be the set of $q_k \in \sigma_2 \cap B(q_j, r_{\nu(n)}(1-4\varepsilon))$ lying strictly to the right of $q_j$. Let $\tilde{\sigma}_2^+ := \sigma_2^+ \oplus [-\varepsilon r_{\nu(n)}/2, \varepsilon r_{\nu(n)}/2]^2$ and $\tilde{\sigma}_2^- := \sigma_2^- \oplus [-\varepsilon r_{\nu(n)}/2, \varepsilon r_{\nu(n)}/2]^2$ (see Figure 1).

Let $B_\sigma^-$ be the part of $B(q_i, r_{\nu(n)}(1 - 5\varepsilon))$ lying strictly to the left of $Q_i$. Let $B_\sigma^+$ be the part of $B(q_j, r_{\nu(n)}(1 - 5\varepsilon))$ lying strictly to the right of $Q_j$.

Given $\sigma$, define the events $A_\sigma^+$ and $A_\sigma^-$ by

$$A_\sigma^+ := \{\mathcal{Q}^{\mathrm{F}}_{\tau\nu(n+1)}(B_\sigma^+ \setminus \tilde{\sigma}_2^+) = 0\} \cap \{\mathcal{P}^{\mathrm{F}}_{\nu(n+1)}(\tilde{\sigma}_2^+ \oplus B_+(r_{\nu(n)}(1 - 3\varepsilon))) = 0\},$$
$$A_\sigma^- := \{\mathcal{Q}^{\mathrm{F}}_{\tau\nu(n+1)}(B_\sigma^- \setminus \tilde{\sigma}_2^-) = 0\} \cap \{\mathcal{P}^{\mathrm{F}}_{\nu(n+1)}(\tilde{\sigma}_2^- \oplus B_-(r_{\nu(n)}(1 - 3\varepsilon))) = 0\}.$$

See Figure 1 for an illustration of the event $A_\sigma^+$. Note that the events $A_\sigma^+$ and $A_\sigma^-$ are independent.

Suppose that $k$ is such that $Q_k \cap B_\sigma^+ \neq \varnothing$. Then, by the triangle equality,

$$\|q_k - q_j\| \leq r_{\nu(n)}(1 - 5\varepsilon) + \varepsilon r_{\nu(n)} = r_{\nu(n)}(1 - 4\varepsilon). \tag{3.7}$$

Similarly, if $Q_k \cap B_\sigma^- \neq \varnothing$ then $\|q_k - q_i\| \leq r_{\nu(n)}(1 - 4\varepsilon)$.

By our coupling of Poisson processes, for $\nu(n) \leq n' < \nu(n+1)$, we have $\mathcal{P}_{\nu(n)} \subset \mathcal{P}_{n'} \subset \mathcal{P}_{\nu(n+1)}$. Also, if $x \in Q_k$ and $y \in Q_i$ with $\|q_i - q_k\| \leq r_{\nu(n)}(1 - 3\varepsilon)$, then, by the triangle inequality and our condition on $n_1$, we have $\|x - y\| \leq r_{\nu(n)}(1 - \varepsilon) \leq r_{n'}$ for all $n' \in [\nu(n), \nu(n+1))$. Hence, by the argument at (3.7), for any $\sigma \in \mathcal{A}_{n,m}$, we have $E_\sigma \subset A_\sigma^+ \cap A_\sigma^-$.

First we prove (3.5). Take $\sigma \in \mathcal{A}^1_{\nu(n),m}$. Consider just the case where $\sigma$ is near to the left edge of $[0, 1]^2$ (the other three cases are treated similarly). If $\sigma_2^+ = \varnothing$ then $A_\sigma^+ = \{\mathcal{Q}^{\mathrm{F}}_{\tau\nu(n+1)}(B_\sigma^+) = 0\}$,

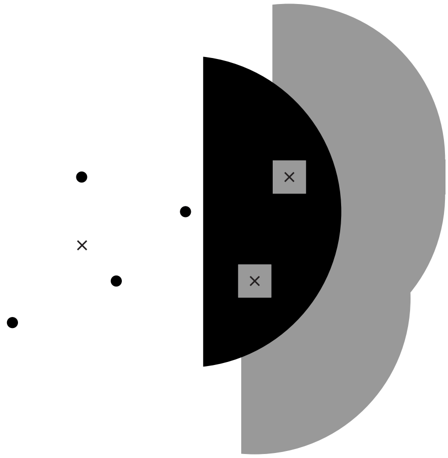

FIGURE 1: The dots are the points of $\sigma_1$, and the crosses are the points of $\sigma_2$. The grey squares are the set $\tilde{\sigma}_2^+$ (since $\varepsilon = \varepsilon_n < \frac{1}{99}$, they should really be smaller). The event $A_\sigma^+$ says that the black region contains no points of $\mathcal{Q}^{\mathrm{F}}_{\tau\nu(n+1)}$ and the grey region (partly obscured by the black region) contains no points of $\mathcal{P}^{\mathrm{F}}_{\nu(n+1)}$.

and in this case we have

$$\begin{aligned}\mathbb{P}[A_\sigma^+] &\le \exp\bigl(\tfrac{1}{2} - \tau\nu(n)(\pi(r_{\nu(n)}(1-5\varepsilon))^2 - 2\varepsilon r_{\nu(n)}^2)\bigr)\\ &\le \exp\bigl(\tfrac{1}{2} - \tau\alpha(\log\nu(n))(1-12\varepsilon)\bigr)\\ &\le \nu(n)^{-(1+\varepsilon)/2},\end{aligned} \tag{3.8}$$

where the last inequality comes from (3.2). This proves (3.5) for this case.

Suppose instead that $\sigma_2^+ \neq \varnothing$. Then $\tilde{\sigma}_2^+ \subset \{q_j\} \oplus B_+(r_{\nu(n)}(1-3\varepsilon))$, so that $v_2(\tilde{\sigma}_2^+) \le \pi r_{\nu(n)}^2(1-3\varepsilon)^2/2$. Let $s \in [0,1]$ be chosen such that $v_2(\tilde{\sigma}_2^+) = s^2\pi r_{\nu(n)}^2(1-3\varepsilon)^2/2$. Then, by the Brunn–Minkowski inequality (see, e.g. [6]),

$$v_2(\tilde{\sigma}_2^+ \oplus B_+(r_{\nu(n)}(1-3\varepsilon))) \ge \frac{\pi r_{\nu(n)}^2}{2}(1-3\varepsilon)^2(1+s)^2,$$

and also $v_2(B_\sigma^+) \ge \pi r_{\nu(n)}^2((1-5\varepsilon)^2 - 2\varepsilon)/2$, so that

$$\begin{aligned}\mathbb{P}[A_\sigma^+] &\le \exp(-\tau\nu(n)v_2(B_\sigma^+ \setminus \tilde{\sigma}_2^+) - \nu(n)v_2(\tilde{\sigma}_2^+ \oplus B_+(r_{\nu(n)}(1-3\varepsilon))))\\ &\le \exp\bigl(-\tfrac{1}{2}\nu(n)\pi r_{\nu(n)}^2[\tau((1-5\varepsilon)^2 - 2\varepsilon - s^2(1-3\varepsilon)^2) + (1+s)^2(1-3\varepsilon)^2]\bigr)\\ &\le \exp\bigl(-\tfrac{1}{2}\alpha(\log\nu(n))g_\tau(s)\bigr),\end{aligned}$$

where we set $g_\tau(s) := (\tau+1+2s)(1-12\varepsilon) + s^2(1-3\varepsilon)^2(1-\tau)$. If $\tau \le 1$ then $g_\tau(s)$ is minimised over $s \in [0,1]$ at $s = 0$. If $\tau > 1$ then $g_\tau(\cdot)$ is concave, so its minimum over $[0,1]$ is achieved at $s = 0$ or $s = 1$; also in this case $g_\tau(1) \ge (3+\tau)(1-12\varepsilon) + 1 - \tau$. Hence, using (3.2) and (3.3), we obtain

$$\begin{aligned}\mathbb{P}[A_\sigma^+] &\le \exp\bigl(-\tfrac{1}{2}\alpha(\log\nu(n))\min\{(1+\tau)(1-12\varepsilon), 4 - 12\varepsilon(3+\tau)\}\bigr)\\ &\le \nu(n)^{-(1+\varepsilon)/2},\end{aligned} \tag{3.9}$$

completing the proof of (3.5).

Now we prove (3.4). If $\sigma \in \mathcal{A}^2_{n,m}$ then $\mathbb{P}[A^+_\sigma] \le \nu(n)^{-(1+\varepsilon)/2}$ by (3.8) and (3.9), and $\mathbb{P}[A^-_\sigma] \le \nu(n)^{-(1+\varepsilon)/2}$ similarly. Therefore, $\mathbb{P}[E_\sigma] \le \mathbb{P}[A^+_\sigma \cap A^-_\sigma] \le \nu(n)^{-1-\varepsilon}$, completing the proof of (3.4).

Finally, to prove (3.6), let $\sigma \in \mathcal{A}^0_{n,m}$. Assume that $\sigma$ is near the lower-left corner of $[0,1]^2$ (the other cases are treated similarly). First suppose that $\sigma_2^+ = \varnothing$. Then $\mathbb{P}[E_\sigma] \le \mathbb{P}[\mathcal{Q}^{\mathrm{F}}_{\tau\nu(n+1)}(B^+_\sigma) = 0]$ and since the upper half of $B^+_\sigma$ is contained in $[0,1]^2$, in this case

$$\begin{aligned}\mathbb{P}[E_\sigma] &\le \exp\bigl(-\tau\nu(n)\pi r^2_{\nu(n)}\bigl[\tfrac14(1-5\varepsilon)^2 - \tfrac12\varepsilon\bigr]\bigr)\\ &\le \nu(n)^{-\alpha\tau(1-12\varepsilon)/4}\\ &\le \nu(n)^{-(1+\varepsilon)/4}.\end{aligned} \tag{3.10}$$

Now suppose that $\sigma_2^+ \ne \varnothing$. Let $q_\ell$ be the last element (in the lexicographic order) of $\sigma_2^+$. Then

$$\begin{aligned}\mathbb{P}[E_\sigma] &\le \mathbb{P}[\mathcal{P}^{\mathrm{F}}_{\nu(n+1)}(\{q_\ell\} \oplus B_+(r_{\nu(n)}(1-3\varepsilon))) = 0]\\ &\le \exp\bigl(\tfrac14 - \nu(n)\pi r^2_{\nu(n)}(1-3\varepsilon)^2\bigr)\\ &\le \nu(n)^{-\alpha(1-6\varepsilon)/4}\\ &\le \nu(n)^{-1/20},\end{aligned}$$

where, for the last inequality, we used the facts that $\alpha > \frac14$ and $\varepsilon < \frac{1}{99}$. Together with (3.10) this demonstrates (3.6).

For $m, n \in \mathbb{N}$ and $r > 0$, let $\mathcal{K}_{n,m}(r)$ be the class of bipartite point sets $(\mathcal{X}, \mathcal{Y})$ in $[0,1]^2$ such that $G(\mathcal{X}, \mathcal{Y}, r)$ has at least one component, the vertex set of which has projection onto $\mathcal{L}_n$ of order $m$ and contains at least one element of $\mathcal{X}$.

**Lemma 3.6.** *Let $m \in \mathbb{N}$. Almost surely, for all but finitely many $n \in \mathbb{N}$, we have $(\mathcal{P}^{\mathrm{F}}_{n'}, \mathcal{Q}^{\mathrm{F}}_{\tau n'}) \notin \mathcal{K}_{\nu(n),m}(r_{n'})$ for all $n' \in \mathbb{N} \cap [\nu(n), \nu(n+1))$.*

*Proof.* By Lemmas 3.3 and 3.5, for $n \ge n_1$, we have

$$\begin{aligned}\mathbb{P}\Bigl[\bigcup_{\nu(n)\le n' < \nu(n+1)} \{(\mathcal{P}^{\mathrm{F}}_n, \mathcal{Q}^{\mathrm{F}}_{\tau n}) \in \mathcal{K}_{\nu(n),m}(r_{n'})\}\Bigr] &\le \sum_{\sigma \in \mathcal{A}_{\nu(n),m}} \mathbb{P}[E_\sigma]\\ &\le |\mathcal{A}^2_{\nu(n),m}| \times \nu(n)^{-(1+\varepsilon)} + |\mathcal{A}^1_{\nu(n),m}| \times \nu(n)^{-(1+\varepsilon)/2} + |\mathcal{A}^0_{\nu(n),m}| \times \nu(n)^{-1/20}.\end{aligned}$$

Using Lemma 3.4 and the definition $\nu(n) := n^{\lceil 4/\varepsilon_0 \rceil}$, and recalling that $\varepsilon = \varepsilon_n \ge \varepsilon_0$ as described just after (3.3), this probability is $O(n^{-2})$, so it is summable in $n$. Then the result follows by the Borel–Cantelli lemma.

**Lemma 3.7.** (See [6, Lemma 9.1].) *For any two closed connected subsets $A$ and $B$ of $[0,1]^2$ with union $A \cup B = [0,1]^2$, the intersection $A \cap B$ is connected.*

Given $n \in \mathbb{N}$, let $k(n)$ be the choice of $k \in \mathbb{N}$ satisfying $\nu(k) \le n < \nu(k+1)$. Also, given $K \in \mathbb{N}$, let $F_K(n)$ be the event that $G(\mathcal{P}^{\mathrm{F}}_n, \mathcal{Q}^{\mathrm{F}}_{\tau n}, r_n)$ has two or more components with projections onto $\mathcal{L}_{\nu(k(n))}$ of order greater than $K$.

**Lemma 3.8.** *There exists $K \in \mathbb{N}$ such that, with probability 1, the event $F_K(n)$ occurs for only finitely many $n$.*

*Proof.* Suppose that $F_K(n)$ occurs. Then there exist distinct components $U = (U_1, U_2)$ and $V = (V_1, V_2)$ in $G(\mathcal{P}_n^{\mathrm{F}}, \mathcal{Q}_{\tau n}^{\mathrm{F}}, r_n)$, both with projections onto $\mathcal{L}_{\nu(k(n))}$ of order greater than $K$. Let $U'$ be the union of closed Voronoi cells in $[0, 1]^2$ (relative to $\mathcal{P}_n \cup \mathcal{Q}_{\tau n}$) of vertices of $U$, and let $V'$ be the union of closed Voronoi cells in $[0, 1]^2$ of vertices of $V$.

The interior of $U'$ and the interior of $V'$ are disjoint subsets of $[0, 1]^2$, and we now show that they are connected sets. Suppose that $X \in U_1$ and $Y \in U_2$ with $\|X - Y\| \le r_n$; then we claim that the entire line segment $[X, Y]$ is contained in the interior of $U'$. Indeed, let $z \in [X, Y]$, and suppose that $z$ lies in the closed Voronoi cell of some $W \in \mathcal{P}_n^{\mathrm{F}} \cup \mathcal{Q}_{\tau n}^{\mathrm{F}}$. If $W \in \mathcal{P}_n^{\mathrm{F}}$ then

$$\|W - Y\| \le \|W - z\| + \|z - Y\| \le \|X - z\| + \|z - Y\| = \|X - Y\| \le r_n,$$

so $W \in U$. Similarly, if $W \in \mathcal{Q}_{\tau n}^{\mathrm{F}}$ then $\|W - X\| \le r_n$, so again $W \in U$. Hence, the interior of $U'$ is connected, and likewise for $V'$.

Let $\tilde{V}$ be the closure of the component of $[0, 1]^2 \setminus U'$, containing the interior of $V'$, and let $\tilde{U}$ be the closure of $[0, 1]^2 \setminus \tilde{V}$ (essentially, this is the set obtained by filling in the holes of $U'$ that are not connected to $V'$).

Then $\tilde{U}$ and $\tilde{V}$ are closed connected sets, whose union is $[0, 1]^2$. Therefore, by Lemma 3.7, the set $\partial U := \tilde{U} \cap \tilde{V}$ is connected. Note that $\partial U$ is part of the boundary of $U'$ (it is the 'exterior boundary' of $U'$ relative to $V'$).

Let $T$ be the set of cube centres $q_i \in \mathcal{L}_{\nu(k(n))}$ such that $Q_i \cap (\partial U) \neq \varnothing$. Then $T$ is $*$-connected in $\mathcal{L}_{\nu(k(n))}$, i.e. for any $x, y \in T$, there is a path $(x_0, x_1, \dots, x_k)$ with $x_0 = x$, $x_k = y$, $x_i \in \mathcal{L}_{\nu(k(n))}$, and $\|x_i - x_{i-1}\|_\infty = \varepsilon r_{\nu(k(n))}$ for $1 \le i \le k$ (here $\varepsilon = \varepsilon_{\nu(k(n))}$).

Also, for each $q_i \in T$, we claim that $\mathcal{P}_n(Q_i)\mathcal{Q}_{\tau n}(Q_i) = 0$. Indeed, suppose on the contrary that $\mathcal{P}_n(Q_i)\mathcal{Q}_{\tau n}(Q_i) > 0$. Then all points of $(\mathcal{P}_n \cup \mathcal{Q}_{\tau n}) \cap Q_i$ lie in the same component of $G(\mathcal{P}_n^{\mathrm{F}}, \mathcal{Q}_{\tau n}^{\mathrm{F}}, r_n)$. If they are all in $U$ then $Q_i$ and all neighbouring $Q_j$ (including diagonal neighbours) are contained in $U'$. If all points of $(\mathcal{P}_n \cup \mathcal{Q}_{\tau n}) \cap Q_i$ are not in $U$ then $Q_i$ and all neighbouring $Q_j$ (including diagonal neighbours) are disjoint from $U'$. Therefore, $(\partial U) \cap Q_i = \varnothing$.

We now prove the isoperimetric inequality

$$|T| \ge \left(\frac{K}{2}\right)^{1/2}. \tag{3.11}$$

To see this, define the *width* of a nonempty closed set $A \subset [0, 1]^2$ to be the maximum difference between $x$-coordinates of points in $A$, and the *height* of $A$ to be the maximum difference between $y$-coordinates of points in $A$.

We claim that either the height or the width of $\partial U$ is at least $(K/2)^{1/2}\varepsilon r_{\nu(k(n))}$. Indeed, if not then $\partial U$ is contained in some square of side $(K/2)^{1/2}\varepsilon r_{\nu(k(n))}$, and then either $U'$ or $V'$ is contained in that square, so either $U$ or $V$ is contained in that square, contradicting the assumption that the projections of $U$ and of $V$ onto $\mathcal{L}_{\nu(k(n))}$ have order greater than $K$. For example, if the projection of $U$ has order greater than $K$ then at least one of $U_1$ and $U_2$, say $U_1$, has projection of order greater than $K/2$, and then the union of squares of side $\varepsilon r_{\nu(k(n))}$ centred at vertices in the projection of $U_1$ has total area greater than $(K/2)\varepsilon^2 r_{\nu(k(n))}^2$, so is not contained in any square of side $(K/2)^{1/2}\varepsilon r_{\nu(k(n))}$. Thus, the claim holds, and so (3.11) follows by the $*$-connectivity of $T$.

For $\nu, m \in \mathbb{N}$, let $\mathcal{A}'_{\nu,m}$ be the set of $*$-connected subsets of $\mathcal{L}_\nu$ with $m$ elements. By a similar argument as used in the proof of Lemma 3.4 (see also [6, Lemma 9.3]), there are finite

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constants $\gamma$ and $C$ such that, for all $\nu, m \in \mathbb{N}$,

$$|\mathcal{A}'_{\nu,m}| \le C\left(\frac{\nu}{\log \nu}\right)\gamma^m. \tag{3.12}$$

Set $\phi_n := \mathbb{P}[\mathcal{P}_n(Q_i)\mathcal{Q}_{\tau n}(Q_i) = 0]$; this does not depend on $i$. By the union bound and (3.1),

$$\begin{aligned}\phi_n &\le \exp(-n(\varepsilon r_{\nu(k(n))})^2) + \exp(-\tau n(\varepsilon r_{\nu(k(n))})^2)\\ &\le 2\exp\left(-(\tau \wedge 1)\varepsilon^2 \frac{\alpha}{\pi}\frac{n \log \nu(k(n))}{\nu(k(n))}\right)\\ &\le 2\nu(k(n))^{-(\tau\wedge 1)\varepsilon^2\alpha/\pi}\\ &\le 3n^{-(\tau\wedge 1)\varepsilon^2\alpha/\pi},\end{aligned}$$

where the last inequality holds for all large enough $n$. Using (3.11) and (3.12), we obtain

$$\mathbb{P}[F_K(n)] \le \sum_{m \ge (K/2)^{1/2}} C\left(\frac{\nu(k(n))}{\log \nu(k(n))}\right)\gamma^m \phi_n^m \le 2Cn(3\gamma n^{-\varepsilon^2\alpha(\tau\wedge 1)/\pi})^{(K/2)^{1/2}},$$

which is summable in $n$ provided $K$ is chosen so that $\varepsilon^2\pi^{-1}\alpha(\tau \wedge 1)(K/2)^{1/2} > 3$. The result then follows by the Borel–Cantelli lemma.

*Proof of Theorem 3.1.* Choose $K \in \mathbb{N}$ as in Lemma 3.8. Writing 'i.o.' for 'for infinitely many $n$' (i.e. infinitely often), we have

$$\mathbb{P}[\mathcal{G}^1(n, \tau, r_n) \notin \mathcal{K} \text{ i.o.}] \le \left(\sum_{m=1}^{K} \mathbb{P}[(\mathcal{P}_n^{\mathrm{F}}, \mathcal{Q}_{\tau n}^{\mathrm{F}}) \in \mathcal{K}_{\nu(k(n)),m}(r_n) \text{ i.o.}]\right) + \mathbb{P}[F_K(n) \text{ i.o.}].$$

By Lemmas 3.6 and 3.8, this is 0.

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